

# Introduction to Symplectic Geometry : Lecture 12

September 29, 2021

## Recall : A Lie group theoretic viewpoint

- Lemma : Suppose a Lie group  $G$  acts transitively on a manifold  $M$ . Suppose for a point  $m \in M$  the isotropy group is  $G_m$ , i.e.

$$G_m = \{g \in G : gm = m\}.$$

Then the map

$$G/G_m \rightarrow M, \quad gG_m \mapsto gm$$

is well-defined and is a diffeomorphism.

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$$\Omega \in \Lambda^2 \mathbb{R}^{2n}$$

$\Omega$  : non-degenerate

Space of linear symplectic structures  $\subset$  open  $\Lambda^2 \mathbb{R}^{2n}$

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$$f(\mathbb{R}^{2n}) = \{J \in \text{End}(\mathbb{R}^{2n}) \mid J^2 = -\text{Id}\}$$

$-\text{Id}$  is a regular value of  
 $\text{End}(\mathbb{R}^{2n}) \rightarrow \text{End}(\mathbb{R}^{2n}), A \mapsto A^2$

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- The space of metrics on  $\mathbb{R}^n$  is  $GL(n, \mathbb{R})/O(n, \mathbb{R}) \simeq GL^+(n, \mathbb{R})/SO(n, \mathbb{R})$ .

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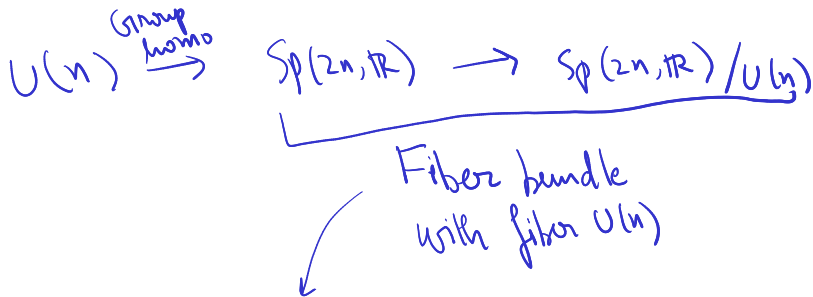
$$H : \mathrm{Sp}(2n, \mathbb{R})/U(n) \times [0, 1] \rightarrow \mathrm{Sp}(2n, \mathbb{R})/U(n)$$

$$H(x, 0) = x$$

$$H(x, 1) = * \\ + x$$

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$U(n)$ -Principal bundle  
 $\mathrm{Sp}(2n, \mathbb{R})$  topologically  $U(n) \times \mathrm{Sp}(2n, \mathbb{R})/U(n)$

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Last time : symplectic vector bundle  
can be given a compatible complex  
structure, and consequently a Hermitian  
metric (i.e. a  $U(n)$ -structure).

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*Alternate viewpoint*

- The contractibility of  $\mathrm{Sp}(2n, \mathbb{R})/U(n)$  implies : A symplectic vector bundle  $(E, \omega) \rightarrow M$  has a  $U(n)$ -structure. The structure group can be reduced from  $\mathrm{Sp}(2n, \mathbb{R})$  to  $U(n)$ .

Hermitian metric

$$\omega(\cdot, J\cdot) + i\omega(\cdot, \cdot)$$

$E$  : symplectic vector bundle

vector bundle  $E$  with a Hermitian metric

transition fns  
 $U_a \cap U_b \rightarrow \mathrm{Sp}(2n, \mathbb{R})$

trans fns  
 $U_a \cap U_b \rightarrow U(n, \mathbb{C})$

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- In a similar way,  $GL^+(n, \mathbb{R})$  deformation retracts to  $SO(n, \mathbb{R})$ .

← Reason?

Space of metrics is contractible (°: it is convex)  
IS  
 $GL^+(n, \mathbb{R}) / SO(n, \mathbb{R})$

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- The contractibility of  $\mathrm{Sp}(2n, \mathbb{R})/U(n)$  implies : A symplectic vector bundle  $(E, \omega) \rightarrow M$  has a  $U(n)$ -structure. The structure group can be reduced from  $\mathrm{Sp}(2n, \mathbb{R})$  to  $U(n)$ . (We have seen a more direct proof though!)
- In a similar way,  $GL^+(n, \mathbb{R})$  deformation retracts to  $SO(n, \mathbb{R})$ . (Reason : The space of metrics on  $\mathbb{R}^n$  is contractible.)

