Transfinite considerations

V.S. Sunder
Institute of Mathematical Sciences
Chennai, India
sunder@imsc.res.in

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Overview

1. Posets

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2. Zorn's Lemma - statement and applications

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3. Cardinalty of sets - order and infinities

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3. Cardinalty of sets - order and infinities
4. Well-ordering - ordinals and cardinals, transfinite induction

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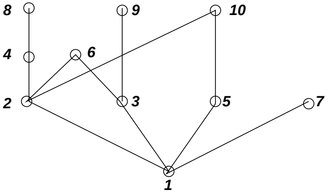
3. Another example of a poset which is not totally ordered is given by $X = \{1, 2, ..., 9, 10\}$, with $x \leq y \Leftrightarrow x | y$. The order is best illustrated by a **directed graph** as follows:



An element $\omega \in X$ is said to be a **maximal element** if $x \in X, \omega \leq x \Rightarrow x = \omega$. Note that 'maximal' is not the same as 'largest'. In the last example, 6,7,8,9 and 10 are all maximal elements, while only 10 is the largest element.

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Our graphical portrayal of posets shows that finite posets clearly have maximal elements (reason: keep going up as long as you can, and the assumed finiteness ensures your quest will eventually end in success). It also shows there may be problems with infinite posets.

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Counterexample: Let X be the collection of all subsets of $\mathbb{N}$ with infinite complements. For any $\omega \in X$, set $x = \omega \cup \{n\}$ for some $n \notin \omega$, then $x \in X, \omega \leq x$ and $\omega \neq x$. Thus X has no maximal element.

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Zorn's lemma is a vital ingredient of every mathematician's tool-kit; it is essential to prove existence of (i) maximal ideals in commutative rings, (ii) bases in vector spaces, (iii) *orthonormal* bases in Hilbert spaces, ...
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Any infinite but locally finite tree has an infinite path.

Cardinality

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Sets X and Y are said to have the *same cardinality* if there exists a bijective correspondencwe between them, i.e., if there exists a function $f : X \to Y$ which is 1-1 (i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$) and onto (i.e., $y \in Y \Rightarrow \exists x \in X$ such that $y = f(x)$).

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When this happens, we say $|X| = |Y|$. Thus, we have not defined $|X|$, but identified when $|X| = |Y|$. More generally, say $|X| \leq |Y|$ if there exists a 1-1 function $f : X \to Y$. (Thus, $|X| \leq |Y|$ if and only if there exists a subset $Y_0 \subset Y$ such that $|X| = |Y_0|$.)

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It is easy to see that $\leq$ is a reflexive and transitive relation. That it is anti-symmetric is the content of the so-called *Schroeder-Bernstein theorem* - whose proof amounts to showing that if you are given that there exist 1-1 functions $f : X \to Y$ and $g : Y \to X$ then you can construct a bijection, say F between X and Y.

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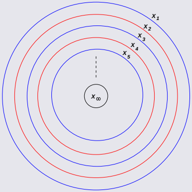
Proof.

We are given that there exist 1-1 functions $f : X \rightarrow Y$ and $g : Y \rightarrow X$ and need to construct a bijection, say F between X and Y.
Let $h = g \circ f$ and define

$$X_n = \begin{cases} X & \text{if } n = 0 \\ g(Y) & \text{if } n = 1 \\ h(X_{n-2}) & \text{if } n \geq 2 \\ \cap_{k=1}^{\infty} X_k & \text{if } n = \infty \end{cases}$$



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(contd.)Notice that $X_0 \supset X_1 \supset X_2 \supset \cdots$, and that

$$X = \left(\coprod_{n=0}^{\infty}((X_{2n} \setminus X_{2n+1})\right) \coprod \left(\coprod_{n=0}^{\infty}((X_{2n+1} \setminus X_{2n+2})\right) \coprod X_{\infty}$$

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Notice that the equation

$$F(x) = \begin{cases} g^{-1}(x) & \text{if } x \in X_\infty \\ f(x) & \text{if } x \notin X_\infty \end{cases}$$

defines the desired bijection F of X onto Y. □

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(1) $\Rightarrow$ (2) Note that $|\mathbb{N}| = |\mathbb{N} \setminus \{1\}|$. If $f : \mathbb{N} \to X$ is 1-1, set $X_0 = X \setminus \{f(1)\}$ $(= f(\mathbb{N} \setminus \{1\}) \coprod (X \setminus f(\mathbb{N})))$.

(2) $\Rightarrow$ (1) If $x_1 \in X \setminus X_0$, inductively define $x_{N+1} = f(x_n)$ and notice that $\mathbb{N} \ni n \mapsto x_n \in X$ is an injective map. $\square$

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In fact, these are equivalent to requiring that there exists a partition $X = X_1 \coprod X_2$ such that $|X| = |X_1| = |X_2|$ (provided X is not empty).

