Introduction to Symplectic Geometry : Lecture 10

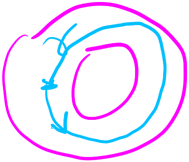
September 15, 2021

$\pi_1(U_0 \cap U_1)$

$\to \pi_1(Sp(\mathbb{R}^{2n}), A_0)$

$\to \pi_1(Sp(\mathbb{R}^{2n}), A_1)$

$\psi_{10} : U_0 \cap U_1 \longrightarrow Sp(\mathbb{R}^{2n})$ $\qquad A \in Sp(\mathbb{R}^{2n})$



$A \cdot \psi_{10}(\gamma(0))$

$\neq \psi_{10}(\gamma(0)) A$

$\tilde{\psi}_{10} := A \psi_{10} A^{-1}$

$\pi_1(U_0 \cap U_1, z) \longrightarrow \pi_1(Sp(\mathbb{R}^{2n}), \_\_)$

$\psi_{10}(z)$

Recall : the group of symplectomorphisms

- Last time we showed : Let (M, ω) be a compact symplectic manifold. The tangent space $T_{\mathrm{Id}} \operatorname{Symp}(M, \omega)$ is linearly isomorphic to $\{\alpha \in \Omega^1(M) : d\alpha = 0\}$.

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- A vector field $v \in \operatorname{Vect}(M)$ is a **symplectic vector field** if $d(i_v\omega)$ is closed. The flow of a time-dependent family v_t of symplectic vector fields is a symplectomorphism in $\operatorname{Symp}_0(M, \omega)$.

$L_v \omega = 0$

Hamiltonian diffeomorphisms

- Let M be a symplectic manifold. A **Hamiltonian vector field** is a vector field $v \in \mathrm{Vect}(M)$ such that $i_v\omega$ is exact.
- A Hamiltonian vector field v is generated by a Hamiltonian function $H : M \to \mathbb{R}$ by the condition $dH = i_v\omega$.

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- Thus a Hamiltonian diffeomorphism is given by a time-dependent **Hamiltonian function** $H_t : M \to \mathbb{R}$, $t \in [0, 1]$, and the generating vector field is

$$i_{v_t}\omega = dH_t.$$

Hamiltonian diffeomorphisms

Egs in $\mathrm{Symp}(M,\omega) \setminus \mathrm{Symp}_0(M,\omega)$?

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- If M is simply connected, $\mathrm{Symp}_0(M,\omega) = \mathrm{Ham}(M,\omega)$,

$H^1(M)=0$

$v \in \mathrm{Vect}(M)$

$i_v\omega$ closed $\Rightarrow$ $i_v\omega$ is exact

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- If M is simply connected, $\mathrm{Symp}_0(M, \omega) = \mathrm{Ham}(M, \omega)$, otherwise they are different.

Arnold's conjecture

Conjecture

Let (M, ω) be a compact symplectic manifold, and let

$$H_t : M \to \mathbb{R}, \quad \underline{t \in \mathbb{R}/\mathbb{Z}}$$

$H_t = H_{t+1}$

be a time-dependent Hamiltonian function. Assume that ~~the~~ all fixed points of the time one flow $\phi : M \to M$ are non-degenerate. Then,

$$\#\,\mathrm{Fix}(\phi) \geq \sum_{i=1}^{\dim(M)} \mathrm{rank}(H^i(M)).$$

$p \in \mathrm{Fix}(\phi)$

p is non-degenerate fixed pt

$\iff d\phi_p : T_pM \to T_pM$ does not have 1 as an eigenvalue

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p is a non deg fixed point of ϕ

$\Rightarrow$ p is an isolated fixed point

Remarks about Arnold's conjecture

- The result does not hold if 'Hamiltonian diffeomorphism' is replaced by a 'symplectomorphism isotopic to the identity'.

$\mathbb{R}\times S^1$, $dt\wedge d\theta$

$\phi_\tau(t,\theta) := (t+\tau,\theta)$

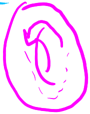
$\phi_0 = Id$

No fixed pts

$\phi_\tau \in Symp_0$

$(\phi_\tau \sim \phi_0)$ Isotopic

$T^2 = ((\mathbb{R}\times S^1)/\mathbb{Z}, dt\wedge d\theta)$



$\phi_\tau \in Symp_0(T^2)$. does not have fixed points

Remarks about Arnold's conjecture

- The result does not hold if 'Hamiltonian diffeomorphism' is replaced by a 'symplectomorphism isotopic to the identity'.
- The result is easy to see if the Hamiltonian function is required to be time-independent, i.e. $H_t = H$.

