Introduction to Symplectic Geometry : Lecture 14

October 4, 2021

Recall : Hamiltonian group actions on symplectic manifolds

- Suppose a Lie group G acts smoothly on a manifold M.

Recall : Hamiltonian group actions on symplectic manifolds

- Suppose a Lie group G acts smoothly on a manifold M. The action is 'generated' by the vector fields $\xi_M \in \mathrm{Vect}(M)$, $\xi \in \mathfrak{g}$

$$\xi_M(m) := \frac{d}{dt} e^{t\xi} m|_{t=0}.$$

Recall : Hamiltonian group actions on symplectic manifolds

- Suppose a Lie group G acts smoothly on a manifold M. The action is 'generated' by the vector fields $\xi_M \in \mathrm{Vect}(M)$, $\xi \in \mathfrak{g}$

$$\xi_M(m) := \frac{d}{dt} e^{t\xi} m|_{t=0}.$$

- The map

$$\mathfrak{g} \to \mathrm{Vect}(M), \quad \xi \mapsto \xi_M$$

is $\mathbb{R}$-linear

Recall : Hamiltonian group actions on symplectic manifolds

- Suppose a Lie group G acts smoothly on a manifold M. The action is 'generated' by the vector fields $\xi_M \in \mathrm{Vect}(M)$, $\xi \in \mathfrak{g}$

$$\xi_M(m) := \frac{d}{dt} e^{t\xi} m|_{t=0}.$$

- The map

$$\mathfrak{g} \to \mathrm{Vect}(M), \quad \xi \mapsto \xi_M$$

is $\mathbb{R}$-linear and G-equivariant, i.e. $(\mathrm{Ad}_g \xi)_M = dL_g(\xi_M)$.

Recall : Hamiltonian group actions on symplectic manifolds

- Suppose a Lie group G acts smoothly on a manifold M. The action is 'generated' by the vector fields $\xi_M \in \mathrm{Vect}(M)$, $\xi \in \mathfrak{g}$

$$\xi_M(m) := \frac{d}{dt} e^{t\xi} m|_{t=0}.$$

- The map

$$\mathfrak{g} \to \mathrm{Vect}(M), \quad \xi \mapsto \xi_M$$

is $\mathbb{R}$-linear and G-equivariant, i.e. $(\mathrm{Ad}_g\, \xi)_M = dL_g(\xi_M)$.
- Now suppose M is a symplectic manifold and G acts via Hamiltonian diffeomorphisms. Also assume $G \subset GL(n, \mathbb{R})$ and G is compact.

Recall : Hamiltonian group actions on symplectic manifolds

- Suppose a Lie group G acts smoothly on a manifold M. The action is 'generated' by the vector fields $\xi_M \in \mathrm{Vect}(M)$, $\xi \in \mathfrak{g}$

$$\xi_M(m) := \frac{d}{dt} e^{t\xi} m|_{t=0}.$$

- The map

$$\mathfrak{g} \to \mathrm{Vect}(M), \quad \xi \mapsto \xi_M$$

is $\mathbb{R}$-linear and G-equivariant, i.e. $(\mathrm{Ad}_g\, \xi)_M = dL_g(\xi_M)$.
- Now suppose M is a symplectic manifold and G acts via Hamiltonian diffeomorphisms. Also assume $G \subset GL(n, \mathbb{R})$ and G is compact.
- Then, for any $\xi \in \mathfrak{g}$, $i_{\xi_M}\omega$ is exact, so there exists $\mu_\xi : M \to \mathbb{R}$ such that $d\mu_\xi = -i_{\xi_M}\omega$.

Recall : Hamiltonian group actions on symplectic manifolds

- Suppose a Lie group G acts smoothly on a manifold M. The action is 'generated' by the vector fields $\xi_M \in \mathrm{Vect}(M)$, $\xi \in \mathfrak{g}$

$$\xi_M(m) := \frac{d}{dt} e^{t\xi} m|_{t=0}.$$

- The map

$$\mathfrak{g} \to \mathrm{Vect}(M), \quad \xi \mapsto \xi_M$$

is $\mathbb{R}$-linear and G-equivariant, i.e. $(\mathrm{Ad}_g\, \xi)_M = dL_g(\xi_M)$.
- Now suppose M is a symplectic manifold and G acts via Hamiltonian diffeomorphisms. Also assume $G \subset GL(n, \mathbb{R})$ and G is compact.
- Then, for any $\xi \in \mathfrak{g}$, $i_{\xi_M}\omega$ is exact, so there exists $\mu_\xi : M \to \mathbb{R}$ such that $d\mu_\xi = -i_{\xi_M}\omega$.
- It is reasonable to require that the map

$$\mathfrak{g} \to C^\infty(M), \xi \mapsto \mu_\xi$$

is linear and equivariant.

