Universal skein theory for finite depth subfactor planar algebras

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Joint work with Srikanth Tupurani

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- WHEN is a planar algebra said to be of finite depth ?
- WHY presentations/skein theories for planar algebras ?
- HOW is the main theorem proved ?

What is a planar algebra ? I Tangles and composition

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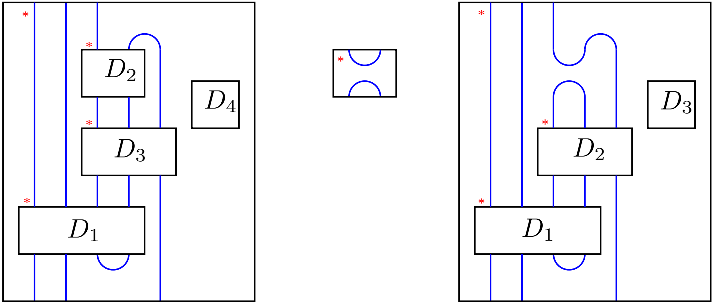
Here are some examples of planar tangles.



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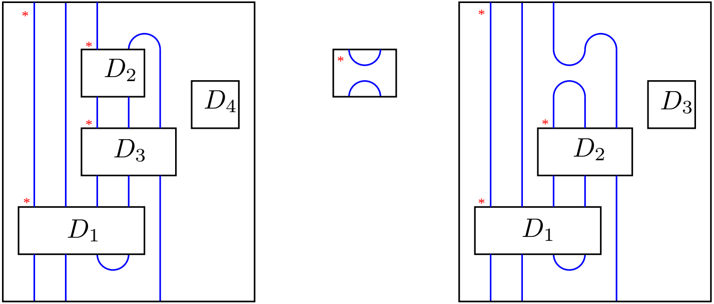

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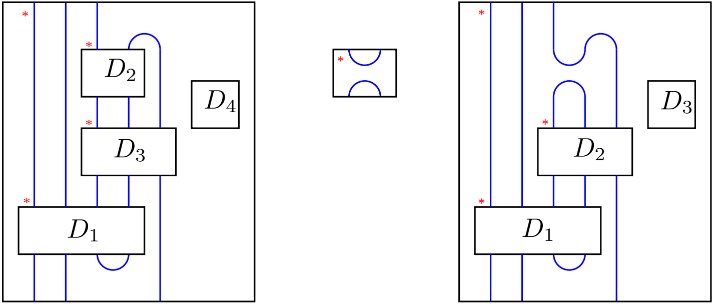

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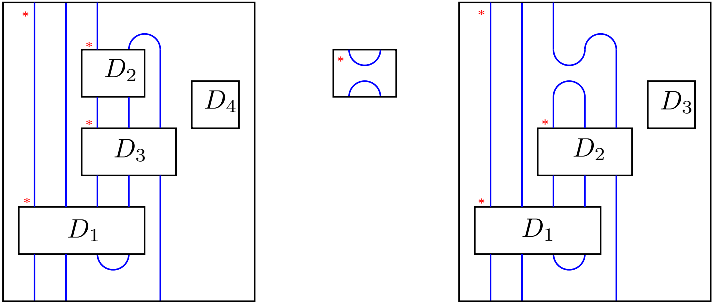

The first tangle, say T, is a 3-tangle with internal boxes of colour 4,2,3 and 0. The second, say S, is a 2-tangle with no internal boxes. Tangles may be composed. The third tangle is denoted $T \circ_{D_2} S$.

What is a planar algebra ? II Definition and proposition

Planar algebra

A planar algebra P is a collection of vector spaces $\{P_n\}_{n=0,1,2,\cdots}$ together with maps Z_T for every planar tangle T satisfying compatibility with composition.

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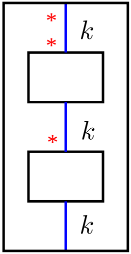
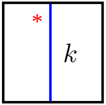
Proposition

For a planar algebra P and each k, the vector space P_k acquires an associative algebra structure for the action of the tangle M^k with a unit given by the tangle 1^k and algebra homomorphism $P_k \rightarrow P_{k+1}$ given by I^{k+1}.

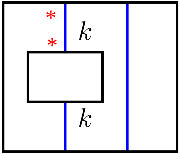
What is a planar algebra ? III Elementary tangles



M^k



1^k



I^{k+1}

The letters adjacent to the strings represent the number of times the string is cabled.

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Jones' theorem (1999)

Every finite index extremal II_1-subfactor yields a subfactor planar algebra in a natural way. All subfactor planar algebras arise in this manner.