$H_t = H$ What are some fixed points?

$p \in M$ $dH(p) = 0$ $i_v\omega = dH$

$\Rightarrow v(p) = 0$

ϕ = time 1 flow of v

$\Rightarrow \phi(p) = p$

Morse theory $\Rightarrow$ $\#\mathrm{crit}(H) \geq \sum_{i=0}^{\dim M} \mathrm{rk}(H^i(M))$

$\mathrm{Ham}(M, \omega) \neq \mathrm{Symp}_0(M, \omega)$

- Result: Let $M = \mathbb{R} \times S^1$ be the cylinder with the standard symplectic form $dt \wedge d\theta$. Let $\phi : M \to M$ be a Hamiltonian diffeomorphism. Let $C := \{0\} \times S^1$.

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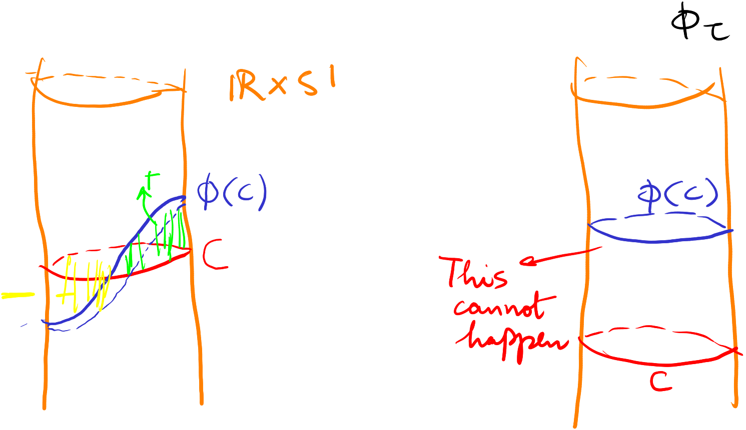

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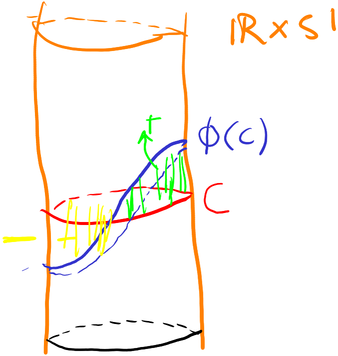

Signed area b/w Blue and Red

:= Area b/w Blue & Black

− Area b/w Red & Black.

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- We may replace H_t by $\overline{H}_t$ such that
  1. $\cup_t \operatorname{supp}(\overline{H}_t)$ is compact,
  2. and if $\overline{\phi}_t$ is the Hamiltonian flow generated by $\overline{H}_t$, then,

$$\overline{\phi}_t(C) = \phi_t(C).$$

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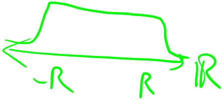

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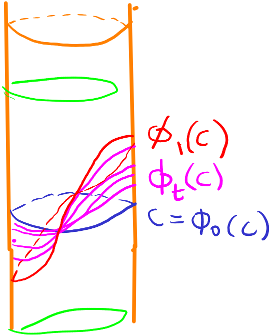

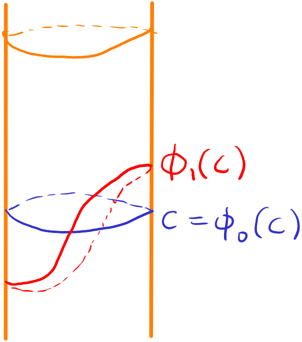

let $\eta : \mathbb{R} \times S^1 \to [0,1]$ be a bump function

$\eta \equiv 1$ on $\bigcup_{t \in [0,1]} \phi_t(C)$

$\operatorname{supp} \eta$ is compact.

Then, $\overline{H}_t := \eta H_t$

Proof continued

Lets assume $\bigcup_{t \in [0,1]} \mathrm{supp}(H_t)$ is compact

- Choose a loop $C' = \{t\} \times S^1$ lying outside $\mathrm{supp}(H)$.

