

$$\left\{ \begin{array}{l} (S^2, \omega) \times (S^2, \omega) \xrightarrow{f} S^2 \times S^2 \\ (x, y) \mapsto (y, x) \end{array} \right. \quad f \in \text{Symplect}_0(S^2 \times S^2)$$

Introduction to Symplectic Geometry : Lecture 11

September 20, 2021



$S^1 \times S^1$

$$(0, \alpha) \mapsto (0, \alpha, 0, 2)$$

Symplectomorphism group

See 9.5

McDuff-Salamon

J-holomorphic curves

Recall : Complex structures on symplectic vector spaces

- A compatible complex structure on a symplectic vector space is a linear map

$$J : V \rightarrow V \quad J^2 = -\text{Id},$$

and $\omega(v, Jv) > 0$ for all non-zero $v \in V$, and $\omega(v, w) = \omega(Jv, Jw)$.

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that varies smoothly with x .

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- On a manifold M a fiberwise compatible complex structure on the tangent space

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is called a compatible **almost complex structure**.

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Contractibility of the space of complex structures

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- Result : $\mathcal{J}(V, \omega)$ is contractible.

Contractibility of the space of complex structures

- Proof, Step 1 : There exists a continuous map

$$r : \text{Met}(V) \rightarrow \mathcal{J}(V, \omega).$$

such that $r(g_J) = J$ for all $J \in \mathcal{J}(V, \omega)$.

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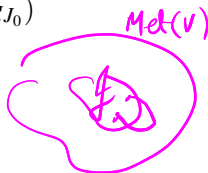
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- Step 2 : Fix a $J_0 \in \mathcal{J}(V, \omega)$. For any $J \in \mathcal{J}(V, \omega)$, the path

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lies in $\mathcal{J}(V, \omega)$ and connects J to J_0 .



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lies in $\mathcal{J}(V, \omega)$ and connects J to J_0 . The path varies continuously with J, t . Therefore, $\mathcal{J}(V, \omega)$ deformation retracts to a point.

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$$\text{let } A : V \rightarrow V$$

$$g(Av, w) = \omega(v, w)$$

$$J := \underbrace{(-A^2)^{-1/2}}_{A^*A} A$$

$$J^2 = -\text{Id}$$

$$\phi \in \text{Sp}(V)$$

$$g \mapsto \phi^* g := g(\phi \cdot, \phi \cdot)$$

$$\text{Note } \omega(\phi v, \phi w) = \omega(v, w)$$

$$A \mapsto \phi^{-1} A \phi$$

$$J \mapsto \phi^{-1} J \phi = \phi^* J$$

Contractibility of the space of complex structures

$$\text{Met}(V) \rightarrow \mathcal{f}(V, \omega)$$

- Result : Let $r : \mathcal{J}(V, \omega) \rightarrow \text{Met}(V)$ be the map in the proof of contractibility. For any $\phi \in \text{Sp}(V)$, $r(\phi^*g) = \phi^*(r(g))$.

$$g \in \text{Met}(V) \quad \phi^*g(v, w) := g(\phi(v), \phi(w)) \\ v, w \in V$$

$$J: V \rightarrow V \quad \phi^*J(v) := \phi^{-1}J(\phi(v))$$

$$(\phi^*J)^2 = -\text{Id}$$

Contractibility of the space of complex structures

$$\text{Met}(V) \rightarrow \mathcal{G}(V, \omega)$$

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$$\phi^*g = g(\phi \cdot, \phi \cdot), \quad \phi^*J = \phi^{-1}J\phi.$$

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$$\phi^*g = g(\phi\cdot, \phi\cdot), \quad \phi^*J = \phi^{-1}J\phi.$$

- Thus r is $\text{Sp}(2n)$ -equivariant.

Fiber-wise complex structures on a symplectic vector bundle

- Result : Let $(E, \omega) \rightarrow M$ be a symplectic vector bundle. Then the set of ω -compatible fiber-wise ~~almost~~ complex structures, denoted by $\mathcal{J}(E, \omega)$, is non-empty and contractible.