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A slight variation of this argument can be used to show that $|\mathbb{N}| \lneqq |\mathbb{R}|$.

Well-ordering theorem

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2. **The well-ordering theorem:** Every non-empty set can be well-ordered,

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The smallest infinite ordinal is ω, the 'type' of $\mathbb{N}$ with its natural ordering. (Notice that any finite totally ordered set is well-ordered, as is $\mathbb{N}$, so that

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Before we can see more ordinal numbers, it will be desirable to digress into the algebra of ordinal numbers. Ordinal numbers can be added, multiplied, etc., although these operations are not commutative!

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The sum is a little more delicate. The *disjoint union* of X and Y is the union of copies of them which have no intersection; a formal way to do this is to set

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If $(X_j, \leq_j), j = 1, 2$ are posets, their Cartesian product acquires the *reverse dictionary ordering*, thus:

$$(x_1, x_2) \leq (y_1, y_2) \Rightarrow \begin{cases} y_1 \leq y_2 & \text{if } y_1 \neq y_2 \\ x_1 \leq x_2 & \text{if } y_1 = y_2 \end{cases}$$

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And their disjoint union acquires a partial order if we demand that $x_1 \leq x_2 \; \forall x_j \in X_j$ and that the new order restricts on X_j to $\leq_j$.

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4. Show that
   - $3 + \omega = \omega \neq \omega + 3$
   - $3 \cdot \omega = \omega \neq \omega \cdot 3$

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3. Deduce from the previous problems that if $\alpha_j, j = 1, 2$ are ordinal numbers, then the sum $\alpha_1 + \alpha_2$ and the product $\alpha_1 \cdot \alpha_2$ are well-defined.
4. Show that
   - $3 + \omega = \omega \neq \omega + 3$
   - $3 \cdot \omega = \omega \neq \omega \cdot 3$
5. If α, β, γ are ordinal numbers, show that
   - $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$
   - $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$

Some facts on ordinals

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- A well-ordered set B is said to be a **contrinuation** of a well-ordered set A, if A is order-isomorphic to some initial segment of B; if this happens and if α and β denote the order-types of A and B respectively, we shall say that $\alpha \leq \beta$.

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- It is true that if α, β are any two ordinals, then either $\alpha < \beta$ or $\alpha = \beta$ or $\beta < \alpha$

Mathematicians discovered long ago that there logical pitfalls and landmines around if one speaks loosely of things like ‘the set of all sets’. (*Reason:* If you want to allow such gadget as $\mathcal{A} = \{A : A \notin A\}$, you run into the fallacy that both possibilities $\mathcal{A} \in \mathcal{A}$ and $\mathcal{A} \notin \mathcal{A}$ lead to contradictions.

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Thus, while one should shy away from talking of the ‘set of all ordinals’, one may talk of the set of all ordinals which are less than a fixed ordinal; and since one can take as large an ordinal, say Ω, as one may care to, it makes sense to talk of the set of all ordinals that are less than any fixed ordinal, however large.

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If you are not put off by this bit of 'transfinite trickery', you will be ready to accept that a 'listing' of the ordinals looks like this:

$$1, 2, 3, \cdots, \omega, \omega+1, \omega+2, \omega+3, \cdots, \omega+\omega = \omega\cdot 2, \omega\cdot 2+1, \cdots$$

$$\omega\cdot 3, \cdots, \omega^2, \cdots, \omega^3, \cdots, \omega^\omega, \cdots, \omega^{\omega^\omega}, \cdots$$

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To make sense of α^β for ordinal numbers α, β, one needs, as in the case of sums and products, to show that if A and B are posets, then (i) there is a natural poset structure on the set A^B of all functions from B into A, (ii) the isomorphism type of A^B only depends on those of A and B, and (iii) A^B is well-ordered if A and B are.

We are finally in a position to define a **cardinal number** as an ordinal number , say α such that if A is any well-ordered set of type α, and if B is any well-ordered set of type β, say, then $|A| = |B| \Rightarrow \alpha \leq \beta$.

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Thus, a cardinal number is the smallest ordinal number in the set of all ordinal numbers of the same cardinality. The italicised paragraph in the last slide ensures that there is no logical problem about this definition. We could also have defined a cardinal number as the set of all ordinal numbers of a fixed cardinality, rather than as the smallest element of that set.

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Unlike ordinals, addition and multiplication are commutative operations on cardinal numbers. In fact, if α, β are cardinal numbers, of which at least one is infinite, then

$$\alpha + \beta = \alpha \cdot \beta = \max\{\alpha, \beta\}.$$

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Transfinite Induction

If A is any well-ordered set, there is an 'induction' mechanism available, as follows:

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This can be used to verify the validity of some statement for every ordinal nuber!