From the Temperley Lieb algebra to non-crossing partitions

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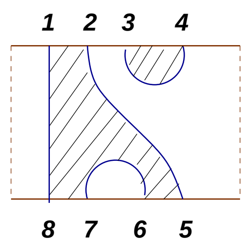
A **Kauffman diagram** is an isotopy class of a planar (i.e., non-crossing) arrangement of n curves in a box with their ends tied to $2n$ marked points on the boundary; an example, with $n = 4$ is illustrated below:



The collection of such diagrams will be denoted by $\mathcal{K}_n$.

Proposition 1:

$$|\mathcal{K}_n| = \frac{1}{n+1}\binom{2n}{n}$$

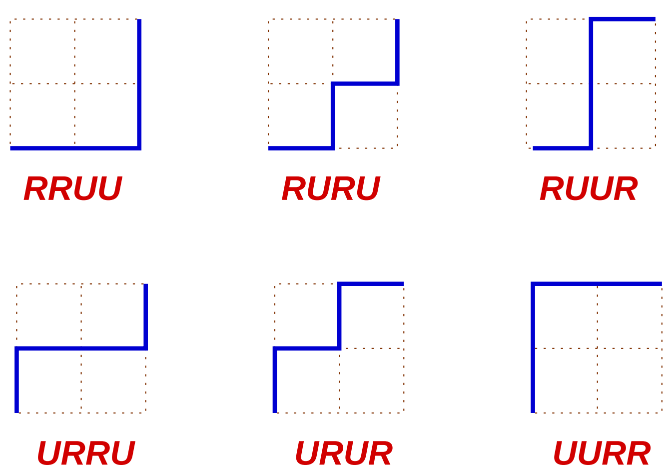
$\square$

We shall indicate a proof of this identity (taken from [GHJ]) below.

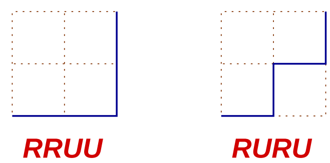
For $x, y \in \mathbb{R}^2$ such that $x_i \leq y_i$ for $i = 1, 2$, let $P(x, y)$ denote the collection of all 'walks' γ from x to y, in which each step is of unit length, and is to the right (R) or up (U). It is clear that

$$|P(x,y)| = \binom{y_1 - x_1 + y_2 - x_2}{y_1 - x_1} .$$

We will primarily be interested in $P((0,0),(n,n))$. For instance, we see that $P((0,0),(2,2))$ is as follows:



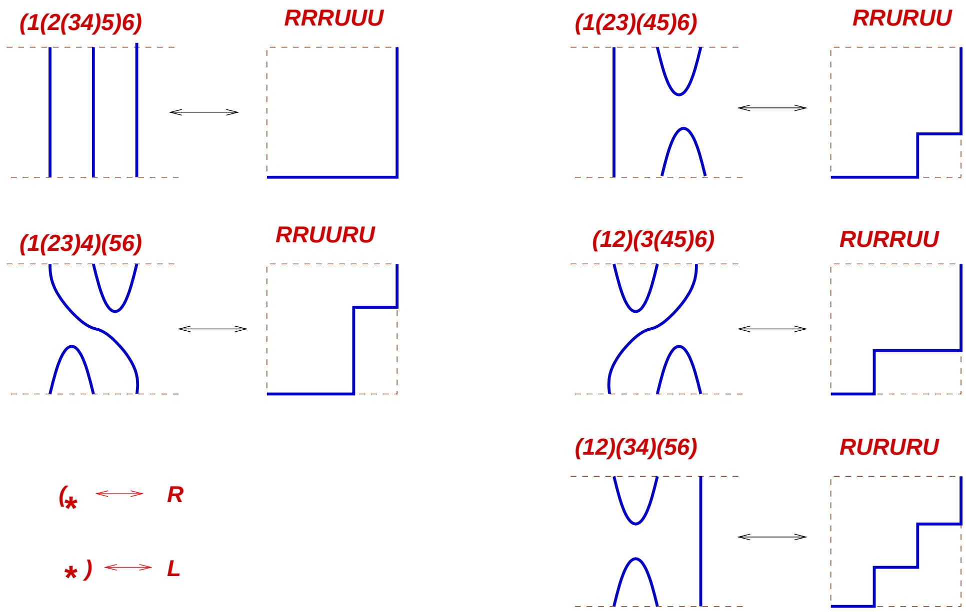
Let $P_g((0,0),(n,n))$ consist of those paths which do not cross the main diagonal (- i.e., every initial segment has at least as many R's as U's.) Thus, $P((0,0),(2,2))$ is as follows:



