Homework 7. March 9, 2018.

1. Define

$$F : \mathbb{R}^2 \to \mathbb{R}^3, \quad (x, y) \mapsto (x, y, xy).$$

   Suppose u, v, w are coordinates on $\mathbb{R}^3$. Express $F_{*,p}\frac{\partial}{\partial x}$, $F_{*,p}\frac{\partial}{\partial y}$ as a linear combination of $\frac{\partial}{\partial u}|_{F(p)}$, $\frac{\partial}{\partial v}|_{F(p)}$, $\frac{\partial}{\partial w}|_{F(p)}$.

2. There are two coordinate systems on $\mathbb{R}^2\backslash\{0\}$. One is the Cartesian coordinate system (x, y), and the other is the polar coordinate system (r, θ) related to the former as

$$x = r\cos\theta, \quad y = r\sin\theta.$$

   Express the tangent vectors $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$ in terms of the vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

3. In the last homework, you showed that the unit sphere S^2 is a smooth manifold.
   (a) Show that the antipodal map $a : S^2 \to S^2$ is a smooth map.
   (b) We define projective space as the quotient $\mathbb{RP}^2 := S^2/\{x \sim a(x)\}$. Provide a smooth atlas on $\mathbb{RP}^2$ such that the projection $S^2 \to \mathbb{RP}^2$ is smooth.
   (c) Show that the projection $\mathbb{R}^3\backslash\{0\} \to \mathbb{RP}^2$ is a smooth map.

4. (Height function on the torus)
   (a) Show that the level set

$$T := \{((x^2 + y^2)^{\frac{1}{2}} - 2)^2 + z^2 = 1\}$$

   is a submanifold of $\mathbb{R}^3$.
   (b) Show that there is a diffeomorphism $S^1 \times S^1 \to T$.
   (c) Find the critical points of the map

$$T \to \mathbb{R}, \quad (x, y, z) \mapsto y.$$

   Observe that this is the height function of the standing torus that we discussed in class.