Universal skein theory for finite depth subfactor planar algebras

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Joint work with Srikanth Tupurani

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### Theorem

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- which may be chosen in $P_{k+1}$ (but not necessarily in $P_k$).
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- WHAT is a planar algebra?
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- **WHEN** is a planar algebra said to be of finite depth?
- **WHY** presentations/skein theories for planar algebras?
- **HOW** is the main theorem proved?
What is a planar algebra? I Tangles and composition
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Here are some examples of planar tangles.

![Planar Tangle Diagrams]

Vijay Kodiyalam (IMSc)  Skein theory for planar algebras  Chennai, August 2010
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What is a planar algebra? Tangles and composition

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The first tangle, say $T$, is a 3-tangle with internal boxes of colour 4,2,3 and 0. The second, say $S$, is a 2-tangle with no internal boxes. Tangles may be composed. The third tangle is denoted $T \circ_{D_2} S$. 
Planar algebra

A planar algebra $P$ is a collection of vector spaces $\{P_n\}_{n=0,1,2,\ldots}$ together with maps $Z_T$ for every planar tangle $T$ satisfying compatibility with composition.
What is a planar algebra? II Definition and proposition

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For our example $T$, the map $Z_T : P_4 \otimes P_2 \otimes P_3 \otimes P_0 \rightarrow P_3$. 
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Proposition

For a planar algebra \( P \) and each \( k \), the vector space \( P_k \) acquires an associative algebra structure for the action of the tangle \( M^k \) with a unit given by the tangle \( 1^k \) and algebra homomorphism \( P_k \rightarrow P_{k+1} \) given by \( I^{k+1} \).
What is a planar algebra? III Elementary tangles

The letters adjacent to the strings represent the number of times the string is cabled.
Which planar algebras are subfactor planar algebras?
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Jones’ theorem (1999)

Every finite index extremal $II_1$-subfactor yields a subfactor planar algebra in a natural way. All subfactor planar algebras arise in this manner.
When is a planar algebra said to be of finite depth?
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The following tangles are the Jones projection tangles (for $n \geq 2$).

$$E^n = \begin{array}{c}
\ast \\
n - 2
\end{array}$$
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Define $E_n \in P_n$ by $E_n = Z_{E^n}(1)$. The $E_n$ are scaled Jones projections.

**Finite depth**

A planar algebra $P$ is said to be of finite depth if there is a $k \in \mathbb{N}$ such that $1_{k+1} \in P_k E_{k+1} P_k$. The least such $k$ is said to be the depth.
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For a subfactor planar algebra, finite depth is equivalent to finiteness of the principal graphs of the subfactor.
Given a label set $L = \bigsqcup_k L_k$ the universal planar algebra on $L$, denoted $P(L)$, is the planar algebra with $P(L)_k$ being the vector space with basis all $L$-labelled $k$-tangles. There is an obvious planar algebra structure.
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In any planar algebra $P$ there is a notion of a planar ideal. For a subset $R \subseteq P(L)$, if the planar ideal that it generates is $I(R)$, the quotient planar algebra $P(L)/I(R)$ is denoted $P(L, R)$ and $(L, R)$ is said to present the quotient. Such a presentation is also known as a skein theory for the planar algebra.
Why presentations/skein theories? II Examples

• Lnd 2002: Group planar algebra
• KdyLndSnd 2003: Kac algebra planar algebra
• MrrPtrSny 2008: $D_{2n}$ planar algebra
• Ptr 2009: Haagerup planar algebra
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Let $P$ be a subfactor planar algebra of finite depth $k$. Then,
- $P$ has a finite presentation
- with a single generator
- which may be chosen in $P_{k+1}$ (but not necessarily in $P_k$).
Step I: Description of generators and relations
Given a planar algebra $P$ of finite depth $k$, let $B$ be a basis of $P_k$ and set $L = L_k = B$. These will be the generators of our presentation.
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**Templates**

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**Templates**

A template is an ordered pair $S \Rightarrow T$ of tangles of the same colour.

