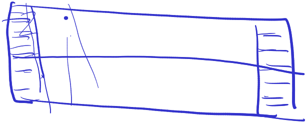
Introduction to Symplectic Geometry : Lecture 14

October 4, 2021

$V =$ 

$\Phi_{UV} = \mathrm{Id}$ on $U \cap V$

$\phi := \Phi_{UV}^{-1}$

on $U \cap V$

$E|_V \simeq V \times \mathbb{C}^n$

$\downarrow$

To change triv

$\phi : V \to \underline{GL(n, \mathbb{C})}$

$\Phi_{UV} \xrightarrow{\text{replace by}} \Phi_{UV}\,\phi^{-1}$

$= \Phi_{UV}$

Recall : Hamiltonian group actions on symplectic manifolds

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is $\mathbb{R}$-linear

Recall : Hamiltonian group actions on symplectic manifolds

$G \subseteq GL(n, \mathbb{R})$, G compact.

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is $\mathbb{R}$-linear and G-equivariant, i.e. $(\mathrm{Ad}_g\, \xi)_M = dL_g(\xi_M)$.

$t \mapsto e^{t\xi}$ Group homo

$e^{t\xi} := 1 + \xi + \frac{\xi^2}{2!} + \dots$

1-parameter subgroup

with $\frac{d}{dt} e^{t\xi}\Big|_{t=0} = \xi.$

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$\mathrm{Ad}_g \xi = ?$

$g \in G$

$c_g : G \longrightarrow G$

$h \mapsto g h g^{-1}$

$c_g(Id) = Id$

$\xi \in \mathfrak{g}$

As matrices

$g \in GL(n, \mathbb{R})$ $\xi \in \mathfrak{gl}(n, \mathbb{R})$

$\mathrm{Ad}_g \xi = g \xi g^{-1}$

$\mathrm{Ad}_g := dc_g : \underbrace{T_{Id} G}_{\simeq \mathfrak{g}} \longrightarrow \underbrace{T_{Id} G}_{\simeq \mathfrak{g}}$

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$$\big(dL_g(\xi_M)\big)_{gm}$$

$$dL_g : T_m M \to T_{gm} M$$
$$\xi_M(m) \mapsto dL_g(\xi_M(m))$$

G-action on M induces

$$G \circlearrowright \mathrm{Vect}(M) \qquad X \xmapsto{g \in G} dL_g(X)$$

$$\left(dL_g(\xi_M)\right)_{gm}$$

$$dL_g : T_m M \longrightarrow T_{gm} M$$

$$\xi_M(m) \mapsto \underline{dL_g(\xi_M(m))}$$

$$\xi \in \mathfrak{g}$$

$$\frac{d}{dt} e^{t\xi} m \Big|_{t=0} \longleftrightarrow \xi_M(m) \xmapsto{\ dL_g\ } \frac{d}{dt} g e^{t\xi} m \Big|_{t=0}$$

$$= \frac{d}{dt} (g e^{t\xi} g^{-1})(gm) \Big|_{t=0}$$

$$= \frac{d}{dt} \left(e^{Ad_g t\xi}\right)(gm) \Big|_{t=0}$$

$$= (Ad_g \xi)_M (gm)$$

<u>Rmk</u>: $\boxed{e^{Ad_g \xi} = g(e^{\xi})g^{-1}}$

$$LHS = 1 + g\xi g^{-1} + \frac{(g\xi g^{-1})^2}{2!} + \cdots$$

$$= g\left(1 + \xi + \frac{\xi^2}{2!} + \cdots\right) g^{-1} = g e^{\xi} g^{-1}$$

MATRIX DEF of $\exp : \mathfrak{g} \longrightarrow G$

$\xi \in \mathfrak{g}$

$$\boxed{e^{\xi} = 1 + \xi + \frac{\xi^2}{2!} + \cdots\cdots} \in G$$

Def of exp via flow

$\xi \in T_{Id} G$ $\quad$ $\xi_G \in \mathrm{Vect}(G)$ $\rightarrow$ Left-invariant

$c : (-\varepsilon, \varepsilon) \to G$ $\quad \rightsquigarrow$ Integral curve of ξ_G

$c(0) = Id \quad c'(0) = \xi$

$$e^{t\xi} := c(t)$$

$\xi \mapsto Ad_g \xi$ $\qquad\qquad$ $c(t) \longmapsto g\, c(t)\, g^{-1}$

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- It is reasonable to require that the map

$$\mathfrak{g} \to C^\infty(M), \xi \mapsto \mu_\xi$$

is linear and equivariant.

