Homework 10. April 6, 2018.

1. Define a 2-form Ω on $\mathbb{R}^3$ by $\Omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$.
   (a) Compute Ω in spherical coordinates (ρ, ϕ, θ) defined by $(x, y, z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$.
   (b) Compute $d\Omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.
   (c) Compute the restriction $\Omega|_{S^2} = \iota^*\Omega$, using coordinates (ϕ, θ), on the open subset where these coordinates are defined.
   (d) Show that Ω_{S^2} is nowhere zero.

2. In each of the following problems, $g : M \to N$ is a smooth map between manifolds M and N, and ω is a differential form on N. In each case, compute $g\omega$ and $d\omega$, and verify by direct computation that $g(d\omega) = d(g\omega)$.
   (a) $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) := g(s, t) := (st, e^t)$. $\omega = xdy$.
   (b) $g : \{(r, \theta) : r > 0\} \to \mathbb{R}^2$ given by $(x, y) = (r\cos\theta, r\sin\theta)$; $\omega = dy \wedge dx$.
   (c) $g : \mathbb{R}^2 \to \mathbb{R}^3$ by $(x, y, z) = g(\theta, \phi) = ((\cos\phi + 2)\cos\theta, (\cos\phi + 2)\sin\theta, \sin\phi)$; $\omega = ydz \wedge dx$.
   (d) $g : \{(r, \theta, \phi) : r > 0\} \to \mathbb{R}^3$ given by $(x, y, z) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$; $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$.

3. Suppose $V \subset U \subset M$ are open sets in the manifold M such that the closure of V is compact and contained in U. Show that there is a bump function $\eta : M \to [0, 1]$ supported in U and equal to 1 on V.

4. Suppose $F : M \to N$ is a diffeomorphism of manifolds. Suppose $c : (-\epsilon, \epsilon) \to M$ is an integral curve of the vector field $X \in \mathrm{Vect}(M)$. Show that $F \circ c$ is an integral curve of $F \circ c$.

5. Suppose g is a Riemannian metric on a manifold M, and $F : M \to \mathbb{R}$ is a smooth function. The *gradient vector field* of F, denoted by $\operatorname{grad} F$, is defined by the condition

$$dF(v) = g(\operatorname{grad} F, v), \quad \forall v \in \mathrm{Vect}(M).$$

   (a) Justify why $\operatorname{grad} F$ is well-defined.
   (b) Suppose $c(t)$ is an integral curve of the vector field $\operatorname{grad} F$. Compute $\frac{d}{dt}F(c(t))$.