Fiber-wise complex structures on a symplectic vector bundle

- Result : Let $(E, \omega) \rightarrow M$ be a symplectic vector bundle. Then the set of ω -compatible fiber-wise almost complex structures, denoted by $\mathcal{J}(E, \omega)$, is non-empty and contractible.
- The result is proved by constructing

$$r : \underline{\text{Met}(E)} \rightarrow \mathcal{J}(E, \omega)$$

such that $r(g_J) = J$ for any $J \in \mathcal{J}(E, \omega)$.

$\text{Met}(E) :=$ space of fiberwise metrics

$$g_1, g_2 \in \text{Met}(E)$$

$$(1-t)g_1 + tg_2 \in \text{Met}(E)$$

$\text{Met}(E)$ is convex

Fiber-wise complex structures ..

- We will now construct

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- Choose an open cover $M = \cup_i U_i$ and trivializations

$$E|U \simeq (U, (\mathbb{R}^{2n}, \omega_{std}))$$

- Given $g \in \text{Met}(E)$, apply the map r fiber-wise. *on U*

*g_x is
a metric on \mathbb{R}^{2n}*

*↓
 \mathcal{J}_U fiberwise compatible
complex str on $E|U$*

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$$(U_0 \times \mathbb{R}^{2n}) \rightsquigarrow (U_1 \times \mathbb{R}^{2n})$$

$$\Phi_\alpha : U_0 \cap U_1 \rightarrow \text{Sp}(\mathbb{R}^{2n})$$

$$x \in U_0 \cap U_1$$

$$U_0 \times \mathbb{R}^{2n}$$

$$U_1 \times \mathbb{R}^{2n}$$

$$E_x \cong \mathbb{R}^{2n}$$

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$$\Phi_{01}(x) \in Sp(\mathbb{R}^{2n})$$

$$g_x$$

$$\Phi_{01}(x)^* g_x$$

Since π is
 $Sp(2n)$ -equiv

$$J_x = \mathcal{L}(g_x)$$

$$\Phi_{01}(x)^* J_x$$

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- The complex structure $r(g)$ on different charts are consistent and therefore glue.

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We have shown : $\mathcal{J}(E, \omega)$ is non-empty
 $\mathcal{J}(E, \omega)$ is contractible

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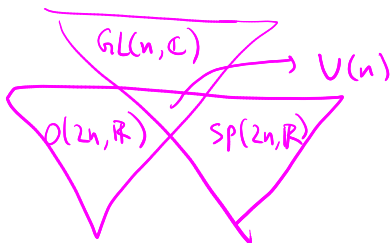
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- Given $g \in \text{Met}(E)$, apply the map r fiber-wise.
- The complex structure $r(g)$ on different charts are consistent and therefore glue. Reason : On fibers r is $\text{Sp}(2n, \mathbb{R})$ -equivariant.
- We have thus shown : On a symplectic manifold, the space of compatible almost compatible structures is non-empty and contractible.

Observation

$$\underline{GL(n, \mathbb{C})} \cap Sp(2n, \mathbb{R}) = Sp(2n, \mathbb{R}) \cap \underline{O(2n, \mathbb{R})} = \underline{O(2n, \mathbb{R})} \cap GL(n, \mathbb{C}) = U(n)$$

HW



Observation

$$GL(n, \mathbb{C}) \cap Sp(2n, \mathbb{R}) = Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R}) = O(2n, \mathbb{R}) \cap GL(n, \mathbb{C}) = U(n)$$

Homework.

A Lie group theoretic viewpoint

- Lemma : Suppose a Lie group G acts transitively on a manifold M . Suppose for a point $m \in M$ the isotropy group is G_m , i.e.

$$G_m = \{g \in G : gm = m\}.$$

Then the map

$$G/G_m \rightarrow M, \quad gG_m \mapsto gm$$

is well-defined and is a diffeomorphism.

Suppose the action of G_m on G by left mult is free and proper. Then G/G_m is a manifold.