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$\mathfrak{g}^\vee := \text{dual of } \mathfrak{g}$

$$\mu : M \to \mathfrak{g}^\vee, \quad \text{such that} \quad \underline{\underline{\mu_\xi}} = \langle \mu, \xi \rangle : M \to \mathbb{R}$$

$\xi \in \mathfrak{g}$

$m \mapsto \langle \mu(m), \xi \rangle$

Equivariance of (1) implies $\mu(gm) = \operatorname{Ad}_g^* \mu(m)$, where

$$\langle \operatorname{Ad}_g^* \phi, \xi \rangle := \langle \phi, \operatorname{Ad}_{g^{-1}} \xi \rangle, \quad \forall \phi \in \mathfrak{g}^\vee, \xi \in \mathfrak{g}.$$

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Suppose G acts on (M, ω) via Hamiltonian diffeomorphisms.

i.e. $L_g : M \longrightarrow M$ $\quad g \in G$
$\qquad m \longmapsto gm$

is a Hamiltonian diffeomorphism.

Then it is 'natural to expect' that the action has a moment map.

Moment map

- Definition : The action of a group G on (M, ω) is Hamiltonian if there exists a map
$$\mu : M \to \mathfrak{g}^\vee$$
satisfying

(Hamiltonian)

1. $d\mu_\xi = -i_{\xi_M}\omega$ for all $\xi \in \mathfrak{g}$, where $\mu_\xi := \langle \mu, \xi \rangle : M \to \mathbb{R}$,
2. (Equivariance) and $\mu(gm) = \mathrm{Ad}_g^* \mu(m)$ for all $g \in G$, $m \in M$.

$\mu_\xi := \langle \mu, \xi \rangle : M \longrightarrow \mathbb{R}$ $\rightsquigarrow$ Hamiltonian function whose vector field is ξ_M

$$\begin{array}{ccc} \mu : M & \longrightarrow & \mathfrak{g}^\vee \\ \circlearrowleft & & \circlearrowleft \ \mathrm{Ad}^* \\ G & & G \end{array}$$

$\rightarrow$ Dualization of Ad

Moment map

$L : V \to W \qquad L^* : W^* \longrightarrow V^*$

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  2. (Equivariance) and $\mu(gm) = \mathrm{Ad}_g^* \mu(m)$ for all $g \in G$, $m \in M$.

$g \in G \qquad \xi \in \mathfrak{g}$

$\eta \in \mathfrak{g}^\vee \qquad \langle \underbrace{\mathrm{Ad}_g^* \eta}_{\in \, \mathfrak{g}^\vee}, \xi \rangle := \langle \eta, \mathrm{Ad}_{g^{-1}} \xi \rangle$

$G \overset{\mathrm{Ad}^*}{\curvearrowright} \mathfrak{g}^\vee$ is a left action. i.e. $\mathrm{Ad}_{gh}^* = \mathrm{Ad}_g^* \circ \mathrm{Ad}_h^*$

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- If G is Abelian, the equivariance is same as invariance : $\mu(gm) = \mu(m)$.

$\mathrm{Ad}_g = \mathrm{Id} \quad \forall g \in G$

μ is G-invariant

OR μ is const on G orbits

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When is $\mu : M \to \mathbb{R}$ a moment map for an S^1-action

Under what condition does \mu generate a Hamiltonian S^1-action?

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Given $\mu : M \to \mathbb{R}$, let ξ be a generator of $\mathrm{Lie}(S^1) \simeq \mathfrak{g}$

$d\mu_\xi = -i_{\xi_M}\omega \to$ we get ξ_M from this equation

ξ_M generates an S^1-action if time 2π flow $= \mathrm{Id}_M$

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Is μ constt on S^1-orbits?

$$\underbrace{L_{\xi_M}\mu_\xi}_{\parallel} = 0 \,? \qquad \text{OR} \qquad \xi_M(\mu_\xi) = 0 \,?$$

$$i_{\xi_M} d\mu_\xi = i_{\xi_M}(-i_{\xi_M}\omega)$$
$$= -\omega(\xi_M, \xi_M) = 0$$

H : Hamiltonian fn

v_H : v.f. $\qquad -i_{v_H}\omega = dH$

$$L_{v_H} H = 0$$

$$\|$$

$$i_{v_H} dH$$

$$\|$$

$$\omega(v_H, v_H)$$

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- If G is Abelian, the equivariance is same as invariance : $\mu(gm) = \mu(m)$.
- Example $G = S^1$: A function $\mu : M \to \mathbb{R}$ is a moment map for an S^1-action if for the vector field $v \in \mathrm{Vect}(M)$ defined by $i_v\omega = d\mu$, the flow is 2π-periodic for all points on M.

