Planar depth and planar subalgebras

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We consider the notion of planar depth of a planar algebra, viz., the smallest $n$ for which the planar algebra is generated by its ‘$n$-boxes’. We establish a simple result which yields a sufficient condition, in terms of the principal graph of the planar algebra, for the planar depth to be bounded by $k$. This suffices to determine the planar depth of the $E_6$, $E_8$ and the $\text{III}_1$ subfactors.

We then consider a planar subalgebra of the ‘group planar algebra’ which is naturally associated with a group $\Theta$ of automorphisms of the given group $G$. We show that this planar algebra corresponds to the ‘subgroup-subfactor’ associated with the inclusion $\Theta \subset \langle G \times \Theta \rangle$ (given by the semi-direct product extension). We conclude with a discussion of the planar depth of this planar algebra $P^\Theta$ in some examples.

Key Words: operator algebras, subfactor, planar algebra, standard invariant.

1. INTRODUCTION

Jones defined the notion of a planar algebra in [6], where he showed, using Popa’s characterisation in [10] of the so-called $\lambda$-lattices, that (spherical, connected $C^*$-) planar algebras are in bijection with standard invariants of extremal subfactors. In particular, each planar algebra is the planar algebra associated with at least one subfactor.

Given a planar algebra $P = \{P_n\}$, let us define $P^n$ to be the planar subalgebra (of $P$) generated by $P_n$; then we have a tower

$$P^0 \subseteq P^1 \subseteq P^2 \subseteq \cdots \subseteq P^n \subseteq P^{n+1} \subseteq \cdots \cup_{n=0}^\infty P^n = P \quad (1.1)$$
of planar subalgebras of $P$. (Note that $P^0$ is nothing but the Temperley-Lieb planar algebra $TL$ - with the same $\delta$ as the given planar algebra.) It is fairly easy to see that this tower will stabilise at a finite stage provided the initial planar algebra $P$ has finite depth'. Of course, this finite depth condition is far from necessary for the tower to stabilise; for instance, the Temperley-Lieb planar algebra $TL$ - cf. [6] - satisfies $TL = (TL)^k, \forall k \geq 0$ (since it is generated by its zero boxes) although $TL$ has infinite depth in the ‘generic case’. We shall call the smallest integer for which the tower (1.1) stabilises the planar depth; i.e. $P$ has planar depth $k$, $0 \leq k \leq \infty$ if $P = P^k \Rightarrow l \geq k$. Since ‘planar depth $\leq$ depth’, it is natural to ask what the planar depth of a finite-depth planar algebra can be.

In section 2, we prove a fairly simple fact (cf. Corollary 2.2.2 and Proposition 2.2.1) that suffices to determine the planar depth of some planar algebras of finite depth.

The first author (in [9]) gave a presentation of the planar algebra $P(G)$ of the group subfactor corresponding to the fixed-points of an outer action of a finite group $G$ on a II$_1$ factor. He then considered the scenario of a finite group $\Theta$ acting on the group $G$ as group-automorphisms. He showed (Theorem 9 of [9]) that the planar algebra that is generated by the elements in $P(G)_2 \equiv CG$ which are fixed by (the linear extension of) $\Theta$, is strictly smaller than $P(G)$. He discussed this subplanar algebra for several examples.

In section 3, we begin by observing that if $\Theta$ acts on $G$ as above, then there is a natural associated action of $\Theta$ on the planar algebra $P(G)$. The invariants of this action, call it $P^\Theta$ (the group $G$ and the action of $\Theta$ on it will be fixed once and for all), yield a planar subalgebra of $P(G)$. We see that the ‘subplanar algebra’ of the last sentence of the previous paragraph is just what we call $(P^\Theta)^2$.

We show that $P^\Theta$ can in fact be identified with the planar algebra associated with the subgroup-subfactor corresponding to the subgroup $\Theta$ of the semi-direct product $G \rtimes \Theta$ and in particular has finite depth. This finite depth statement implies, as observed earlier, that $P^\Theta$ has finite planar depth. We conclude by discussing some examples that illustrate several possible features of the tower (1.1).