Proof continued

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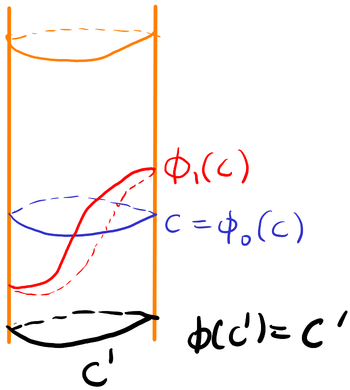

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$\mathrm{Ham}(M, \omega) \neq \mathrm{Symp}_0(M, \omega)$

- We have proved the following : Let $M = \mathbb{R} \times S^1$ be the cylinder with the standard symplectic form $dt \wedge d\theta$. Let $\phi : M \to M$ be a Hamiltonian diffeomorphism. Let $C := \{0\} \times S^1$. Then the signed area between C and $\phi(C)$ is zero.

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- There exists $\phi \in \mathrm{Symp}_0(M)$ such that the signed area between C and $\phi(C)$ is non-zero.

Complex structures on symplectic vector spaces

- A **complex structure** on a real vector space V is a linear map

$$J : V \to V, \quad \text{satisfying} \quad J^2 = -\,\mathrm{Id}\,.$$

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$i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad \partial_x \mapsto \partial_y$

$\partial_y \mapsto -\partial_x$

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- Additionally, a tame complex structure is **compatible** if

$$\text{(Compatible)} \quad \omega(v, w) = \omega(Jv, Jw) \quad \forall v, w \in V.$$

Complex structures on symplectic vector spaces

- A compatible complex structure gives a positive inner product

Complex structures on symplectic vector spaces

- A compatible complex structure gives a positive inner product $(v, w) \mapsto \omega(v, Jw)$. $\underset{\text{compatibility}}{=}$ $\omega(Jv, J^2 w)$

$= \omega(w, Jv) \mapsto (w, v)$

Tameness $\Rightarrow (v, v) > 0$ if $v \neq 0$

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  $\omega(w, Jv)$

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- On a manifold M a fiberwise complex structure on the tangent space

$$J_x : T_xM \to T_xM, \quad J_x^2 = -\operatorname{Id}$$

is called an **almost complex structure**.

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- On a symplectic manifold (M,ω) **tame and compatible almost complex structures** are defined in the obvious way.
- We will show later : on symplectic manifolds tame and compatible almost complex structures exist.

Contractiblity of the space of complex structures

- Let (V, ω) be a symplectic vector space, and let $\mathcal{J}(V, \omega)$ be the space of ω-compatible almost complex structures.

Claim: $\mathcal{J}(V,\omega)$ is non-empty.

Pf $\exists$ basis $e_1, \ldots e_n, f_1 \ldots f_n$

$$\omega = \sum_{i=1}^{n} e_i^* \wedge f_i^*$$

$$Je_i := f_i \qquad Jf_i = -e_i$$

$\boxed{\Xi}$

Contractiblity of the space of complex structures

- Let (V, ω) be a symplectic vector space, and let $\mathcal{J}(V, \omega)$ be the space of ω-compatible almost complex structures.
- Result : $\mathcal{J}(V, \omega)$ is contractible.

Contractiblity of the space of complex structures

- Proof technique : Let $\mathcal{M}(V)$ be the space of metrics on V.

Contractiblity of the space of complex structures

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- There is an inclusion

$g_J(v, w) := \omega(v, Jw)$

$$\psi : \mathcal{J}(V, \omega) \to \mathcal{M}(V), \quad \underline{J} \mapsto \psi(J) := \underline{g_J} := \omega(\cdot, J\cdot).$$

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- Proof technique : Let $\mathcal{M}(V)$ be the space of metrics on V.
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$$\psi : \mathcal{J}(V,\omega) \to \mathcal{M}(V), \quad J \mapsto \psi(J) := g_J := \omega(\cdot, J\cdot).$$

- $\mathcal{M}(V)$ is convex,

$g_0, g_1 \in \mathcal{M}(V) \Rightarrow (1-t)g_0 + tg_1 \in \mathcal{M}(V$
$\forall t \in [0,1]$

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- $\mathcal{M}(V)$ is convex, hence contractible.