It is an easy exercise to verify that

$$|\mathcal{K}_n| = |P_g((0,0),(n,n))|.$$

The bijection is illustrated below, for $n = 3$:



Proof of Proposition 1:

We need to show that

$$|P_g((0,0),(n,n))| = \frac{1}{n+1}\binom{2n}{n}.$$

Note - by a shift - that $|P_g((0,0),(n,n))| = |P_g((1,0),(n+1,n))|$, and that the right side counts the ('good') paths in $P((1,0),(n+1,n))$ which do not meet the main diagonal. Consider the set $P_b((1,0),(n+1,n))$ of ('bad') paths which do cross the main diagonal. The point is that any path in $P_b((1,0),(n+1,n))$ is of the form $\gamma = \gamma_1 \circ \gamma_2 \in P_b((1,0),(n+1,n))$, where $\gamma_1 \in P((1,0),(j,j)), \gamma_2 \in P((j,j),(n+1,n))$, and (j,j) is the 'first point' where γ touches the main diagonal. Define $\widetilde{\gamma} = \gamma_1' \circ \gamma_2$, where γ_1' is the reflection of γ_1 about the main diagonal. This yields a bijection

$$P_b((1,0),(n+1,n)) \ni \gamma \leftrightarrow \widetilde{\gamma} \in P((0,1),(n+1,n))$$

Hence

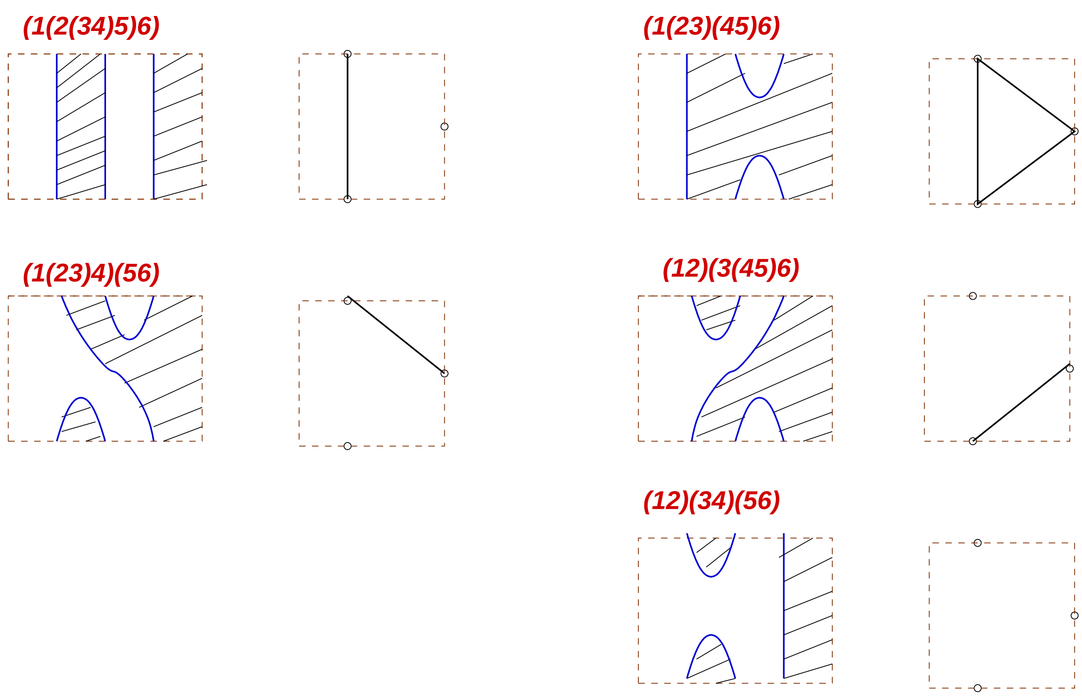
$$
\begin{aligned}
&|P_g((1,0),(n+1,n))| \\
&\quad = |P((1,0),(n+1,n))| - |P_b((1,0),(n+1,n))| \\
&\quad = |P((1,0),(n+1,n))| - |P((0,1),(n+1,n))| \\
&\quad = \binom{2n}{n} - \binom{2n}{n+1} \\
&\quad = \frac{1}{n+1}\binom{2n}{n} ,
\end{aligned}
$$