2. A CRITERION FOR ESTIMATING PLANAR DEPTH

We shall use the terminology of [6] for planar algebras. Thus a planar algebra $P$ is a tower $\{P_n : n = 0, 1, 2, \ldots\}$ of finite-dimensional $C^*$-algebras equipped with an action of the coloured operad $P$ of labelled tangles. We shall, as in [6], use the term ‘$k$-boxes’ to denote elements of $P_k$. 
As stated in the introduction, if $P = \{P_n\}$ is a planar algebra, we shall denote by $P^n$ the planar subalgebra of $P$ generated by $P_n$. We shall denote by $P^n_k$ the vector space of $k$-boxes in $P^n$.

**Proposition 2.2.1.** Suppose the principal graph $\Gamma$ of $P$ satisfies the following condition for some $n \in \mathbb{N}$:

($\dagger$) each vertex at distance $(n+1)$ from * is adjacent to a unique vertex at distance $n$ from *, and these latter vertices (= neighbours) are all distinct.

Then $P^n = P^{n+1}$.

**Proof:** The proof relies on an inspection of the tower

$$P^n_k \subset P^{n+1}_k \subseteq P^{n+1},$$

showing that the condition ($\dagger$) implies that the second inclusion is actually an equality, and thus $P^n = P^{n+1}$.

We introduce some notation for the Bratteli diagrams for the inclusions in (2):

(i) Denote by $X$, $Y$, and $Z$ the vertex set for $P^n_k$, $P^{n+1}_k$, and $P^{n+1}_k$, respectively.

(ii) Define the set $B = \{(x,y) : x \in X$ is connected to $y \in Y$ in the Bratteli diagram.$\}$

$$\bigsqcup \{(y,z) : y \in Y$ is connected to $z \in Z$ in the Bratteli diagram.$\},$$

where $\bigsqcup$ denotes the disjoint union.

(iii) For sets $S$ and $T$ we shall write $(S,T) = Id$ if there is a bijection $f : S \to T$ such that for all $s \in S, t \in T$

$$(s,t) \in B \iff t = f(s).$$

(iv) Since the inclusion $P^n = P^n_k \subset P^{n+1}_k = P^{n+1}$ is part of a Jones tower, there is a natural partition of $X$ and $Z$ into ‘old’ and ‘new’ vertices (see [4] for this terminology) which we write as $X = X^O \bigsqcup X^N$ and $Z = Z^O \bigsqcup Z^N$.

(v) The inclusion $P^n_k \subset P^{n+1}_k$ is also part of a Jones tower and thus $X$ and $Y$ have a natural partition into ‘old’ and ‘new’ vertices as well. Furthermore, this partition for $X$ is the same as the partition in (v) above since the two Jones towers are identical up to $P^n_k = P^n$. We denote the partition of $Y$ by $Y = Y^O \bigsqcup Y^N$. 

With this notation, showing equality between \( P_{n+1}^n \) and \( P_{n+1}^{n+1} \) is equivalent to showing \( (Y, Z) = Id \). We shall accomplish this by showing the four statements:

\[
(Y \times Z) = Id,
(Y \times Z \cap B = \emptyset, (Y \times Z \cap B = \emptyset.
\]

(3)

We have the following facts about the Bratteli diagram:

(a) \( (\{y\} \times Z) \cap B \neq \emptyset \) for all \( y \in Y \).
(b) The nature of the partition of \( X, Y \) and \( Z \) into old and new vertices implies that

\[
\begin{align*}
(\text{i}) & \hspace{1cm} (X \cap Z) \cap B = \emptyset, \\
(\text{ii}) & \hspace{1cm} \text{for all } y \in Y, (X \cap \{y\}) \cap B \neq \emptyset, \\
(\text{iii}) & \hspace{1cm} \text{there is no } y \in Y \text{ such that} \\
& \hspace{1cm} (X \cap \{y\}) \cap B \neq \emptyset \text{ and } (\{y\} \times Z) \cap B \neq \emptyset.
\end{align*}
\]

(c) It follows from (bii) and (biii) that \( (Y \cap Z) \cap B = \emptyset \).
(d) Since both \( Y \) and \( Z \) correspond to the set of minimal central projections in the ‘basic construction ideal’ \( P_\alpha \cap P_n \) (with \( \alpha \in P_{n+1} \) denoting the \( n \)-th Jones projection), we have \( (Y \cap Z) = Id \) and \( (Y \cap Z \cap B = \emptyset \).