Hamiltonian group actions on symplectic manifolds

- Now suppose M is a symplectic manifold and G acts via Hamiltonian diffeomorphisms.

Hamiltonian group actions on symplectic manifolds

- Now suppose M is a symplectic manifold and G acts via Hamiltonian diffeomorphisms.
- Then, for any $\xi \in \mathfrak{g}$, $i_{\xi_M}\omega$ is exact, so there exists $\mu_\xi : M \to \mathbb{R}$ such that $d\mu_\xi = -i_{\xi_M}\omega$.

Hamiltonian group actions on symplectic manifolds

- Now suppose M is a symplectic manifold and G acts via Hamiltonian diffeomorphisms.
- Then, for any $\xi \in \mathfrak{g}$, $i_{\xi_M}\omega$ is exact, so there exists $\mu_\xi : M \to \mathbb{R}$ such that $d\mu_\xi = -i_{\xi_M}\omega$.
- It is reasonable to require that the map

$$\mathfrak{g} \to C^\infty(M), \quad \xi \mapsto \mu_\xi \tag{1}$$

is linear and equivariant.

Hamiltonian group actions on symplectic manifolds

- Now suppose M is a symplectic manifold and G acts via Hamiltonian diffeomorphisms.
- Then, for any $\xi \in \mathfrak{g}$, $i_{\xi_M}\omega$ is exact, so there exists $\mu_\xi : M \to \mathbb{R}$ such that $d\mu_\xi = -i_{\xi_M}\omega$.
- It is reasonable to require that the map

$$\mathfrak{g} \to C^\infty(M), \quad \xi \mapsto \mu_\xi \tag{1}$$

is linear and equivariant. (Equivariance means $\mu_{Ad_g \xi} = \mu_\xi \circ L_{g^{-1}}$.)

Hamiltonian group actions on symplectic manifolds

- Now suppose M is a symplectic manifold and G acts via Hamiltonian diffeomorphisms.
- Then, for any $\xi \in \mathfrak{g}$, $i_{\xi_M}\omega$ is exact, so there exists $\mu_\xi : M \to \mathbb{R}$ such that $d\mu_\xi = -i_{\xi_M}\omega$.
- It is reasonable to require that the map

$$\mathfrak{g} \to C^\infty(M), \quad \xi \mapsto \mu_\xi \tag{1}$$

  is linear and equivariant. (Equivariance means $\mu_{Ad_g \xi} = \mu_\xi \circ L_{g^{-1}}$.)
- Linearity of (1) implies that we may rewrite (1) as

$$\mu : M \to \mathfrak{g}^\vee, \quad \text{such that} \quad \mu_\xi = \langle \mu, \xi \rangle.$$

Hamiltonian group actions on symplectic manifolds

- Now suppose M is a symplectic manifold and G acts via Hamiltonian diffeomorphisms.
- Then, for any $\xi \in \mathfrak{g}$, $i_{\xi_M}\omega$ is exact, so there exists $\mu_\xi : M \to \mathbb{R}$ such that $d\mu_\xi = -i_{\xi_M}\omega$.
- It is reasonable to require that the map

$$\mathfrak{g} \to C^\infty(M), \quad \xi \mapsto \mu_\xi \tag{1}$$

is linear and equivariant. (Equivariance means $\mu_{Ad_g\xi} = \mu_\xi \circ L_{g^{-1}}$.)
- Linearity of (1) implies that we may rewrite (1) as

$$\mu : M \to \mathfrak{g}^\vee, \quad \text{such that} \quad \mu_\xi = \langle \mu, \xi \rangle.$$

Equivariance of (1) implies $\mu(gm) = \mathrm{Ad}_g^* \, \mu(m)$, where

$$\langle \mathrm{Ad}_g^* \, \phi, \xi \rangle := \langle \phi, \mathrm{Ad}_{g^{-1}} \xi \rangle, \quad \forall \phi \in \mathfrak{g}^\vee, \xi \in \mathfrak{g}.$$

Moment map

- Definition : The action of a group G on (M, ω) is Hamiltonian if there exists a map
$$\mu : M \to \mathfrak{g}^{\vee}$$
satisfying
  1. $d\mu_{\xi} = -i_{\xi_M}\omega$ for all $\xi \in \mathfrak{g}$, where $\mu_{\xi} := \langle \mu, \xi \rangle : M \to \mathbb{R}$,
  2. (Equivariance) and $\mu(gm) = \mathrm{Ad}_g^* \mu(m)$ for all $g \in G$, $m \in M$.

