Introduction to Symplectic Geometry : Lecture 13

October 1, 2021

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The action is transitive and the isotropy group at $(0, \ldots, 0, 1)$ is $U(n-1)$.

$$U(n-1) \to \underbrace{U(n) \to S^{2n-1}}$$

principal $U(n-1)$-bundle

Eventually $\pi_1(U(n)) = \pi_1(U(n-1)) = \mathbb{Z}$

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  The action is transitive and the isotropy group at $(0, \ldots, 0, 1)$ is $U(n-1)$.
- So, $U(n)/U(n-1) = S^{2n-1}$ and $U(n) \to S^{2n-1}$ is a principal $U(n-1)$-bundle.

What is $\pi_1(\mathrm{Sp}(2n, \mathbb{R}))$?

- In a similar way, we have $SU(n)/SU(n-1) = S^{2n-1}$, and $SU(n) \to S^{2n-1}$ is an $SU(n-1)$-principal bundle.

$$SU(n-1) \longrightarrow SU(n) \longrightarrow S^{2n-1}$$

principal $SU(n-1)$- bundle

$n=2$ $\quad SU(1) = \{Id\}$ $\quad \Rightarrow SU(2) \simeq S^3$

simply connected

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- By induction, $SU(n)$ is simply connected.
- Topologically $U(n) \simeq SU(n) \times S^1$ (Homework). Therefore, $\pi_1(U(n)) = \mathbb{Z}$.

A characteristic class for complex vector bundles on $\mathbb{P}^1$

- Definition : A **complex vector bundle** $E \to M$ is a vector bundle with a fiberwise linear complex structure

$$J_x : E_x \to E_x, \quad J_x^2 = -\operatorname{Id}.$$

- By the same discussion as for symplectic vector bundles, complex vector bundles over $\mathbb{P}^1$ are given by a transition function $\Phi_{01} : U_0 \cap U_1 \to GL(n, \mathbb{C})$.

$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$

$U_0 = \mathbb{C}$ $\qquad$ $U_1 = \mathbb{P}^1 \setminus \{0\}$

$$\Phi_{01}(x) := \left(\Phi_0 \circ \Phi_1^{-1}\right)\Big|_{\{x\} \times \mathbb{C}}$$

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- A bundle $E \to \mathbb{P}^1$ given by a transition function Φ_{01} is determined up to isomorphism by the map

$$[\Phi_{01}] : \pi_1(U_0 \cap U_1) \to \pi_1(GL(n, \mathbb{C})) \simeq \mathbb{Z}, \quad 1 \mapsto k$$

The number $k \in \mathbb{Z}$ is the **first Chern number** of the bundle E.

$E \to M$ complex vector bundle $\quad c_1(E) \in H^2(M, \mathbb{Z})$

$M = \Sigma$ cpt oriented surface $\quad$ 1st Chern no. $:= \int_\Sigma c_1(E) \in \mathbb{Z}$

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We have defined the 1st Chern no. for complex vector bundles $E \to \mathbb{P}^1$

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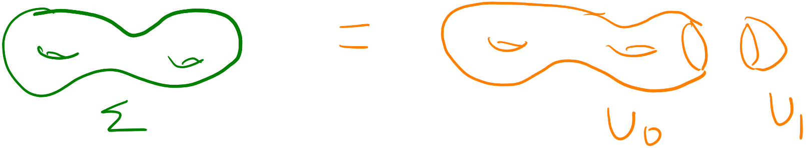

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- Result : Let Σ_0 be a connected oriented surface with non-empty boundary. Any complex vector bundle $E \to \Sigma$ is trivial.

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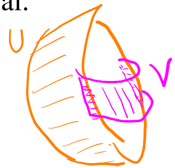
- Result : Let Σ_0 be a connected oriented surface with non-empty boundary. Any complex vector bundle $E \to \Sigma$ is trivial. This result follows from the Claim:
- Claim (Attaching a 1-cell) Suppose U, V are oriented surfaces such that

$$V = (0, 1) \times (0, 1), \quad U \cup V \simeq U + 1\text{-cell attached}.$$

Let $E \to U \cup V$ be a complex vector bundle. If $E|U$ is trivial, $E|(U \cup V)$ is also trivial.