Thus we have shown the last three statements in (3); it remains to show that \( (Y \times Z) = Id \). But given \( (Y \times Z \cap B = \emptyset \) and (a), this is just what condition (f) ensures.

Corollary 2.2.2. Let \( P \) be a planar algebra with associated principal graph \( \Gamma \). Suppose \( \Gamma \) has no double bonds, no vertices of degree greater than 3, and a unique vertex of degree 3. If this degree 3 vertex is of distance \( (k-1) \) from \( * \), then \( P \) has planar depth \( k \) and is generated (as a planar algebra) by one \( k \)-box.

Proof. It follows directly from Proposition 2.2.1 that \( P \) is the planar depth of \( P \) is finite and at most \( k \). On the other hand, a simple dimension argument shows that \( P_k = P_k^{k+1} \) for all \( l \geq k \); hence the planar depth of \( P \) is finite and at most \( k \). Finally, the hypothesis shows that \( \dim P_k = \dim TL_k + 1 \), and this shows that any element of \( P_k \) which is not in \( P_k^{k+1} \) will generate \( P \) as a planar algebra.

Remark 2.2.3. Three remarks are in order here.
(a) Special cases to which this Corollary applies are the cases when $\Gamma$ is $D_{2n}, E_6, E_8$ and the principal graph of the $\frac{5+\sqrt{13}}{2}$ subfactor of $[1]$.  

(b) Of the pair of principal graphs associated to the $\frac{5+\sqrt{13}}{2}$ subfactor of $[1]$, one graph has a unique triple point, while the other has two triple points at different distances from *. (This feature also holds in each case of the hierarchy of pairs of graphs listed (see [5], case (2)) by Haagerup as other possible finite principal graphs of subfactors of index less than $3 + \sqrt{3}$.)  

Since the planar depth of a subfactor is the same as that of its dual, we see (as illustrated by the graph (in [5]), referred to above, with two triple points) that it is possible for a principal graph to have a triple point at a distance ($k-1$) from * and still have planar depth strictly smaller than $k$.  

(c) As remarked above (and as can be seen from Jones' description of $PM \subset M_1$), a planar algebra and its dual planar algebra have the same planar depth. This is not the case for the usual depth of a subfactor; the $\frac{5+\sqrt{13}}{2}$ subfactor provides an example where the depths of the subfactor and its dual differ. This may be cited as one reason why 'planar depth' is a more natural notion than the usual depth.

3. SOME PLANAR SUBALGEBRAS OF THE GROUP PLANAR ALGEBRA

We shall only be concerned with planar algebras $P$ which come equipped with a 'presentation' (whose symbol $\Phi$ we shall suppress) by a collection $L = \prod_{m=0}^\infty L_k$ of labels. (Again see [6] for notation.)

In fact, following [9], we shall be primarily concerned with the planar algebra $P(G)$, which has a presentation as above with generators given by $L_2 = G$ and $L_k = \emptyset$ for $k \neq 2$, the relation that a simple closed loop (of either orientation) be the scalar $|G|^{\frac{5}{2}}$ and the additional six relations labelled $00, 01, 23, 4$ below (and in Theorem 5 of [9]). We shall assume that we are given a group $\Theta$ and an action $\alpha : \Theta \rightarrow Aut(G)$. Nothing is changed if we replace $\Theta$ by $\Theta/ker \alpha$, so we assume that $\Theta$ acts faithfully, i.e., that $\alpha$ is 1-1.
We consider the map on tangles that replaces the label of each 2-box with the label’s image under \( \theta \in \Theta \). Since \( \theta \) is a group automorphism, it is seen that the set of relations defining \( P(G) \) is unchanged by this map and thus this map defines an automorphism of \( P(G) \) which we shall continue to denote by \( \theta \). It follows that the set \( P^B \) of invariants for this action of \( \Theta \) on \( P(G) \) is a sub-planar algebra of \( P \), and that the set of \( \Theta \)-invariant \( k \)-boxes of \( P(G) \) constitutes precisely the set of \( k \)-boxes of \( P^B \).