thereby proving Prop. 1. □

We now move to another sequence $\{NC_n : n \geq 1\}$ of sets whose cardinalities are also given by the Catalan numbers, where NC_n is the set of *non-crossing partitions*: these being partitions of a set of n marked points on a circle with the property that the convex hulls of any two distinct equivalence classes of the partition are disjoint.

$$|\mathcal{K}_n| = |NC_n| \ ,$$

the bijection being illustrated below for $n = 3$.

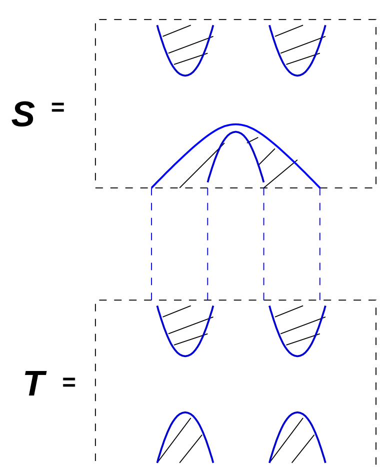


The n points of the element $\tilde{S}$ of NC_n which corresponds to a $S \in NC_n$ may be chosen as points midway between an odd point and the next even point, with the 'black regions' of S determining the equivalence classes of $\tilde{S}$.

The algebras $TL_n(\delta)$: Fix a positive scalar δ (often assumed to be greater than 2, for technical reasons) and define a (complex) algebra $TL_n(\delta)$ with a basis consisting of Kauffman diagrams on $2n$ points, and multiplication defined by the rule

$$ST = \delta^{\lambda(S,T)} U$$

where (1) U is the diagram obtained by concatenation - i.e., identifying the point marked $(2n - j + 1)$ for S with the point marked j for T, for $1 \leq j \leq n$ - and erasing any 'internal loops' formed in the process, and (2) $\lambda(S, T)$ is the number of 'internal loops so erased. For example, we have, if



$$S\,T = \delta\,T$$

$$T^2 = \delta^2\,T$$

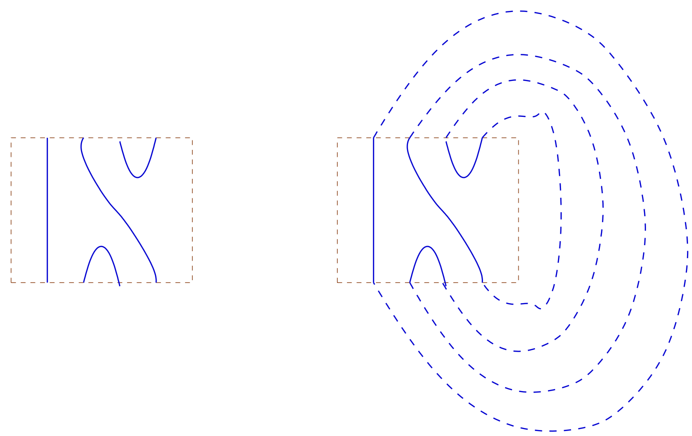
In fact, the algebra $TL_n(\delta)$ is associative (since isotopic diagrams are identified), and even unital - with all the strands of the identity element 'coming straight down' (joining j and $2n - j + 1$).