$U \cap V \simeq$ 2 disjoint disks

$\Phi_{UV} : U \cap V \to GL(n, \mathbb{C})$

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Any map

$\Phi_0 : \mathbb{D} \to GL(n, \mathbb{C})$ can be homotoped to the identity map

$\mathbb{D}$ ↓ disk

$\Phi_1 : \mathbb{D} \longrightarrow GL(n, \mathbb{C})$

$x \longmapsto Id$

Reason : $\mathbb{D}$ is contractible, $GL(n, \mathbb{C})$ is connected

$\therefore$ Φ_{UV} is homotopic to the identity map

Φ_{UV} is homotopic to the identity map

$\Updownarrow$

By changing the trivializations $E|U$ and $E|V$, we can ensure $\Phi_{UV} \equiv Id$

$$\underbrace{\pi_1(U \cap V)}_{= 0} \longrightarrow \pi_1(GL(n, \mathbb{C}))$$

Rmk $E \to S^1$ with fiber $\mathbb{R}^n$ can be non-trivial

$\because GL(n, \mathbb{R})$ is disconnected.

(write down transition functions for the Möbius strip)

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$S^1 =$ 

$U \cup V$

$U \cap V = W_0 \sqcup W_1$

$$\Phi_{UV} : U \cap V \to GL(n,\mathbb{R})$$

E is non-trivial if

$$\Phi_{UV}(W_0) \in GL^+(n,\mathbb{R})$$

$$\Phi_{UV}(W_1) \in GL^-(n,\mathbb{R})$$

A characteristic class for complex vector bundles on $\mathbb{P}^1$

- The same definition also works for any other compact oriented surface Σ : $\Sigma := U_0 \cup U_1$, U_1 is a disk, $U_0 \cap U_1 \simeq S^1 \times (0,1)$.
- Result : Let Σ_0 be a connected oriented surface with non-empty boundary. Any complex vector bundle $E \to \Sigma$ is trivial. This result follows from the Claim:
- Claim (Attaching a 1-cell) Suppose U, V are oriented surfaces such that

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  Let $E \to U \cup V$ be a complex vector bundle. If $E|U$ is trivial, $E|(U \cup V)$ is also trivial.
- Proof : Follows from path-connectedness of $GL(n,\mathbb{C})$. The transition function $U \cap V \to GL(n,\mathbb{C})$ is homotopic to the identity map.

A characteristic class for complex vector bundles on $\mathbb{P}^1$

- The first Chern number can also be defined for symplectic vector bundles in the same way.

Way 1 — look $\pi_1(U_0 \cap U_1) \to \pi_1(Sp(2n, \mathbb{R}))$

Way 2 — Equip symp vector bundle $E \to M$ with a compatible complex structure J and compute 1^{st} Chern no. $c_1(E, J)$

Rmk $c_1(E, J)$ is independent of choice of J. ← space of compatible $\mathbb{C}$-structures on E, is contractible.

Holomorphic vector bundles

- Let X be a complex manifold. A **holomorphic vector bundle** $E \to X$ is a complex vector bundle whose transition functions are holomoprhic functions.

$U, V \subseteq X \qquad \Phi_{UV} : U \cap V \to GL(n, \mathbb{C})$

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- Example : $X = \mathbb{P}^1$, $\mathcal{O}_{P^1}(k) :=$ Holomorphic line bundle with first Chern number k. Transition function is $\Phi_{01}(z) = z^k$.

$$\Phi_{01} : U_0 \cap U_1 \longrightarrow GL(1,\mathbb{C})$$

$$U_0 \cap U_1 = \mathbb{C}^\times, \quad GL(1,\mathbb{C}) \cong \mathbb{C}^\times$$

$$z \longmapsto z^k$$

$$\mathcal{O}_{\mathbb{P}^1}(k) := (U_0 \times \mathbb{C}) \amalg (U_1 \times \mathbb{C}) / \sim$$

$\sim$ given by Φ_{01}

Def: M is a complex mfd if it is a smooth mfd with charts

$$\phi_\alpha : U_\alpha \longrightarrow \mathbb{C}^n \quad , \quad M = \bigcup_\alpha U_\alpha$$

and transition functions

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are biholomorphisms.

