Homework 4. February 7th, 2018.

1. (Cell structure on real projective space) Real projective space $\mathbb{RP}^n$ is defined as the quotient $(\mathbb{R}^{n+1}\backslash\{0\})/\mathbb{R}^\times$, where $\mathbb{R}^\times := \mathbb{R}\backslash\{0\}$, i.e. two non-zero vectors $v_1, v_2 \in \mathbb{R}^{n+1}$ are in the same equivalence class if there is a scalar $\lambda \in \mathbb{R}^\times$ such that $\lambda v_1 = v_2$.

   (a) Show that $\mathbb{RP}^n$ is homeomorphic to the quotient of S^n by the relation $x \sim \text{antipode}(x)$.

   (b) Show that there is a homeomorphism $f : \mathbb{RP}^{n-1} \cup_\phi D^n \to \mathbb{RP}^n$, where the attaching map $\phi : \partial D^n \simeq S^{n-1} \to \mathbb{RP}^{n-1}$ is the quotient identifying all pairs of antipodal points on S^{n-1}.

   Hint: Theorem 22.2 of Munkres's Topology might be helpful. In the second part, first write down the maps $\mathbb{RP}^{n-1} \to \mathbb{RP}^n$, $D^n \to \mathbb{RP}^n$, and justify their continuity. The following result might be helpful to prove homeomorphism in part (b): "A continuous bijection $\psi : X \to Y$ between compact Hausdorff spaces is a homeomorphism".

2. (Cell structure on complex projective space) This problem is a complex analogue of Problem (1). Complex projective space $\mathbb{CP}^n$ is defined as the quotient $(\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^\times$, where $\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$, i.e. two non-zero vectors $v_1, v_2 \in \mathbb{C}^{n+1}$ are in the same equivalence class if there is a scalar $\lambda \in \mathbb{C}^\times$ such that $\lambda v_1 = v_2$.

   (a) Define an equivalence relation $\sim$ on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ as follows: for $v_1,\, v_2 \in S^{2n+1}$,

   $$v_1 \sim v_2 \Leftrightarrow \exists \lambda \in \mathbb{C} : \, |\lambda| = 1, \, v_1 = \lambda v_2.$$

   Show that $\mathbb{CP}^n$ is homeomorphic to the quotient $S^{2n+1}/\sim$. The quotient map $S^{2n+1} \to \mathbb{CP}^n$ is called the *Hopf fibration*.

   (b) Show that there is a homeomorphism $f : \mathbb{CP}^{n-1} \cup_\phi D^{2n} \to \mathbb{CP}^n$, where the attaching map $\phi : \partial D^{2n} \simeq S^{2n-1} \to \mathbb{CP}^{n-1}$ is Hopf fibration.

3. Problem 2, p155 Hatcher.

4. Problem 3, p155 Hatcher.

5. Problem 7, p155 Hatcher.