Introduction to Symplectic Geometry : Lecture 15

October 6, 2021

Moment map

- Definition : The action of a group G on (M, ω) is Hamiltonian if there exists a map
$$\mu : M \to \mathfrak{g}^\vee$$
satisfying
  1. $d\mu_\xi = -i_{\xi_M}\omega$ for all $\xi \in \mathfrak{g}$, where $\mu_\xi := \langle \mu, \xi \rangle : M \to \mathbb{R}$,
  2. (Equivariance) and $\mu(gm) = \mathrm{Ad}_g^* \, \mu(m)$ for all $g \in G$, $m \in M$.

$G \circlearrowright (M, \omega)$ $\quad$ $\forall g \quad M \xrightarrow{g} M, \ m \mapsto gm$ is a Ham. diff

$\Rightarrow \quad \forall \xi \in \mathfrak{g}$

$\xi_M(m) := \frac{d}{dt} e^{t\xi} m \Big|_{t=0}$

$\mu_\xi : M \to \mathbb{R}$ $\quad$ $d\mu_\xi = -i_{\xi_M}\omega$ $\longleftrightarrow$ ξ_M is a Hamiltonian vector field

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Co-adjoint action Ad^*

satisfying

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2. (Equivariance) and $\mu(gm) = \mathrm{Ad}^*_g\, \mu(m)$ for all $g \in G$, $m \in M$.

$g \in G$ $\quad \langle \mathrm{Ad}^*_g \eta, \xi \rangle := \langle \eta, \mathrm{Ad}_{g^{-1}} \xi \rangle$

$\eta \in \mathfrak{g}^\vee$

$\xi \in \mathfrak{g}$

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- Example $G = S^1$: A function $\mu : M \to \mathbb{R}$ is a moment map for an S^1-action if for the vector field $v \in \mathrm{Vect}(M)$ defined by $i_v\omega = d\mu$, the flow is 2π-periodic for all points on M.

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- Consider $(\mathbb{C}, \omega = dx \wedge dy)$, and $\mu(x, y) := \frac{1}{2}(x^2 + y^2)$. $= \frac{1}{2}|z|^2 = \frac{1}{2} r^2$

$r\,dr \wedge d\theta$

$\frac{i}{2} dz \wedge d\bar{z}$

$d\mu = r\,dr$

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- Consider $(\mathbb{C}, \omega = dx \wedge dy)$, and $\mu(x, y) := \frac{1}{2}(x^2 + y^2)$. Then $v = \partial_\theta$.

$r\,dr \wedge d\theta$

$\mu = \frac{r^2}{2}$

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- Consider $(\mathbb{C}, \omega = dx \wedge dy)$, and $\mu(x, y) := \frac{1}{2}(x^2 + y^2)$. Then $v = \partial_\theta$.
Thus μ is an S^1-moment map for the action

$$z \xrightarrow{\theta \in S^1} e^{i\theta} z.$$

Moment map

- What about the action

$$S^1 \curvearrowright \mathbb{C}, \quad z \xmapsto{\theta \in S^1} e^{ik\theta} z,$$

for some $k \in \mathbb{Z}$?

$z \overset{\mu}{\longmapsto} \frac{1}{2} k |z|^2 + c$

$d\mu = i_{v_\xi} \omega$

μ is S^1-invariant

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- Lemma : Suppose $(M_1, \omega_1, G, \mu_1)$, $(M_2, \omega_2, G, \mu_2)$ are Hamiltonian G-spaces. Then $(M_1 \times M_2, \omega_1 \oplus \omega_2, G, \mu_1 + \mu_2)$ is a Hamiltonian G-space.

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- The moment map for the diagonal action is

$$S^1 \curvearrowright \mathbb{C}^n, \quad (z_1, \ldots, z_n) \xmapsto{\theta \in S^1} e^{i\theta}(z_1 \ldots, z_n)$$

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Eg :

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is $\mu(z_1, \ldots, z_n) = \frac{1}{2}\sum_i |z_i|^2$.

Eg : $S^1 \curvearrowright \mathbb{C}^2 \quad (z_1, z_2) \xrightarrow{\theta \in S^1} (e^{i\theta} z_1, e^{2i\theta} z_2)$

$\mu(z_1, z_2) = \frac{1}{2}(|z_1|^2 + 2|z_2|^2)$

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- For any $c \in \mathbb{R}$, $\mu(z_1, \ldots, z_n) = \frac{1}{2} \sum_i |z_i|^2 + c$ is also a moment map.