For each \( k = 1, 2, \ldots \) and \( \theta \in \Theta \), let \( \alpha^{(k)}_\theta \in \text{Aut}(G^k) \) be defined by

\[
\alpha^{(k)}_\theta(g_1, g_2, \ldots, g_k) = (\alpha_\theta(g_1), \alpha_\theta(g_2), \ldots, \alpha_\theta(g_k)).
\]

When the context is clear, we shall simply write \( \theta(g_1, \ldots, g_k) \) for what we have defined above as \( \alpha^{(k)}_\theta(g_1, \ldots, g_k) \).

For convenience of reference, we shall gather various simple facts about bases for the spaces \( P(G)_k \) in the form of the following remark.
**Remark 3.3.1.** (a) It was shown in Theorem 6 of [9] that if we let $T(g_1, \cdots, g_{k-1})$ denote the labelled $k$-tangle given by

for $k$ odd, and

for $k$ even, then $\{T(\mathcal{g}) : \mathcal{g} \in G^{k-1}\}$ is an orthonormal basis of $P(G)_k$ (with respect to the inner product given by the natural trace).

(b) We shall find it convenient to use a slightly different basis (which, as we shall see, is actually just a rearrangement of the basis in (a)): define $S(\mathcal{g}), \mathcal{g} \in G^{k-1}$ to be the labelled $k$-tangle given by
for \( k \) odd, and

\[
S(\bar{g} \cdot \bar{g} \cdot \ldots) = \begin{cases} 
\sum_{h \in \mathcal{G}} S(\bar{g}, \ldots, \bar{g}, h, \bar{g}, \ldots, \bar{g}) & \text{if } k \text{ is odd} \\
S(\bar{g}, \ldots, \bar{g}, \bar{g}, \ldots, \bar{g}) & \text{if } k \text{ is even.}
\end{cases}
\]

(f) In what follows, we shall find it convenient to use the notation

\[ [x] = \min\{n \in \mathbb{Z} : x \leq n \} \, . \]

The relations in \( P(G) \) are seen to imply that for arbitrary \( \bar{g}, \bar{h} \in G^{k-1} \), we have:
\[ S(g_1, g_2, \ldots g_{k-1})S(h_1, h_2, \ldots h_{k-1}) \]
\[ = |G|^{|\frac{k}{2}|^{-1}} \left( \prod_{i=2}^{|\frac{k}{2}|} \delta(h_{i-1}, g_{i-1}^{-1}, h_i) \right) \]
\[ \times S(h_1g_1, h_1g_2, \ldots h_1g_{|\frac{k}{2}|}, h_{|\frac{k}{2}|+1}, h_{|\frac{k}{2}|+2} \ldots h_{k-1}) \]

(6)

where \( \delta(a, b) \) is zero unless \( a = b \) in which case it is one.

(g) The action of \( \Theta \) on \( P(G)_k \) maps an orthonormal basis onto itself and consequently yields a unitary representation of \( \Theta \); in particular, the orthogonal projection of \( P(G)_k \) onto \( P^\Theta \) is given by the usual averaging operator. So, if we define \( \Theta S(\vec{g}) = \sum_{\theta \in \Theta} S(\theta(\vec{g})) \), it is seen that \( \{ \Theta S(\vec{g}) : \vec{g} \in G^k/\Theta \} \) is an orthogonal set of vectors, which clearly spans \( P^\Theta \) (where we have written \( \vec{g} \) to denote the orbit of \( \vec{g} \) under \( \Theta \), and \( G^k/\Theta \) to denote the set of all such orbits in \( G^k \)). Finally, it is not hard to deduce from equation (6) and the relations defining \( P(G) \) that if \( \vec{g}, \vec{h} \in G^{k-1} \), then

\[ \Theta S(g_1, g_2, \ldots g_{k-1}) \Theta S(h_1, h_2, \ldots h_{k-1}) \]
\[ = |G|^{|\frac{k}{2}|^{-1}} \sum_{\theta'' \in \Theta} \left( \prod_{i=2}^{|\frac{k}{2}|} \delta(h_i \theta''(g_{\frac{k}{2}|i-1}), h_i) \right) \]
\[ \times \Theta S(h_1 \theta''(g_1), h_1 \theta''(g_2), \ldots h_1 \theta''(g_{|\frac{k}{2}|}), h_{|\frac{k}{2}|+1}, h_{|\frac{k}{2}|+2} \ldots h_{k-1}) \]