Fix g_0

$$\mathcal{M}(V) \times [0,1] \longrightarrow \mathcal{M}(V)$$
$$g, t \mapsto (1-t)g + tg_0$$

is cont in g, t

$\Rightarrow$ $\mathcal{M}(V)$ deformation retracts to a point

Contractiblity of the space of complex structures

- Proof, Step 1 : There exists a continuous map

$$r : \mathrm{Met}(V) \to \mathcal{J}(V, \omega).$$

such that $r(g_J) = J$ for all $J \in \mathcal{J}(V, \omega)$.

(Thinking of $J \mapsto g_J$ as an inclusion $\mathcal{J}(V,\omega) \overset{i}{\underset{r}{\rightleftarrows}} M(V)$, then r is a retraction)

$r \circ i = Id_{\mathcal{J}}$

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- Step 2 : Fix a $J_0 \in \mathcal{J}(V, \omega)$. For any $J \in \mathcal{J}(V, \omega)$, the path

$$[0, 1] \ni t \mapsto \underline{r((1-t)g_J + tg_{J_0})}$$

  lies in $\mathcal{J}(V, \omega)$ and connects J to J_0.

$J \mapsto g_J$

$\mathcal{J} \longrightarrow \mathcal{M}$

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  lies in $\mathcal{J}(V, \omega)$ and connects J to J_0. The path varies continuously with J, t. Therefore, $\mathcal{J}(V, \omega)$ deformation retracts to a point.

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if $g = g_J$

then $A = J$

In general

$A^2 \neq -Id$

Consider $g \in \mathrm{Met}(V)$

$\exists\, A : V \to V$ $\quad g(Av, w) = \omega(v, w)$

$\forall v, w \in V$

$g(v, w) := v^T g w$ $\to$ Matrix

$\omega(v, w) = v^T \omega w$

$A^T \underline{g} = \underline{\omega}$

$A = (g^{-1}\omega)^t$

Contractiblity of the space of complex structures

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Consider $g \in \mathrm{Met}(V$

$\exists\, A : V \to V$ $\qquad g(Av, w) = \omega(v, w)$

$\forall v, w \in V$

* A is g-skew adjoint

$$\underbrace{g(Av, w)}_{=\,\omega(v,w)} = -\underbrace{g(v, Aw)}_{=\,\omega(w,v)}$$

$A^* = -A$

Consider $g \in Met(V$

$\exists A : V \to V$ $\qquad g(Av, w) = \omega(v, w)$

$\forall v, w \in V$

* A is g-skew adjoint $\qquad A^* = -A$

$(Av, w) = -(v, Aw)$

* $A^* A = -A^2$ is g-symmetric and positive definite

$\to g(A^* A v, w) = g(Av, Aw) = g(v, A^* A w)$

$\to g(A^* A v, v) = g(Av, Av) > 0 \quad \forall v \neq 0$

* $J := (A^* A)^{-1/2} A$

If $A = J \Rightarrow \Omega(g_J) = J$

$(-A^2)^{-1/2} A = J$

There is class on Mon 20

No Class on Wed 22
Fri 24
Mon 27

Next Class is on Wed 29

and then Fri 1st Oct

* $A^*A = -A^2$ is g-symmetric and positive definite

$\rightarrow g(A^*Av, w) = g(Av, Aw) = g(v, A^*Aw)$

$\rightarrow g(A^*Av, v) = g(Av, Av) > 0 \quad \forall v \neq 0$

* $(A^*A)^t$ is well-defined $\forall t \in \mathbb{R}$ and it commutes with A.

$\rightarrow$ Commutes with $A \because A^*A = -A^2$

$\rightarrow A^*A = P^{-1} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P \qquad \lambda_i > 0$

$(A^*A)^t = P^{-1} \begin{bmatrix} \lambda_1^t & & 0 \\ & \ddots & \\ 0 & & \lambda_n^t \end{bmatrix} P$

* $A^* A = -A^2$ is g-symmetric and positive definite

$\rightarrow g(A^*Av, w) = g(Av, Aw) = g(v, A^*Aw)$

$\rightarrow g(A^*Av, v) = g(Av, Av) > 0 \quad \forall v \neq 0$

* $(A^*A)^t$ is well-defined $\forall t \in \mathbb{R}$ and it commutes with A.

* $J = (A^*A)^{-1/2} A \Rightarrow \quad J^2 = -Id$

$(-A^2)^{1/2} A$