Moment map

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for some $k \in \mathbb{Z}$? The moment map is $z \mapsto \frac{k}{2}|z|^2$.
- Lemma : Suppose $(M_1, \omega_1, G, \mu_1)$, $(M_2, \omega_2, G, \mu_2)$ are Hamiltonian G-spaces. Then $(M_1 \times M_2, \omega_1 \oplus \omega_2, G, \mu_1 + \mu_2)$ is a Hamiltonian G-space. Proof : Exercise.
- The moment map for the diagonal action is

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is $\mu(z_1, \ldots, z_n) = \frac{1}{2}\sum_i |z_i|^2$.
- For any $c \in \mathbb{R}$, $\mu(z_1, \ldots, z_n) = \frac{1}{2}\sum_i |z_i|^2 + c$ is also a moment map. For general G, we can add central elements $c \in \text{Lie}(Z(G))$ to the moment map without altering the moment map condition.

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- Proof : $m \in \mu^{-1}(0)$, $g \in G$ implies $\mu(gm) = \mu(m) = 0$.

$Ad_g^* 0 = 0$

Moment map

$t \mapsto e^{t\xi} m$

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- Proof : $m \in \mu^{-1}(0)$, $g \in G$ implies $\mu(gm) = \mu(m) = 0$.
- Remark : Suppose G acts freely on $\mu^{-1}(0)$. Then 0 is regular value of μ.

$gm = m \Rightarrow g = Id$

$m \in \mu^{-1}(0)$ $\quad \xi_M(m) \neq 0$ $\quad \mathfrak{g} \to T_m M$, $\xi \mapsto \xi_M(m)$ is 1-1

$G \to Gm$ is a homeomorphism?

$\xi \mapsto \frac{d}{dt} e^{t\xi} m \big|_{t=0}$

$\mathfrak{g} \to G \quad \xi \mapsto e^{t\xi}$

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$m \in \mu^{-1}(0)$ $\qquad \mathfrak{g} \longrightarrow T_m M \qquad \xi \mapsto \xi_M(m)$ is 1-1

(needs proof)

$\xi \neq 0 \qquad -i_{\xi_M}\omega = d\mu_\xi$

$\xi_M(m) \neq 0 \Rightarrow d\mu_\xi(m) \neq 0 \quad \forall \xi \in \mathfrak{g}$

$d\mu : T_m M \longrightarrow \mathfrak{g}^\vee$ is onto

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$\Downarrow$

$d\mu : T_m M \longrightarrow \mathfrak{g}^\vee$

is onto

Indeed, otherwise there would be a $\xi \in \mathfrak{g}$ for which

$d\mu_\xi(m) = 0$

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- Remark : Suppose G acts freely on $\mu^{-1}(0)$. Then 0 is regular value of μ.
- Proof : For any $x \in \mu^{-1}(0), \xi \in \mathfrak{g} \setminus \{0\}$, $d\mu_\xi(x) \neq 0$ because ..

Rmk. G acts freely on $\mu^{-1}(0)$
Then 0 is a regular value of μ

and so, $\mu^{-1}(0)$ is a smooth manifold

$G \circlearrowright \mu^{-1}(0)$ free G compact

$\mu^{-1}(0)/G$ is a manifold

Symplectic Quotients

- Theorem (Marsden-Weinstein-Meyer) : Let (M, ω, G, μ) be a Hamiltonian G-space. Suppose G is compact and acts freely on $\mu^{-1}(0)$. Then
  - $\overline{M} := \mu^{-1}(0)/G$ is a manifold, and $\pi : \mu^{-1}(0) \to \overline{M}$ is a principal G-bundle.
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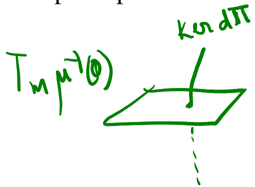
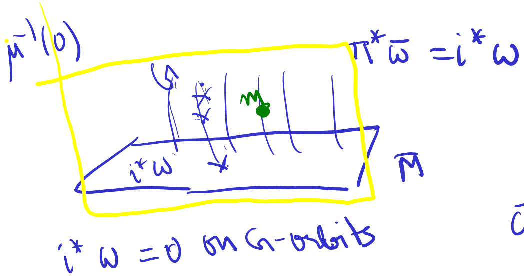
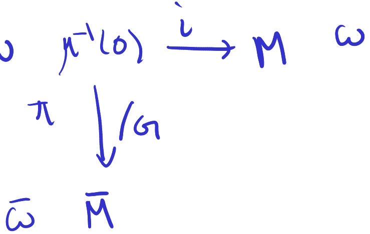
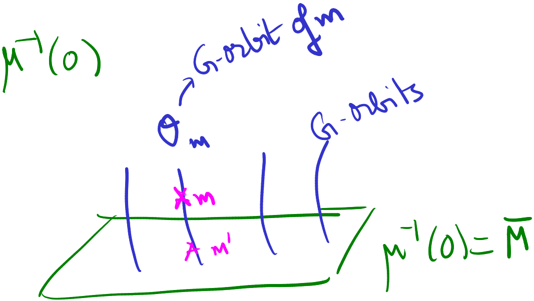
where $i : \mu^{-1}(0) \to M$ is the inclusion map.

