Homework 8. March 17, 2018.

1. In class, we proved that a regular level set of a real-valued function on M is a submanifold of M. Use this result to prove that a regular level set of a smooth map $f : M \to N$ is a submanifold of M.

2. A smooth map between manifolds $f : M \to N$ is a *submersion* if for all points $m \in M$, the pushforward $f_{*,m} : T_m M \to T_{f(m)} N$ is a surjection.

   (a) Show that for a product manifold $M_1 \times M_2$, the projection $\pi : M_1 \times M_2 \to M_1$ is a submersion.

   (b) Suppose $\pi : E \to M$ is a vector bundle. Show that π is a submersion.

   (c) Give an example of a submersion $f : M \to N$ where M and N have the same dimension.

3. *(Submersion theorem)* Suppose $f : M \to N$ is a submersion. Assume $\dim(M) = m$ and $\dim(N) = n$. Show that for any point $p \in M$, there is a chart (U, ϕ) of M centered at p, and a chart (V, ψ) centered at $F(p)$ such that
$$\psi \circ F \circ \phi^{-1}(r_1, \dots, r_m) = (r_1, \dots, r_n).$$
   Hint: The proof is similar to the proof of the regular level set theorem. First choose coordinates $(y_1, \dots, y_n)$ on V and coordinates $(x_1, \dots, x_m)$ on U. We will choose a new chart $(\tilde{U}, \tilde{\phi})$ in a neighbouhood of p and show C^∞-compatibility with U. The first n coordinates in $\tilde{\phi}$ can be $F \circ y_i$. The additional $(m-n)$-coordinates $\tilde{x}_{n+1}, \dots, \tilde{x}_n$ can be chosen to be a linear combination of the coordinates $x_1, \dots, x_m$ such that the matrix $\frac{\partial}{\partial \tilde{x}_i} \in \ker f_{*,p}$.

4. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, and a is a regular value. Then, we know that $M := f^{-1}(a)$ is a submanifold of $\mathbb{R}^n$. For a smooth function $g : \mathbb{R}^n \to \mathbb{R}$, show that the following are equivalent.

   (a) The point $m \in M$ is a critical point of $g|_M$.

   (b) The point $m \in M$ is a critical point of the function $(f, g) : \mathbb{R}^n \to \mathbb{R}^2$.

   Use this to re-do problem 4(c) in Homework 7.

5. The unit sphere S^n is defined by the equation $\sum_{i=1}^{n+1} x_i^2 = 1$ in $\mathbb{R}^{n+1}$. What is the tangent space $T_{(1,0,\dots,0)} S^n$ as a subspace of the tangent space of $\mathbb{R}^{n+1}$?