(7)

**Theorem 3.3.2.** Let \( G, \Theta \) be as above, and let \( G \times \Theta \) denote the semidirect product associated to this group action, and let \( N = R^{G \times \Theta} \subset R^\Theta = M \) denote the associated subgroup-subfactor. Then

\[ P^\Theta \cong P^{N \cap M}. \]

**Proof** For notational convenience let \( P_k = P^\Theta \), \( P_{1,k} = P_{1,i}^\Theta \), \( Q_k = P_{N \cap M}, Q_{1,k} = P_{1,i}^{N \cap M} \). We shall write the elements of \( G \times \Theta \) as ordered pairs \((g, \theta)\) with the usual multiplication \((g_1, \theta_1)(g_2, \theta_2) = (g_1 \theta_1(g_2), \theta_1 \theta_2)\).

We begin by recalling some of the work in [3] which will allow us to describe
the standard invariant of $P^N \subset M$. We let
\[ \frac{1}{|\Theta|} \sum_{\theta \in \Theta} \frac{1}{[1, \theta]} \]
be the projection corresponding to the subgroup $\Theta \subset G \times \Theta$. Define $B_n \in A_{n,n}$ to be the following annular map in $P(G \times \Theta)$:

\[
\begin{array}{c}
q & q & \ldots & q \\
\hline
q & q & \ldots & q \\
\end{array}
\]

for $n$ even,

\[
\begin{array}{c}
q & q & \ldots & q & b \\
\hline
q & q & \ldots & q \\
\end{array}
\]

for $n$ odd.

Define the natural inclusion map
\[ i : B_n(P_n) \to B_{n+1}(P_{n+1}) \]
given by
\[ t \mapsto B_{n+1}(t). \]

We denote this inclusion by $\subset_i$.

Corollary 4.5 of [3] then states that the tower
\[
\begin{array}{c}
Q_0 \subset Q_1 \subset Q_2 \subset \ldots \\
\cup \\
Q_{1,1} \subset Q_{1,2} \subset \ldots \\
\ldots \subset Q_n \ldots \\
\cup \\
\ldots \subset Q_{1,n} \ldots
\end{array}
\]
is isomorphic to the tower:
\[
B_0(P_0(G \times \Theta)) \subseteq \bigcup_{i=1}^{\infty} B_i(P_1(G \times \Theta)) \subseteq \bigcup_{i=1}^{\infty} B_i(P_2(G \times \Theta)) \subseteq \ldots
\]

Using the relations in \( P(G \times \Theta) \) we find that

\[
B_n(S((g_1, \theta_1), (g_2, \theta_2), \ldots, (g_{n-1}, \theta_{n-1})))
\]

\[
= \frac{1}{|\Theta|^n} \sum_{\theta \in \Theta, \gamma \in \Theta^{n-1}} S((\theta(g_1), \gamma_1), (\theta(g_2), \gamma_2), \ldots, (\theta(g_{n-1}), \gamma_{n-1}))
\]

which only depends on the orbit of \((g_1, g_2, \ldots, g_{n-1})\) under \(\Theta\). We shall denote this sum of elements - i.e., \(|\Theta|^n\) times the right side of the above equation - by \(U(\mathbf{f})\); again, note that \(U(\mathbf{f})\) depends only on the orbit \([\mathbf{f}]\) of \(\mathbf{f}\) under \(\Theta\). It follows then that \(\{U(\mathbf{f})\}_{[\mathbf{f}] \in G^{k-1}}\) is an orthogonal basis for \(B_n(P_n(G \times \Theta))\). (It is a complete set since it is the image under \(B_n\) of a basis for \(P_n(G \times \Theta)\). It is an orthogonal set because distinct elements are linear combinations of disjoint subsets of an orthonormal basis of \(P_n(G \times \Theta)\).)