$\mathrm{Lie}(S^1) \simeq i\mathbb{R} \simeq \mathbb{R} \simeq \mathfrak{g}$

$\mathfrak{g}^\vee \simeq \mathbb{R}$

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- Consider $(\mathbb{C}, \omega = dx \wedge dy)$, and $\mu(x, y) := \frac{1}{2}(x^2 + y^2)$. $= \frac{r^2}{2}$ $\quad d\mu = r\,dr$

$d\mu = -i_v\omega \qquad v = -x\,\partial_y + y\,\partial_x = \partial_\theta$

$\omega = r\,dr \wedge d\theta \qquad v = \partial_\theta$

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- Consider $(\mathbb{C}, \omega = dx \wedge dy)$, and $\mu(x, y) := \frac{1}{2}(x^2 + y^2)$. Then $v = \partial_\theta$.

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- Consider $(\mathbb{C}, \omega = dx \wedge dy)$, and $\mu(x, y) := \frac{1}{2}(x^2 + y^2)$. Then $v = \partial_\theta$. Thus μ is an S^1-moment map for the action $\frac{1}{2}|z|^2$

$$z \xmapsto{\theta \in S^1} e^{i\theta} z.$$

Moment map

- What about the action

$$S^1 \curvearrowright \mathbb{C}, \quad z \xmapsto{\theta \in S^1} e^{ik\theta} z,$$

for some $k \in \mathbb{Z}$?

$V = k\partial_\theta$

$\mu(z) = \frac{k}{2}|z|^2$

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  for some $k \in \mathbb{Z}$? The moment map is $z \mapsto \frac{k}{2}|z|^2$.
- Lemma : Suppose $(M_1, \omega_1, G, \mu_1)$, $(M_2, \omega_2, G, \mu_2)$ are Hamiltonian G-spaces. Then $(M_1 \times M_2, \omega_1 \oplus \omega_2, G, \mu_1 + \mu_2)$ is a Hamiltonian G-space. where the G-action is

$$g \cdot (m_1, m_2) := (g m_1, g m_2).$$

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- Lemma : Suppose $(M_1, \omega_1, G, \mu_1)$, $(M_2, \omega_2, G, \mu_2)$ are Hamiltonian G-spaces. Then $(M_1 \times M_2, \omega_1 \oplus \omega_2, G, \mu_1 + \mu_2)$ is a Hamiltonian G-space. Proof : Exercise.
- The moment map for the diagonal action is

$$S^1 \curvearrowright \mathbb{C}^n, \quad (z_1, \ldots, z_n) \xmapsto{\theta \in S^1} e^{i\theta}(z_1 \ldots, z_n)$$

is $\mu : \mathbb{C}^n \to \mathbb{R}$

$(z_1, \ldots, z_n) \mapsto \frac{1}{2} \sum_{i=1}^{n} |z_i|^2$

E_2 $S^1 \curvearrowright \mathbb{C}^2$ $(z_1, z_2) \xmapsto{\theta} (e^{2i\theta} z_1, e^{3i\theta} z_2)$.

Moment map

- What about the action

$$S^1 \curvearrowright \mathbb{C}, \quad z \xmapsto{\theta \in S^1} e^{ik\theta} z,$$

for some $k \in \mathbb{Z}$? The moment map is $z \mapsto \frac{k}{2}|z|^2$.

- Lemma : Suppose $(M_1, \omega_1, G, \mu_1)$, $(M_2, \omega_2, G, \mu_2)$ are Hamiltonian G-spaces. Then $(M_1 \times M_2, \omega_1 \oplus \omega_2, G, \mu_1 + \mu_2)$ is a Hamiltonian G-space. Proof : Exercise.
- The moment map for the diagonal action is

$$S^1 \curvearrowright \mathbb{C}^n, \quad (z_1, \ldots, z_n) \xmapsto{\theta \in S^1} e^{i\theta}(z_1 \ldots, z_n)$$

is $\mu(z_1, \ldots, z_n) = \frac{1}{2}\sum_i |z_i|^2$.

$$S^{2n-1} \simeq \{ \textstyle\sum_i |z_i|^2 = 1 \} \;/\; S^1 \simeq \mathbb{P}^{n-1}$$