**Homework 11. April 18, 2018.**

1. Prove that for a compactly supported vector field on a manifold $M$, flow is defined for all time.
   Hint: Adapt the proof of the fact that the flow of a vector field on a compact manifold is defined for all time.

2. Suppose $f : M \to \mathbb{R}$ is a smooth function and $a \in \mathbb{R}$ is a regular value. Show that $M_a := f^{-1}(-\infty, a]$ is a manifold with boundary.

3. Suppose $f : M \to \mathbb{R}$ is a smooth proper function (i.e. the inverse image of a compact set is compact). Further $a, b \in \mathbb{R}$ are such that all elements in the interval $[a, b]$ are regular values of $f$. Show that the manifolds with boundary $M_a$ and $M_b$ are diffeomorphic.

   Hint: Step 1: Produce a compactly supported vector field $X$ whose integral curves $c(t)$ satisfy $\frac{d}{dt} f(c(t)) = 1$ in the region $f^{-1}[a, b]$. Such a vector field can be produced by multiplying a smooth function to the vector field $\operatorname{grad} f$.

   Step 2: Argue using Problem (1) that the time $(b - a)$ flow of $X$ exists, and it maps $M_a$ to $M_b$.

4. Suppose $M$ is a connected, oriented smooth manifold and there is a diffeomorphism $f : M \to M$ with no fixed points, and for which $f \circ f = \mathrm{Id}$. Define a quotient

   $$N := M/\sim, \quad x \sim f(x) \quad \forall x \in M.$$

   Show that $N$ is orientable if and only if $f$ is orientation-preserving.

5. Use Problem (4) to show that

   (a) $\mathbb{RP}^n$ is orientable if and only if $n$ is odd.

   (b) The Moebius strip is not orientable.