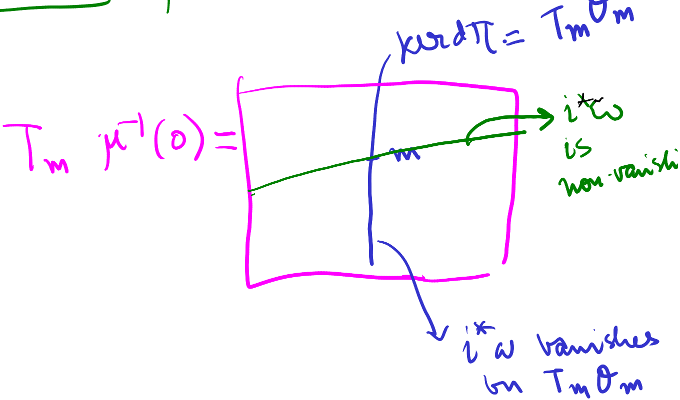








$$\mu^{-1}(0) \xrightarrow{\ \pi\ } M$$



Symplectic Quotients

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with moment map $\mu(z_1, \dots, z_n) = \frac{1}{2}\sum_i |z_i|^2 - \frac{1}{2}$.

Rmk $\mu(z_1, \dots z_n) = \frac{1}{2}\sum |z_i|^2 \to$ 0 is not a reg. value

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  with moment map $\mu(z_1, \ldots, z_n) = \frac{1}{2} \sum_i |z_i|^2 - \frac{1}{2}$. The quotient is $\mathbb{P}^{n-1}$. We have thus showed that the complex projective space has a symplectic form.

$\mu : \mathbb{C}^n \longrightarrow \mathbb{R}$

$$\mu(z_1 \dots z_n) = \frac{1}{2}\sum_i |z_i|^2 - \frac{1}{2}$$

$$\mu^{-1}(0) = \{ \underline{z} \in \mathbb{C}^n \mid \sum_i |z_i|^2 = 1 \}$$

$$\cong S^{2n-1} \subseteq \mathbb{C}^n$$

$$\mu^{-1}(0)/S^1 \cong S^{2n-1}/S^1 = \underline{\mathbb{P}^{n-1}}$$

$\mathbb{P}^{n-1}$ = set of lines through the origin in $\mathbb{C}^n$

A line is $\mathbb{C} \ni \lambda \mapsto \lambda(z_1, \dots, z_n)$

S^{2n-1} intersects every line in a circle and this circle is precisely an S^1-orbit.

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- Theorem (Noether) : A Hamiltonian function $H : M \to \mathbb{R}$ is G-invariant if and only if μ is constant on the trajectories of the Hamiltonian vector field v_H.

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- Noether's theorem implies that one can study the dynamics of v_H on the symplectic quotient $\mu^{-1}(c)/G$.

Proof of Marsden-Weinstein-Meyer

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$\mathcal{O}_p \simeq G$

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Proof of Marsden-Weinstein-Meyer

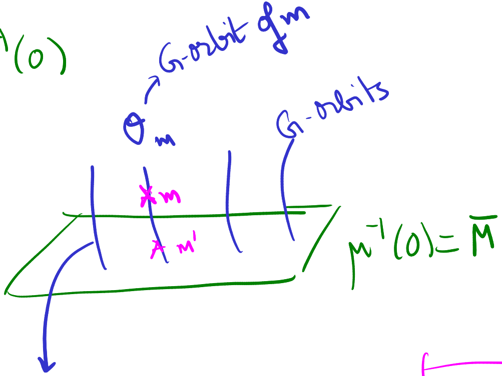
- For any $p \in M$, let $\mathcal{O}_p \subset M$ be the G-orbit of p. If the G-action on p is free, then $T_p\mathcal{O}_p \simeq \mathfrak{g}$.
- Since G is a compact group acting freely on $\mu^{-1}(0)$, the quotient $\mu^{-1}(0)/G$ is a manifold. For $p \in \mu^{-1}(0)$,

$[p] \in \mu^{-1}(0)/G$, $T_{[p]}\,\mu^{-1}(0)/G = T_p\mu^{-1}(0)/T_p\mathcal{O}_p$