In much the same way, each NC_{2n} indexes a basis for an algebra $NC_{2n}(\delta)$ - the only difference being that 'internal loops' are replaced by 'internal components'.

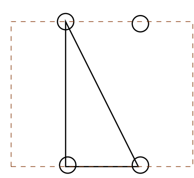
The further ingredient that these algebras come equipped with is a natural pictorially defined trace. Specifically, for $S \in \mathcal{K}_n$ (resp., $\tilde{S} \in NC_{2n}$) define $\tau(S)$ (resp., $\tau(\tilde{S})$ to be δ^c, where c is the number of loops (resp., components) occurring in the diagram obtained by connecting the point marked j to the point marked $2n - j + 1$.

In the example below:

S =



$\tilde{S}$ =



we see that

$$\tau(S) = \delta^2 \ , \ \tau(\tilde{S}) = \delta \ .$$

We have the following result whose statement seems intuitively reasonable/plausible, but where neither the asserted isomorphism nor the proof of the theorem are so intuitively obvious!

Theorem: There exists a trace-preserving algebra isomorphism $\phi : TL_{2n}(\delta) \to NC_n(\delta^2)$; this has the property that

$$\phi(S) = \frac{\tau(S)}{\tau(\tilde{S})}\tilde{S}$$

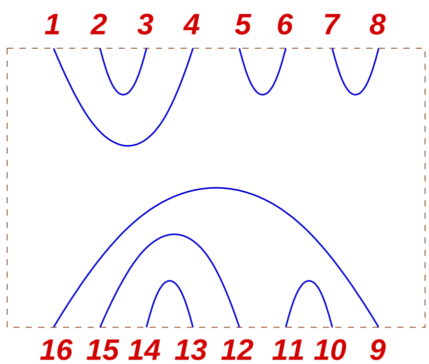
for all Kauffmann diagrams $S \in \mathcal{K}_n$. $\square$

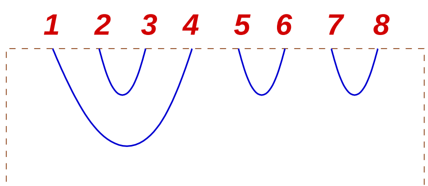
This boils down to proving that, for arbitrary $S, T \in \mathcal{K}_{2n}$, we have

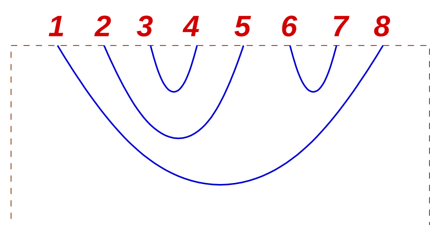
$$\frac{\tau(S)}{\tau(\tilde{S})}\frac{\tau(T)}{\tau(\tilde{T})} = \frac{\tau(ST)}{\tau(\tilde{S}\tilde{T})} \qquad (*) \ .$$

And it turns out that the proof of (*), in turn, can be reduced to that of the special case - of (*) - where neither S nor T has any *through strings*.

Such an S is seen to be determined by an ordered pair (S_+, S_-) where $S_\pm \in \mathcal{K}_n$ - where we think of the $2n$ marked points of $S_\pm$ as being arrayed on one side of the box.

$S =$ 

$S_+ =$ 

$S_- =$ 

We shall think of elements of $\mathcal{K}_n$, such as $S_\pm$, as partitions of $\{1, 2, \cdots, 2n\}$ (where equivalence classes are doubletons); we shall write $S_+ \vee S_-$ for the finest partition which is refined by S_+ as well as S_-. (Thus, in our example above, $S_+ \vee S_-$ is the partition of $\{1, \cdots, 8\}$ containing only one equivalence class.)