A Lie group theoretic viewpoint

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- The space of linear symplectic structures on \mathbb{R}^{2n} is

$$\underline{\omega} = \sum_i e_i^* \wedge f_i^*$$

$e_1 \wedge e_2 + f_1 \wedge f_2$ is symplectic on \mathbb{R}^4
etc

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- The space of linear symplectic structures on \mathbb{R}^{2n} is $GL(2n, \mathbb{R})/Sp(2n, \mathbb{R})$.

$\Omega :=$ Space of symplectic structures on \mathbb{R}^{2n}

$\Omega \cong GL(2n, \mathbb{R})$ is transitive.

$A \in GL(2n, \mathbb{R})$

$\omega \in \Omega$

$$A \cdot \omega := \omega(A^T \cdot, A^T \cdot)$$

$\Omega :=$ Space of symplectic structures on \mathbb{R}^{2n}

$\Omega \curvearrowright GL(2n, \mathbb{R})$ is transitive.

$A \in GL(2n, \mathbb{R}) \quad \omega \in \Omega \quad A \cdot \omega := \omega(A^{-T}, A^{-T})$

* This action is transitive

$\left(\begin{array}{l} \omega_0, \omega_1 \text{ Symp forms on } \mathbb{R}^{2n} \\ \exists A; A \cdot \omega_0 = \omega_1 \end{array} \right)$

* $A \cdot \omega_{\text{std}} = \omega_{\text{std}} \Rightarrow A \in Sp(\mathbb{R}^{2n})$

$\omega \in \Lambda^2 V^*$

$\Omega \subseteq_{\text{open}} \Lambda^2 V^*$

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- The space of linear symplectic structures on \mathbb{R}^{2n} is $GL(2n, \mathbb{R})/Sp(2n, \mathbb{R})$.
- The space of complex structures on \mathbb{R}^{2n} is

$$J: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$$

$2n \times 2n$ matrices

$$J(V) \subseteq$$

$$J(\mathbb{R}^{2n}) \subseteq \text{End}(\mathbb{R}^{2n})$$
$$J(\mathbb{R}^{2n}) \subseteq GL(2n, \mathbb{R})$$

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- The space of complex structures on \mathbb{R}^{2n} is $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$.

* $GL(2n, \mathbb{R}) \curvearrowright \mathcal{J}(\mathbb{R}^{2n})$ is transitive

* $GL(n, \mathbb{C}) \curvearrowright GL(2n, \mathbb{R})$

Result :

$$GL(n, \mathbb{C}) = \left\{ A \in GL(2n, \mathbb{R}) : J_{\text{std}}(A^{-1}) = A(J_{\text{std}}) \right\}$$

\subseteq
 $n \times n$ \mathbb{C} -matrix

we showed \subseteq

Ex: Show \supseteq

Def

$\rightarrow GL(n, \mathbb{C}) :=$ Invertible $n \times n$ complex matrices

$$GL(n, \mathbb{C}) \hookrightarrow GL(2n, \mathbb{R})$$

$$X + iY \longmapsto$$

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$$

X, Y $n \times n$ \mathbb{R} -matrices

Commutator with
 $J_{\text{std}} := \begin{bmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{bmatrix}$

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- The space of complex structures on \mathbb{R}^{2n} is $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$.
- The space of metrics on \mathbb{R}^n is $GL(n, \mathbb{R})/O(n, \mathbb{R}) \simeq GL^+(n, \mathbb{R})/SO(n, \mathbb{R})$.

A Lie group theoretic viewpoint

- The space of compatible complex structures on $(\mathbb{R}^{2n}, \omega_0)$ is

A Lie group theoretic viewpoint

- The space of compatible complex structures on $(\mathbb{R}^{2n}, \omega_0)$ is $\mathrm{Sp}(2n, \mathbb{R})/U(n)$.

$$\mathrm{Sp}(2n, \mathbb{R}) \hookrightarrow \mathcal{J}(\mathbb{R}^{2n}, \omega)_{\mathrm{std}}$$