$\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$

$T_p\mathcal{O}_p \subseteq \ker d\mu_p$

$\mu^{-1}(0)$

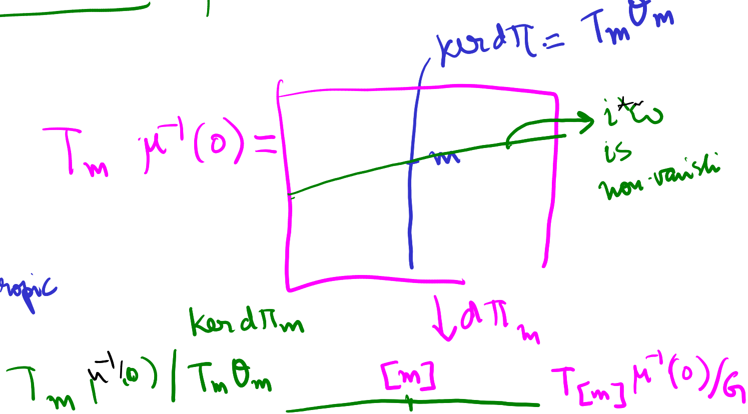


$$\mu^{-1}(0) \xrightarrow{\ \pi\ } M$$

$i^*\omega$ vanishes on $\mathcal{O}_m$

$\Updownarrow$

$T_m\mathcal{O}_m$ is isotropic



Theorem

Let G be a compact Lie group acting freely on a manifold M. Then, M/G is a manifold and

$M \xrightarrow{\pi} M/G$ is a principal G-bundle.

And for any $m \in M$, $[m] := \pi(m) \in M/G$

$$T_{[m]} M/G = T_m M / T_m \mathcal{O}_m$$

$$\text{or } T_m M / \operatorname{Ker} d\pi_m$$

Proof of Marsden-Weinstein-Meyer

- For any $p \in M$, let $\mathcal{O}_p \subset M$ be the G-orbit of p. If the G-action on p is free, then $T_p\mathcal{O}_p \simeq \mathfrak{g}$.
- Since G is a compact group acting freely on $\mu^{-1}(0)$, the quotient $\mu^{-1}(0)/G$ is a manifold. For $p \in \mu^{-1}(0)$,

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- Remark: For $p \in \mu^{-1}(0)$,
  - $T_p\mathcal{O}_p \subset T_p\mu^{-1}(0)$, and

$\mu^{-1}(0)$ is G-invariant
$\mathcal{O}_p \subseteq \mu^{-1}(0)$
$T_p\mathcal{O}_p \subset T_p\mu^{-1}(0)$
$\forall p \in \mu^{-1}(0)$

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- Remark: For $p \in \mu^{-1}(0)$,
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  - $(T_p\mathcal{O}_p)^\omega = T_p\mu^{-1}(0)$.

$T_p\mathcal{O}_p = \{\xi_M(p) : \xi \in \mathfrak{g}\}$

$\omega(\xi_M, v) = 0 = d\mu_\xi(v) = 0$

$v \in (T_p\mathcal{O}_p)^\omega \iff d\mu_\xi(v) = 0 \quad \forall \xi \in \mathfrak{g}$

$\iff d\mu(v) = 0 \iff v \in \operatorname{Ker} d\mu$

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  So, $\mathcal{O}_p$ is isotropic and $\mu^{-1}(0)$ is co-isotropic.

$W \subseteq (V, \omega)$

$W \subseteq W^\omega \Leftrightarrow W$ is isotropic

W^ω is coisotropic

$\because (W^\omega)^\omega = W \subseteq W^\omega$

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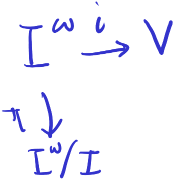

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- Lemma : Let (V, ω) be a symplectic vector space, and $I \subset V$ be isotropic. Then ω descends to a symplectic form $\overline{\omega}$ on I^ω/I, that satisfies $i^*\omega = \pi^*\overline{\omega}$ where $i : I^\omega \to V$ is the inclusion map and $\pi : I^\omega \to I^\omega/I$ is the projection map.

$\mathcal{U}$: contractible

$\bigcup_{t\in[0,1]} (E, J_t) \to \mathcal{U}$

$(E, J_t) \to \Sigma$
$t \in [0,1]$
is a family of $\mathbb{C}$-vector bundles

$\mathcal{E} \to \Sigma \times [0,1]$

Choose a trivialization

$$\mathcal{E} \simeq U \times \mathbb{R}^{2n} \times [0,1]$$

Think of $\bigcup_{t\in[0,1]} (E, J_t) \to M$ as a complex vector bundle $\mathcal{E}$ on $U \times [0,1]$

$$\mathcal{E} \to M \times [0,1]$$

$$\mathcal{E}|_{U \times \{t\}} \simeq (E, J_t)$$