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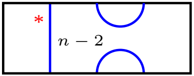
The following tangles are the Jones projection tangles (for $n \geq 2$).

$E^n =$ 

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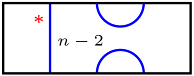

Define $E_n \in P_n$ by $E_n = Z_{E^n}(1)$. The E_n are scaled Jones projections.

Finite depth

A planar algebra P is said to be of finite depth if there is a $k \in \mathbb{N}$ such that $1_{k+1} \in P_k E_{k+1} P_k$. The least such k is said to be the depth.

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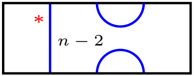
E^n = 

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For a subfactor planar algebra, finite depth is equivalent to finiteness of the principal graphs of the subfactor.

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Given a label set $L = \coprod_k L_k$ the universal planar algebra on L, denoted $P(L)$, is the planar algebra with $P(L)_k$ being the vector space with basis all L-labelled k-tangles. There is an obvious planar algebra structure.

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In any planar algebra P there is a notion of a planar ideal. For a subset $R \subseteq P(L)$, if the planar ideal that it generates is $I(R)$, the quotient planar algebra $P(L)/I(R)$ is denoted $P(L, R)$ and (L, R) is said to present the quotient. Such a presentation is also known as a skein theory for the planar algebra.

Why presentations/skein thories ? II Examples

- Lnd 2002 : Group planar algebra
- KdyLndSnd 2003 : Kac algebra planar algebra
- MrrPtrSny 2008 : D_{2n} planar algebra
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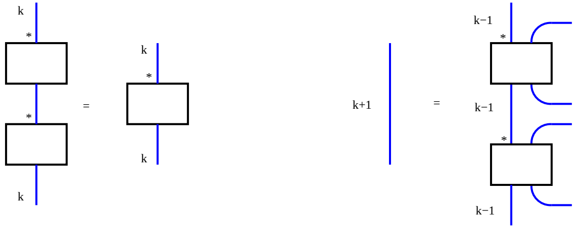
Here are two examples of templates.



We call these the multiplication and depth templates.

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Template holding for (P, B)

If $S \Rightarrow T$ is a template, P is a planar algebra, and $B \subseteq P$, the template is said to hold for (P, B) if the span of Z_S with inputs from B is contained in the span of Z_T with inputs from B.

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To complete Step I, we specify an explicit set of 6 templates that hold for any (P, B) where P is a subfactor planar algebra of finite depth k and B is a basis of P_k. The relations determined by these templates specify a finite subset $R \subseteq P(L)$ where $L = L_k = B$.

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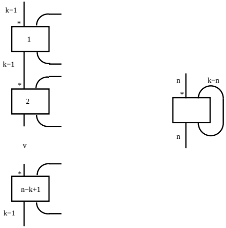
We then show that $P(L, R) \cong P$.

Step II : Sketch of injectivity proof

That there is a map of $P(L,R)$ onto P is clear by choice of the relations. For injectivity we first define a family of tangles T^n as in the figure below.



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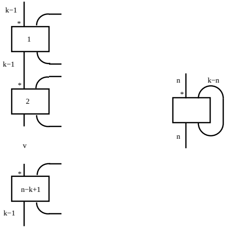

Next, define $\mathcal{T} = \{T^{n_0}_{n_1,\cdots,n_b} : T \circ (T^{n_1}, \cdots, T^{n_b}) \Rightarrow T^{n_0} \text{ for } (P, B)\}$.

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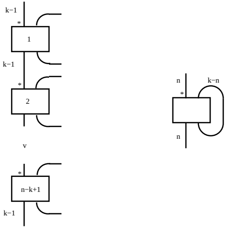

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Injectivity at level $k \Leftrightarrow \mathcal{T} =$ all tangles.

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Let P be a planar algebra for which $1_{k+1} \in P_k E_{k+1} P_k$ for some k. Then for any $m, n \geq k$ there is a natural isomorphism of $P_{k-1} - P_{k-1}$-bimodules

$$P_m \otimes_{P_{k-1}} P_n \to P_{m+n-(k-1)}.$$

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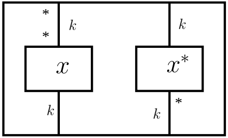
The element $z \in P_{2k}$ defined by



is easily seen to generate P since both x and x^* are in the generated planar algebra.

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Corollary

If P is a subfactor planar algebra of depth k and $2t$ is the even number in $\{k, k+1\}$, then P is generated by a $2t$ box.