Introduction to Symplectic Geometry : Lecture 11

September 20, 2021

Recall : Complex structures on symplectic vector spaces

- A compatible complex structure on a symplectic vector space is a linear map
$$J : V \to V \quad J^2 = -\operatorname{Id},$$
and $\omega(v, Jv) > 0$ for all non-zero $v \in V$, and $\omega(v, w) = \omega(Jv, Jw)$.

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- Result : $\mathcal{J}(V, \omega)$ is contractible.

Contractiblity of the space of complex structures

- Proof, Step 1 : There exists a continuous map

$$r : \mathrm{Met}(V) \to \mathcal{J}(V, \omega).$$

such that $r(g_J) = J$ for all $J \in \mathcal{J}(V, \omega)$.

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- Step 2 : Fix a $J_0 \in \mathcal{J}(V, \omega)$. For any $J \in \mathcal{J}(V, \omega)$, the path

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- Result : Let $r : \mathcal{J}(V, \omega) \to \mathrm{Met}(V)$ be the map in the proof of contractibility. For any $\phi \in \mathrm{Sp}(V)$, $r(\phi^* g) = \phi^*(r(g))$.

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- $$\phi^* g = g(\phi\cdot, \phi\cdot), \quad \phi^* J = \phi^{-1} J \phi.$$
- Thus r is $\mathrm{Sp}(2n)$-equivariant.

Fiber-wise complex structures on a symplectic vector bundle

- Result : Let $(E, \omega) \to M$ be a symplectic vector bundle. Then the set of ω-compatible fiber-wise almost complex structures, denoted by $\mathcal{J}(E, \omega)$, is non-empty and contractible.

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- Result : Let $(E, \omega) \to M$ be a symplectic vector bundle. Then the set of ω-compatible fiber-wise almost complex structures, denoted by $\mathcal{J}(E, \omega)$, is non-empty and contractible.
- The result is proved by constructing

$$r : \mathrm{Met}(E) \to \mathcal{J}(E, \omega)$$

such that $r(g_J) = J$ for any $J \in \mathcal{J}(E, \omega)$.

Fiber-wise complex structures ..

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- Choose an open cover $M = \cup_i U_i$ and trivializations

$$E|U \simeq (U, (\mathbb{R}^{2n}, \omega_{std}).$$

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- The complex structure $r(g)$ on different charts are consistent and therefore glue. Reason : On fibers r is $\mathrm{Sp}(2n, \mathbb{R})$-equivariant.
- We have thus shown : On a symplectic manifold, the space of compatible almost compatible structures is non-empty and contractible.

Observation

$$GL(n,\mathbb{C})\cap\mathrm{Sp}(2n,\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{R})\cap O(2n,\mathbb{R}) = O(2n,\mathbb{R})\cap GL(n,\mathbb{C}) = U(n)$$

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Homework.

A Lie group theoretic viewpoint

- Lemma : Suppose a Lie group G acts transitively on a manifold M. Suppose for a point $m \in M$ the isotropy group is G_m, i.e.

$$G_m = \{g \in G : gm = m\}.$$

Then the map

$$G/G_m \to M, \quad gG_m \mapsto gm$$

is well-defined and is a diffeomorphism.

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- The space of metrics on $\mathbb{R}^n$ is $GL(n,\mathbb{R})/O(n,\mathbb{R}) \simeq GL^+(n,\mathbb{R})/SO(n,\mathbb{R})$.

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