Hamiltonian group actions on symplectic manifolds

- Suppose a Lie group G acts smoothly on a manifold M.

Assume $G \subseteq GL(2n, \mathbb{R})$

- Suppose a Lie group G acts smoothly on a manifold M. The action is 'generated' by the vector fields $\xi_M \in \mathrm{Vect}(M)$, $\xi \in \mathfrak{g}$

$$\xi_M(m) := \frac{d}{dt} e^{t\xi} m|_{t=0}.$$

$\xi \in \mathfrak{g}$ $\quad G \subseteq GL(2n, \mathbb{R})$

$$e^{t\xi} = \exp(t\xi) := 1 + \xi + \frac{\xi^2}{2!} + \cdots$$

$t \mapsto e^{t\xi}$, $\mathbb{R} \to G$ is a homomorphism

s.t. $\frac{d}{dt} e^{t\xi} \Big|_{t=0} = \xi$

$\mathfrak{g} \to \mathrm{Vect}(M)$ $\quad \xi \mapsto \xi_M$

infinitesimal action of G on M

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- The map

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is $\mathbb{R}$-linear

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is $\mathbb{R}$-linear and G-equivariant, i.e. $(\mathrm{Ad}_g \, \xi)_M = dL_g(\xi_M)$.

$g \in G \quad L_g : M \xrightarrow{\text{diffeo}} M, \quad m \mapsto gm$

$dL_g : \mathrm{Vect}(M) \to \mathrm{Vect}(M), \quad v \mapsto dL_g(v$

$G \circlearrowright \mathrm{Vect}(M)$

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$$\xi_m - \frac{d}{dt} e^{t\xi} m \Big|_{t=0}$$

$$dL_g(\xi_m) - \frac{d}{dt} g e^{t\xi} m \Big|_{t=0} \leftrightarrow (\mathrm{Ad}_g \xi)_M$$

$$g e^{t\xi} m = g e^{t\xi} g^{-1}(gm) = e^{t \mathrm{Ad}_g \xi}(gm)$$

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- Now suppose M is a symplectic manifold and G acts via Hamiltonian diffeomorphisms.

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$L_g : M \to M$

- Then, for any $\xi \in \mathfrak{g}$, $i_{\xi_M}\omega$ is exact, so there exists $\mu_\xi : M \to \mathbb{R}$ such that $d\mu_\xi = -i_{\xi_M}\omega$.

$L_{e^{t\xi}} : M \to M$ is a Hamiltonian diffeo $\forall t$

$\iff \xi_M$ is a Hamiltonian vector field

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- Now suppose M is a symplectic manifold and G acts via Hamiltonian diffeomorphisms.
- Then, for any $\xi \in \mathfrak{g}$, $i_{\xi_M}\omega$ is exact, so there exists $\mu_\xi : M \to \mathbb{R}$ such that $d\mu_\xi = -i_{\xi_M}\omega$.
- It is reasonable to require that the map

$$\mathfrak{g} \to C^\infty(M), \xi \mapsto \mu_\xi$$

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is linear and equivariant. (Equivariance means $\mu_{Ad_g\xi} = \mu_\xi \circ L_{g^{-1}}$.)

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$\mathfrak{g} \longrightarrow \mathbb{R}$ linear

$\xi \mapsto \mu_\xi(m)$

$\mu(m) \in \mathfrak{g}^\vee$

is linear and equivariant. (Equivariance means $\mu_{Ad_g\xi} = \mu_\xi \circ L_{g^{-1}}$.)

- Linearity of (1) implies that we may rewrite (1) as

$$\boxed{\mu : M \to \mathfrak{g}^\vee,} \quad \text{such that} \quad \mu_\xi = \langle \mu, \xi \rangle : M \longrightarrow \mathbb{R}$$

Moment map

next class : Monday