Homework 9. March 27, 2018.

1. Let V be a vector space. Show that $v_1 \wedge \cdots \wedge v_k \in \Lambda^k(V)$ is non-zero if and only if $v_1, \ldots, v_k$ are non-zero and linearly independent.

2. Given a linear map $T : V \to V$,

   (a) what is the induced map on the space of k-tensors, i.e. on $V^{\otimes k} \to V^{\otimes k}$?

   (b) What is the induced map on the space of alternating tensors?

   (c) Show that the induced map $T : \Lambda^n(V) \to \Lambda^n(V)$ is multiplication by $\det(T)$.

3. (Change of coordinates) Change of coordinates can be thought of as a diffeomorphism between manifolds. For examaple, let

$$F : \{(r, \theta) | r > 0, \theta \in \mathbb{R}/2\pi\mathbb{Z}\} \to \mathbb{R}^2, \quad (r, \theta) \mapsto (r\cos\theta, r\sin\theta).$$

   Compute $F^*(fdx + gdy)$, where f, g are smooth functions on $\mathbb{R}^2$. (Here, we are also restricting the one-form to $\mathbb{R}^2 \backslash \{0\}$.)

4. (Restriction of a one-form to a hypersurface) An arm of a hyperbola can be parametrized by $\mathbb{R}^2$ as

$$i : \mathbb{R}^2 \to \mathbb{R}^3 \quad (x, y) \mapsto (x, y, \sqrt{1 + x^2 + y^2}).$$

   Compute the pull back under i of a one form $fdX + gdY + hdZ \in \Gamma(\mathbb{R}^3, T^*\mathbb{R}^3)$. Can you find a one-form in the kernel of i^*?

5. (Vector fields)

   (a) Write down a vector field on S^2 that vanishes on the north and south pole, and is non-zero on all other points.

   (b) Write down a non-vanishing vector field on S^3.