Moment map

- Definition : The action of a group G on (M, ω) is Hamiltonian if there exists a map
$$\mu : M \to \mathfrak{g}^\vee$$
satisfying
  1. $d\mu_\xi = -i_{\xi_M}\omega$ for all $\xi \in \mathfrak{g}$, where $\mu_\xi := \langle \mu, \xi \rangle : M \to \mathbb{R}$,
  2. (Equivariance) and $\mu(gm) = \mathrm{Ad}_g^* \, \mu(m)$ for all $g \in G$, $m \in M$.

  μ is known as the **moment map** generating the Hamiltonian action, and (M, ω, G, μ) is called a Hamiltonian G-space.

Moment map

- Definition : The action of a group G on (M, ω) is Hamiltonian if there exists a map
$$\mu : M \to \mathfrak{g}^\vee$$
satisfying
  1. $d\mu_\xi = -i_{\xi_M}\omega$ for all $\xi \in \mathfrak{g}$, where $\mu_\xi := \langle \mu, \xi \rangle : M \to \mathbb{R}$,
  2. (Equivariance) and $\mu(gm) = \mathrm{Ad}_g^* \mu(m)$ for all $g \in G$, $m \in M$.

  μ is known as the **moment map** generating the Hamiltonian action, and (M, ω, G, μ) is called a Hamiltonian G-space.
- If G is Abelian, the equivariance is same as invariance : $\mu(gm) = \mu(m)$.

Moment map

- Definition : The action of a group G on (M, ω) is Hamiltonian if there exists a map
$$\mu : M \to \mathfrak{g}^\vee$$
satisfying
  1. $d\mu_\xi = -i_{\xi_M}\omega$ for all $\xi \in \mathfrak{g}$, where $\mu_\xi := \langle \mu, \xi \rangle : M \to \mathbb{R}$,
  2. (Equivariance) and $\mu(gm) = \mathrm{Ad}_g^* \, \mu(m)$ for all $g \in G$, $m \in M$.

  μ is known as the **moment map** generating the Hamiltonian action, and (M, ω, G, μ) is called a Hamiltonian G-space.
- If G is Abelian, the equivariance is same as invariance : $\mu(gm) = \mu(m)$.
- Example $G = S^1$: A function $\mu : M \to \mathbb{R}$ is a moment map for an S^1-action if for the vector field $v \in \mathrm{Vect}(M)$ defined by $i_v\omega = d\mu$, the flow is 2π-periodic for all points on M.

Moment map

- Definition : The action of a group G on (M, ω) is Hamiltonian if there exists a map
$$\mu : M \to \mathfrak{g}^\vee$$
satisfying
    1. $d\mu_\xi = -i_{\xi_M}\omega$ for all $\xi \in \mathfrak{g}$, where $\mu_\xi := \langle \mu, \xi \rangle : M \to \mathbb{R}$,
    2. (Equivariance) and $\mu(gm) = \mathrm{Ad}_g^* \mu(m)$ for all $g \in G$, $m \in M$.

  μ is known as the **moment map** generating the Hamiltonian action, and (M, ω, G, μ) is called a Hamiltonian G-space.
- If G is Abelian, the equivariance is same as invariance : $\mu(gm) = \mu(m)$.
- Example $G = S^1$: A function $\mu : M \to \mathbb{R}$ is a moment map for an S^1-action if for the vector field $v \in \mathrm{Vect}(M)$ defined by $i_v\omega = d\mu$, the flow is 2π-periodic for all points on M.
- Consider $(\mathbb{C}, \omega = dx \wedge dy)$, and $\mu(x, y) := \frac{1}{2}(x^2 + y^2)$.

Moment map

- Definition : The action of a group G on (M, ω) is Hamiltonian if there exists a map
$$\mu : M \to \mathfrak{g}^\vee$$
satisfying
  1. $d\mu_\xi = -i_{\xi_M}\omega$ for all $\xi \in \mathfrak{g}$, where $\mu_\xi := \langle \mu, \xi \rangle : M \to \mathbb{R}$,
  2. (Equivariance) and $\mu(gm) = \mathrm{Ad}_g^* \mu(m)$ for all $g \in G$, $m \in M$.

  μ is known as the **moment map** generating the Hamiltonian action, and (M, ω, G, μ) is called a Hamiltonian G-space.
- If G is Abelian, the equivariance is same as invariance : $\mu(gm) = \mu(m)$.
- Example $G = S^1$: A function $\mu : M \to \mathbb{R}$ is a moment map for an S^1-action if for the vector field $v \in \mathrm{Vect}(M)$ defined by $i_v\omega = d\mu$, the flow is 2π-periodic for all points on M.
- Consider $(\mathbb{C}, \omega = dx \wedge dy)$, and $\mu(x, y) := \frac{1}{2}(x^2 + y^2)$. Then $v = \partial_\theta$.

