Introduction to Symplectic Geometry : Lecture 12

September 29, 2021

Recall : A Lie group theoretic viewpoint

- Lemma : Suppose a Lie group G acts transitively on a manifold M. Suppose for a point $m \in M$ the isotropy group is G_m, i.e.

$$G_m = \{g \in G : gm = m\}.$$

Then the map

$$G/G_m \to M, \quad gG_m \mapsto gm$$

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- In a similar way, $GL^+(n, \mathbb{R})$ deformation retracts to $SO(n, \mathbb{R})$. (Reason : The space of metrics on $\mathbb{R}^n$ is contractible.)

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- Question : Does a real vector bundle with oriented fibers of rank $2n$ always have fiber-wise symplectic form?
- Answer : For obstructions look at the cosets $GL^+(2n, \mathbb{R})/Sp(2n, \mathbb{R})$. The group is not contractible. So one may be able to produce vector bundles without a fiber-wise symplectic structures.

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  The action is transitive and the isotropy group at $(0, \ldots, 0, 1)$ is $U(n-1)$.
- So, $U(n)/U(n-1) = S^{2n-1}$ and $U(n) \to S^{2n-1}$ is a principal $U(n-1)$-bundle.

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