Here are two examples of templates.

We call these the multiplication and depth templates.
Step I : Description of generators and relations II
Template holding for \((P, B)\)

If \(S \Rightarrow T\) is a template, \(P\) is a planar algebra, and \(B \subseteq P\), the template is said to hold for \((P, B)\) if the span of \(Z_S\) with inputs from \(B\) is contained in the span of \(Z_T\) with inputs from \(B\).
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If a template holds for \((P, B)\) it gives relations in \(P(B)\).
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To complete Step I, we specify an explicit set of 6 templates that hold for any \((P, B)\) where \(P\) is a subfactor planar algebra of finite depth \(k\) and \(B\) is a basis of \(P_k\). The relations determined by these templates specify a finite subset \(R \subseteq P(L)\) where \(L = L_k = B\).
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To complete Step I, we specify an explicit set of 6 templates that hold for any \((P, B)\) where \(P\) is a subfactor planar algebra of finite depth \(k\) and \(B\) is a basis of \(P_k\). The relations determined by these templates specify a finite subset \(R \subseteq P(L)\) where \(L = L_k = B\).

We then show that \(P(L, R) \cong P\).
Step II : Sketch of injectivity proof

That there is a map of $P(L, R)$ onto $P$ is clear by choice of the relations. For injectivity we first define a family of tangles $T^n$ as in the figure below.
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![Diagram of tangles](image)

Next, define $\mathcal{T} = \{ T^{n_0}_{n_1, \ldots, n_b} : T \circ (T^{n_1}, \ldots, T^{n_b}) \Rightarrow T^{n_0} \text{ for } (P, B) \}$. 
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Injectivity at level $k \iff \mathcal{T} = \text{all tangles}$.
Step III : Consequences of templates
Consequences

Given a set of templates, consider the smallest set containing them and closed under transitivity and composition on the outside. Each element of this set is said to be a consequence of those of the original set.
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To show that \(\mathcal{T}\) contains all tangles, it suffices to see that it is closed under composition (which is obvious by definition) and that it contains a basic set of generating tangles.
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To show that \(T\) contains all tangles, it suffices to see that it is closed under composition (which is obvious by definition) and that it contains a basic set of generating tangles.

Proposition

\(T\) contains a set of generating tangles.
Step IV: Finish of injectivity proof
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**Proposition**

Let $P$ be a planar algebra for which $1_{k+1} \in P_k E_{k+1} P_k$ for some $k$. Then for any $m, n \geq k$ there is a natural isomorphism of $P_{k-1} - P_{k-1}$-bimodules

$$P_m \otimes_{P_{k-1}} P_n \rightarrow P_{m+n-(k-1)}.$$
Step V : Single generation
Suppose that $P$ is a subfactor planar algebra of depth $k$. Certainly, it is generated as a planar algebra by $P_k$. Since $P_k$ is a finite-dimensional $C^*$-algebra it is singly generated by say, $x$, which we may assume has a non-zero trace.
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The element $z \in P_{2k}$ defined by

\[
\begin{array}{c}
\ast \ \ k \\
\ast \ \ k
\end{array}
\begin{array}{c}
x \\
k
\end{array}
\begin{array}{c}
x^* \\
k
\end{array}
\begin{array}{c}
\ast \ \ k \\
k
\end{array}
\]

is easily seen to generate $P$ since both $x$ and $x^*$ are in the generated planar algebra.
Step VI: Can we improve the $2k$?
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**Proposition**

Let $A$ be a finite dimensional complex semisimple algebra and $S$ an involutive anti-automorphism of $A$. Then there is an $a \in A$ such that $a$ and $Sa$ generate $A$. 
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Corollary

If $P$ is a subfactor planar algebra of depth $k$ and $2t$ is the even number in $\{k, k + 1\}$, then $P$ is generated by a $2t$ box.