Moment map

- Definition : The action of a group G on (M, ω) is Hamiltonian if there exists a map
$$\mu : M \to \mathfrak{g}^{\vee}$$
satisfying
  1. $d\mu_\xi = -i_{\xi_M}\omega$ for all $\xi \in \mathfrak{g}$, where $\mu_\xi := \langle \mu, \xi \rangle : M \to \mathbb{R}$,
  2. (Equivariance) and $\mu(gm) = \mathrm{Ad}_g^* \, \mu(m)$ for all $g \in G$, $m \in M$.

  μ is known as the **moment map** generating the Hamiltonian action, and (M, ω, G, μ) is called a Hamiltonian G-space.
- If G is Abelian, the equivariance is same as invariance : $\mu(gm) = \mu(m)$.
- Example $G = S^1$: A function $\mu : M \to \mathbb{R}$ is a moment map for an S^1-action if for the vector field $v \in \mathrm{Vect}(M)$ defined by $i_v\omega = d\mu$, the flow is 2π-periodic for all points on M.
- Consider $(\mathbb{C}, \omega = dx \wedge dy)$, and $\mu(x, y) := \frac{1}{2}(x^2 + y^2)$. Then $v = \partial_\theta$. Thus μ is an S^1-moment map for the action
$$z \xrightarrow{\theta \in S^1} e^{i\theta} z.$$

Moment map

- What about the action

$$S^1 \curvearrowright \mathbb{C}, \quad z \xmapsto{\theta \in S^1} e^{ik\theta} z,$$

for some $k \in \mathbb{Z}$?

Moment map

- What about the action

$$S^1 \curvearrowright \mathbb{C}, \quad z \xmapsto{\theta \in S^1} e^{ik\theta} z,$$

  for some $k \in \mathbb{Z}$? The moment map is $z \mapsto \frac{k}{2}|z|^2$.
- Lemma : Suppose $(M_1, \omega_1, G, \mu_1)$, $(M_2, \omega_2, G, \mu_2)$ are Hamiltonian G-spaces. Then $(M_1 \times M_2, \omega_1 \oplus \omega_2, G, \mu_1 + \mu_2)$ is a Hamiltonian G-space.

Moment map

- What about the action

$$S^1 \curvearrowright \mathbb{C}, \quad z \xmapsto{\theta \in S^1} e^{ik\theta} z,$$

for some $k \in \mathbb{Z}$? The moment map is $z \mapsto \frac{k}{2}|z|^2$.

- Lemma : Suppose $(M_1, \omega_1, G, \mu_1)$, $(M_2, \omega_2, G, \mu_2)$ are Hamiltonian G-spaces. Then $(M_1 \times M_2, \omega_1 \oplus \omega_2, G, \mu_1 + \mu_2)$ is a Hamiltonian G-space. Proof : Exercise.

Moment map

- What about the action

$$S^1 \curvearrowright \mathbb{C}, \quad z \overset{\theta \in S^1}{\longmapsto} e^{ik\theta} z,$$

  for some $k \in \mathbb{Z}$? The moment map is $z \mapsto \frac{k}{2}|z|^2$.
- Lemma : Suppose $(M_1, \omega_1, G, \mu_1)$, $(M_2, \omega_2, G, \mu_2)$ are Hamiltonian G-spaces. Then $(M_1 \times M_2, \omega_1 \oplus \omega_2, G, \mu_1 + \mu_2)$ is a Hamiltonian G-space. Proof : Exercise.
- The moment map for the diagonal action is

$$S^1 \curvearrowright \mathbb{C}^n, \quad (z_1, \ldots, z_n) \overset{\theta \in S^1}{\longmapsto} e^{i\theta}(z_1 \ldots, z_n)$$

  is

Moment map

- What about the action

$$S^1 \curvearrowright \mathbb{C}, \quad z \xmapsto{\theta \in S^1} e^{ik\theta} z,$$

for some $k \in \mathbb{Z}$? The moment map is $z \mapsto \frac{k}{2}|z|^2$.

- Lemma : Suppose $(M_1, \omega_1, G, \mu_1)$, $(M_2, \omega_2, G, \mu_2)$ are Hamiltonian G-spaces. Then $(M_1 \times M_2, \omega_1 \oplus \omega_2, G, \mu_1 + \mu_2)$ is a Hamiltonian G-space. Proof : Exercise.
- The moment map for the diagonal action is

$$S^1 \curvearrowright \mathbb{C}^n, \quad (z_1, \dots, z_n) \xmapsto{\theta \in S^1} e^{i\theta}(z_1 \dots, z_n)$$

is $\mu(z_1, \dots, z_n) = \frac{1}{2} \sum_i |z_i|^2$.