Analogous to equation (6), we find - arguing this time in the group \(G \times \Theta\) - that if \(\mathbf{f}, \mathbf{h} \in G^{k-1}\), then

\[
U(g_1, g_2, \ldots, g_{k-1})U(h_1, h_2, \ldots, h_{k-1})
\]

\[
= |G|^{\frac{k}{2} - 1} |\Theta|^{k-1} \sum_{\theta^* \in \Theta} \left( \prod_{i=2}^{k-1} \delta(h_i, \theta^*(g_{k+1-i}), h_i) \right)
\]

\[
\times U(h_1, \theta^*(g_1), h_1, \theta^*(g_2), \ldots, h_1, \theta^*(g_{\frac{k}{2}+1}), h_1, \theta^*(g_{\frac{k}{2}+2}), \ldots, h_{k-1})
\]

(8)

Define \(\beta_k : P_k \to B_k(P_k(G \times \Theta))\) by

\[
\beta_k(\Theta(S(\mathbf{f}))) = |\Theta|^{-k} U(\mathbf{f}) , \quad \mathbf{f} \in G^{k-1}.
\]

(9)

By the foregoing remarks, \(\beta_k\) maps an orthogonal basis of \(P_k\) onto an orthogonal basis of \(B_k(P_k(G \times \Theta))\) and is thus a well defined bijection of vector spaces. To establish the isomorphism of towers we only need to verify that, for all \(k\),
1. \( \beta_k \) is a homomorphism,
2. \( \beta_k(P_{1,k}) = B_k(P_{1,k}(G \times \Theta)) \),
3. \( \beta_k|_{P_{n-1}} = i \circ \beta_{k-1} \).

It is a straightforward consequence of equations (6), (7) and the carefully
chosen constant in Definition (9) that \( \beta_k \) indeed preserves multiplication
and is thus a homomorphism.

The second assertion above is a consequence of Remark 3.3.1(d), and the
fact that \( \{U(\eta) : g_1 = 1, [\eta] \in G^{k-1}/\Theta \} \) is a basis for \( B_k(P_{1,k}(G \times \Theta)) \) (the
proof of which fact is analogous to that of Remark 3.3.1(d)).

Finally, the third assertion above follows from Remark 3.3.1(e).

\[ \square \]

In the sequel, we shall economise on parentheses and write \( P^{\Theta;\eta} \) for
\( (P^\Theta)^\eta \).

**Corollary 3.3.3.** Suppose there exists \( \eta^0 \in G^{l-1} \) such that the map-
ing
\[ \Theta \ni \theta \mapsto \theta(\eta^0) \in G^{l-1} \]
is injective. Then \( P^\Theta = P^{\Theta;\eta^0} \).

**Proof** If \( k \geq 2l - 1 \) and \( A, B \in P_k \), define \( \Pi(A, B) \) to be the element of
\( P_{2k-2l+1} \) given by the following tangle:

![Tangle Diagram]

It follows from the group relations that
\[
\Pi(T(g_1, \ldots, g_{k-1}), T(h_1, \ldots, h_{k-1}))
= |G|^{k-1} \left( \prod_{i=1}^{l-1} \delta(g_{k-i} h_i, 1) \right) T(g_1, \ldots, g_{k-l}, h_i, \ldots, h_{k-1}) .
\]

(10)

Let us write (as in Remark 3.3.1(g))
\[
\Theta T(\mathcal{F}) = \sum_{\theta \in \Theta} T(\theta(\mathcal{F})) .
\]

Then, for arbitrary \( \mathcal{F} \in G^{2k-2l+1} \), we find, using (9), that
\[
\Pi(\Theta T(g_1, \ldots, g_{k-l}, g_0^0, \ldots, g_{l-1}^0), \Theta T((g_{l-1}^0)^{-1}, (g_{l-2}^0)^{-1}, \ldots, (g_0^0)^{-1}, g_{k-l+1}, \ldots, g_{2k-2l+1}))
= |G|^{k-1} \sum_{\theta, \theta'} \left( \prod_{i=1}^{l} \delta(\delta(g_i^0 h_i \theta^0((g_{l-i}^0)^{-1}), 1) \right)
\times T(\theta(g_1), \theta(g_2), \ldots, \theta(g_{k-l}), \theta(g_{k-l+1}), \ldots, \theta^0(g_{2k-2l+1})) .
\]

(11)