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$GL(2n, \mathbb{R}) / Sp(2n, \mathbb{R})$   
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Structure group of real vector  
bundles with a fiberwise orientation  
 $GL^+(n, \mathbb{R})$

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- Question : Does a real vector bundle with oriented fibers of rank  $2n$  always have fiber-wise symplectic form?
- Answer : For obstructions look at the cosets  $GL^+(2n, \mathbb{R})/Sp(2n, \mathbb{R})$ . The group is not contractible. So one may be able to produce vector bundles without a fiber-wise symplectic structures.

# What is $\pi_1(\mathrm{Sp}(2n, \mathbb{R}))$ ?

- Same as  $\pi_1(U(n, \mathbb{C}))$ .

$\mathrm{Sp}(2n, \mathbb{R}) \xrightarrow{\text{def. retracts}} U(n, \mathbb{C})$

Recall that the isomorphism classes of symplectic vector bundles on  $\mathbb{P}^1$

is given by the map

$$\pi_1(U_0 \cap U_1) \rightarrow \boxed{\pi_1(\mathrm{Sp}(2n, \mathbb{R}))}$$

Here  $\mathbb{P}^1 = U_0 \cup U_1$   
 $\downarrow \quad \downarrow$   
charts

$$U_0 \cap U_1 \cong S^1 \times (0, 1)$$

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- Same as  $\pi_1(U(n, \mathbb{C}))$ .
- Consider the action of  $U(n)$  on the unit sphere

$$S^{2n-1} := \{z \in \mathbb{C}^n : |z|^2 = \sum_i |z_i|^2 = 1\}. \quad \subseteq \mathbb{C}^n$$

\* transitive

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The action is transitive and the isotropy group at  $(0, \dots, 0, 1)$  is  $U(n-1)$ .



$$U(n-1) \rightarrow U(n)$$
$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{A} & 0 \\ 0 & 1 \end{bmatrix}$$

$$A \in U(n-1)$$

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- So,  $U(n)/U(n-1) = S^{2n-1}$  and  $U(n) \rightarrow S^{2n-1}$  is a principal  $U(n-1)$ -bundle.

$$U(n-1) \rightarrow U(n) \rightarrow S^{2n-1}$$

└──────────────────────────┘  
 $U(n-1)$ -principal bundle

# What is $\pi_1(\mathrm{Sp}(2n, \mathbb{R}))$ ?

- In a similar way, we have  $SU(n)/SU(n-1) = S^{2n-1}$ , and  $SU(n) \rightarrow S^{2n-1}$  is an  $SU(n-1)$ -principal bundle.

$$\star \boxed{SU(n-1) \rightarrow SU(n) \rightarrow S^{2n-1}}$$

$\star SU(n) \subset S^{2n-1}$  is transitive

$\star$  stabilizer of a point is  $SU(n-1)$

$$SU(1) = \{\mathrm{Id}\} \rightarrow SU(2) \xrightarrow{\cong} S^3$$

$$\pi_1(SU(2)) = 0$$

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- Result : If in a fibration  $F \rightarrow E \xrightarrow{\pi} B$ , both the base  $B$  and the fiber  $F$  are simply connected, then the total space is simply connected.

$$\text{Let } \gamma: S^1 \rightarrow E$$

$$\pi(\gamma) \text{ in } B$$

$$\exists \delta: \mathbb{D}^2 \rightarrow B$$

$$\delta|_{\partial\mathbb{D}^2} = \pi \circ \gamma$$

$$E/\mathbb{D}^2 \simeq \mathbb{D}^2 \times F \supseteq \gamma$$

so  $\gamma$  bounds a disk in  $E$

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$$\begin{array}{c} F \\ \downarrow \\ SU(n-1) \end{array} \rightarrow SU(n) \rightarrow \begin{array}{c} B \\ \downarrow \\ S^{2n-1} \end{array}$$

By ind-hypp  $SU(n-1)$  is simply connected

$\Rightarrow SU(n)$  is simply connected

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- Result : If in a fibration  $F \rightarrow E \rightarrow B$ , both the base  $B$  and the fiber  $F$  are simply connected, then the total space is simply connected.
- (As an aside, can we say the following : Let  $P \rightarrow B$  be a fiber bundle with fiber  $F$ . If  $B$  is simply connected, then  $\pi_1(F) \simeq \pi_1(P)$ ?)

Hopf fibration

$$S^1 \rightarrow S^3 \rightarrow S^2$$

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- By induction,  $SU(n)$  is simply connected.
- Topologically  $U(n) \simeq SU(n) \times S^1$  (Homework). Therefore,  $\pi_1(U(n)) = \mathbb{Z}$ .