One more piece of notation: given $B \in \mathcal{K}_n$, we shall write $|B|$ for the number of classes in B and $\tilde{B}$ for $B \vee B_0$, where

$$B_0 = \{\{1, 2\}, \{3, 4\}, \cdots \{2n - 1, 2n\}\}.$$

The key lemma turns out to be the following 'linearisation result':

Lemma:

$$
\begin{aligned}
& 2(|X \vee Y| - 2|\tilde{X} \vee \tilde{Y}|) \\
& \quad = \; |X| - 2|\tilde{X}| + |Y| - 2|\tilde{Y}|
\end{aligned}
$$

for all $X, Y \in \mathcal{K}_n$. $\square$

This is because the assertion (*) translates - in case neither S nor T has through strings - to the assertion that

$$
\begin{aligned}
& (|S_- \vee T_+| - 2|\tilde{S_-} \vee \tilde{T_+}| \\
& \qquad + |S_+ \vee T_-| - 2|\tilde{S_+} \vee \tilde{T_-}| \\
& \quad = \; (|S_- \vee S_+|) - 2(|\tilde{S_-} \vee \tilde{S_+}|) \\
& \qquad + (|T_- \vee T_+|) - 2(|\tilde{T_-} \vee \tilde{T_+}|)
\end{aligned}
$$

which is seen, by our 'linearisation result', to indeed be true.

Since $|X| = n \, \forall X \in \mathcal{K}_n$, our linearisation lemma may be restated thus:

$$\begin{aligned} & |X \vee Y| - 2|\tilde{X} \vee \tilde{Y}| \\ = \; & n - |\tilde{X}| - |\tilde{Y}| \end{aligned}$$

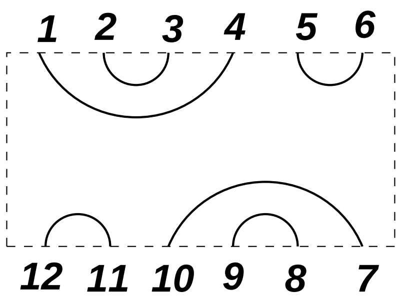
for all $X, Y \in \mathcal{K}_n$.

Instead of spelling out a detailed proof of this result, we shall simply:

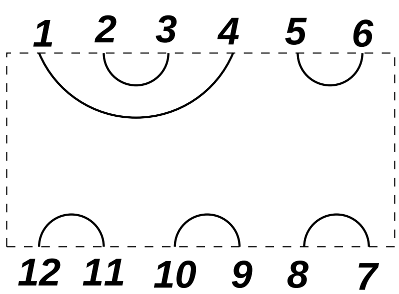
(a) state that 'one half' of this assertion is a consequence of the Euler characteristic, and

(b) illustrate the assertion above with an example.

Consider the example given by



= **X** = {{1,4},{2,3},{5,6} {12,11},{10,7},{9,8}}



= **Y** = {{1,4},{(2,3},{5,6} {12,11},{10,9},{8,7}}

Here, $n = 6$, and we see that

$X \bigvee Y = \{\{1,4\},\{2,3\},\{5,6\},\{12,11\},\{10,9,8,7\}\}$

while

$\widetilde{X} \bigvee \widetilde{Y} = \widetilde{X} = \{\{1,2,3,4,5,6,7,8,9,10,11,12\}\}$

and

$$\widetilde{Y} = \{\{1,2,3,4,12,11,10,9\},\{5,6,7,8\}\}$$

so the equation to be proved reads:

$$5 - 2 \cdot 1 = 6 - 1 - 2$$

Further details can be found in our paper [KS] below:

References:

[GHJ] F. Goodman, P. de la Harpe and V.F.R. Jones, *Coxeter graphs and towers of algebras,* MSRI Publ., 14, Springer, New York, 1989.

[KS] Vijay Kodiyalam and V.S. Sunder, *Temperley-Lieb and Non-crossing Partition planar algebras*, to appear in a Conference Proceedings to be published by AMS, in the 'Contemporary Math.' series. (may also be found on my home-page "http://www.imsc.res.in/~sunder/")