Since \( \theta \mapsto \theta(\mathcal{F}) \) is 1-1, it follows that
\[
\prod_{i=1}^{l} \delta(\delta(g_i^0 h_i \theta^0((g_{l-i}^0)^{-1}), 1) = \delta(\delta(\mathcal{F}^0), \theta^0(\mathcal{F}^0)) = \delta(\theta, \theta') ,
\]
and thus the right side of (10) simplifies to
\[
|G|^{k-1} \sum_{\theta} T(\theta(g_1), \ldots, \theta(g_{2k-2l+1}) = |G|^{k-1} \Theta T(\mathcal{F}^0) .
\]

Since \( \{\Theta T(\mathcal{F})\} \) is a basis for \( P_{2k-2l+1} \), we have shown \( P_{\Theta; 2l} = P_{\Theta; 2l+1} = P_{\Theta; 2l+3} = P_{\Theta; 2l+5} = \ldots = P_{\Theta} \).

**Example 3.3.4.** We consider a few examples.

(a) Let \( \Theta = Z_2 \) act on \( Z_n \) 'by inversion'; thus, the involutory automorphism corresponding to the non-trivial element of \( \Theta \) is given by \( \tau(x) = -x \forall x \in Z_n \). It follows from 'the Mackey machine' - see [4] or [8], for instance - that the principal graph (for \( P_{\Theta} \) and hence for the subgroup-subfactor corresponding to the inclusion \( Z_2 \subset Z_n \times Z_2 \)) is given thus:
Case (i): $n$ is odd

$n-1)/2$ vertices

Case (ii): $n$ is even

$(n-2)/2$ vertices

In both cases, it is clear that $P^0;1 = TL \neq P^0;2$, while it follows from Proposition 2.2.1 that $P^0;3 = P^0;4 = R^0$ if $n$ is odd. (Of course, the case $n = 3$ must be discussed separately, since we get $P^0 = TL$ in this case.) Further, an argument given in [9] (see the pictorial identity within the example $G = D_{2n+1}$ towards the end) shows that $P^0;2 = P^0;3$ for all $n$. In particular, the planar depth of $P^0$ is two, if $n$ is odd.

On the other hand, it turns out that for even $n$, the planar depth of $P^0$ is four, as we show now. In view of the remarks of the last paragraph, it will suffice to show that $P^0;2 \neq P^0$ in this case. Since $P^0$ has finite depth, it will suffice to show that $P^0;2$ is the free product of $P(Z_2)$ and $P(Z_{n/2})$ (since free products necessarily have infinite depth). (See [2] for the definition and these facts about free products - or free compositions, as they are called there - of subfactors; also see [6] for free products in the planar algebra context.)

We know from Theorem 3.3.2 that $P^0 = P^{N \cup M}$, where $N = R^{Z \times Z_2}$ and $M = R^{Z_2}$. Then, it follows from the analysis of [3] that the free
product $P^{N \cap Q} \ast P^{Q \cap M}$ is contained in $P^{N \cap M}$. Let $H$ denote the centre of $\mathbb{Z}_n \times \mathbb{Z}_3$, so that $H = \mathbb{Z}_2 \times \mathbb{Z}_2'$ and let $Q = R^H$, so that $N \subset Q \subset M$ is an intermediate subfactor. Since the subgroup $H$ is normal in $\mathbb{Z}_n \times \mathbb{Z}_2$, and since

$$(R^A \subset R^H) \cong (R^{A/H} \subset R)$$

whenever $B$ is a normal subgroup of a finite group $A$, we find that $P^{N \cap Q} \cong P((\mathbb{Z}_n \times \mathbb{Z}_2)/(\mathbb{Z}_2 \times \mathbb{Z}_2)) \cong P(\mathbb{Z}_n/2)$ and that $P^{Q \cap M} \cong P(\mathbb{Z}_2)$. We may conclude that

$$\dim P^{N \cap M}_2 = \dim P^{Q \cap M}_2$$

$$= \text{no. of orbits of } \Theta \text{ in } \mathbb{Z}_n$$

$$= \frac{n-2}{2} + 2$$

$$= n/2 + 2 - 1$$

$$= (\dim P^{N \cap Q}_2) + (\dim P^{Q \cap M}_2) - 1.$$ 

It is proved in [3] that this equality allows us to conclude that $P^{\Theta;2} = P^{N \cap Q} \ast P^{Q \cap M}$, as desired.

(b) Let $\Theta = \mathbb{Z}_3$ act on $\mathbb{Z}_2 \times \mathbb{Z}_2$, with the generator of $\mathbb{Z}_3$ cyclically permuting the three non-trivial elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. It again follows from the Mackey machine that the principal graph is given by the extended Coxeter diagram $E_6$; and we may conclude from Corollary 2.2.2 that

$$P^{\Theta;k} = \begin{cases} 
TL & \text{if } k \leq 2 \\
\rho^\Theta & \text{if } k > 2 
\end{cases}$$

(c) The action of $\mathbb{Z}_3$ on $\mathbb{Z}_2 \times \mathbb{Z}_2$ in Example (b) above, can be extended to an action of $S_3$ (with the transpositions interchanging pairs of non-trivial elements); in this case, $P^{\Theta}$ has principal graph given by the extended Coxeter diagram $E_7$; and we may conclude from Corollary 2.2.2 that

$$P^{\Theta;k} = \begin{cases} 
TL & \text{if } k \leq 3 \\
\rho^\Theta & \text{if } k > 3 
\end{cases}$$

(d) Consider $\Theta = \mathbb{Z}_3$ acting on $G = \mathbb{Z}_7$ with the generator of $\mathbb{Z}_3$ acting as the map $x \mapsto x^2$. In this case, we find that the principal graph is given by:

```
*----*
|    |
|    |
|    |
*----*
```
We can deduce from Proposition 2.2.1 that $P^G:3 = P^G:4 = P^G$. We can easily see that $P^G:1 = TL \neq P^G:2$; we shall now proceed to show that $P^G:2 = P^G:3$.

Let $X$ (resp., $Y$) denote the sum of the 2-boxes labelled by the members of $[1] = \{1, 2, 4\}$ (resp., $[3] = \{3, 6, 5\}$). If we let $Z$ denote the 2-box labelled by 0, then it is clear that $\{Z, X, Y\}$ is a basis for $P^G:2$.

If $z, w \in G$, let us write $((z, w))$ for the element (of $P(G)_3$) obtained if we substitute $z$ and $w$ respectively, for $E$ and $F$ in the picture given by

\[
\begin{array}{c}
((E, F)) = \\
E
\end{array}
\]

and $[z, w] = \sum_{\theta \in \Theta} ((\theta(z), \theta(w)))$. The definitions imply - by considering the $\Theta$-orbits in $G \times G$ - that $P^G_3$ is linearly spanned by the set

\[
\{(0,0), [0,1], [0,3]\} \cup \{[1, w] : w \in G\} \cup \{[3, w] : w \in G\}.
\]

Next we write $(A, B, C, D)$ for the value of the following picture (where $A, B, C, D \in P^G_2$):

\[
\begin{array}{c}
A \quad B \\
G \quad D
\end{array}
\]

and we have the following identities:

\[
\begin{align*}
(Z, Z, Z) &= \sqrt{T}[0,0] \\
(Z, Z, X) &= \sqrt{T}[0,1] \\
(Z, Z, Y) &= \sqrt{T}[0,3]
\end{align*}
\]
(X, Z, Y, Z) = \sqrt{T} [1, 0]
(X, X, Z, Z) = \sqrt{T} [1, 1]
(X, X, X, Z) = \sqrt{T} [1, 2]
(X, Y, X, Z) = \sqrt{T} ([1, 3] + [1, 5])
(X, X, Y, Z) = \sqrt{T} [1, 4]
(X, Y, X, X) = \sqrt{T} (2[1, 0] + [1, 2] + [1, 4] + [1, 5] + [1, 6])
(X, Y, Y, Z) = \sqrt{T} [1, 6].

These identities show that \([0, x], [1, x] \in P^{\frac{3}{2}}_3\) for all \(x \in G\). Similar computations show that \([3, x] \in P^{\frac{3}{2}}_3\) for all \(x \in G\). Thus, we have shown that \(P^{\frac{3}{2}}_3 \subset P^{\frac{3}{2}}_3\), so \(P^{\frac{3}{2}}_3 \subset P^{\frac{3}{2}}_3\), and \(P^{\frac{3}{2}}_3 = P^{\frac{3}{2}}_3\), as desired.

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**REFERENCES**