ATM lectures

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1 Vector bundles

Throughout X denotes a compact Hausdorff space.

Definition 1.1 *A vector bundle over X is a topological space E together with a surjection $p : E \to X$ such that*

1. *For $x \in X$, $E_x := p^{-1}(x)$ has a finite dimensional vector space structure,*
2. *E is locally trivial i.e.*

   For $x \in X$, there exists an open set $U_x \ni x$, $n_x \geq 0$ and a homeomorphism $h_x : p^{-1}(U_x) \to U_x \times \mathbb{C}^{n_x}$ such that h_x is fibre-wise linear and $\pi_1 \circ h_x = p$.

Remark 1.2 *If E is a vector bundle over X then $dim(E_x)$ is locally constant.*

Example 1.3 *Let $E := X \times \mathbb{C}^n$. E is called the trivial bundle of rank n.*

Example 1.4 *Let M be a smooth manifold. Then TM, the tangent bundle is a real bundle and one can complexify it to get a complex vector bundle.*

Example 1.5 *Let $Gr(n, k) := \{k-$ dimensional subspaces of $\mathbb{C}^n\}$. Topologise $Gr(n, k)$ by identifying it with projections in $M_n(\mathbb{C})$ with trace k. Then $Gr(n, k)$ is a compact Hausdorff space. Let*

$$E := \{(p, v) \in Gr(n, k) \times \mathbb{C}^n : pv = v\}$$

Then E is a vector bundle over $Gr(n, k)$.

Example 1.6 *Let $p \in M_n(C(X))$ be a projection. We can think of p as a continuous projection valued map from X to $M_n(C(X))$. Let*

$$E := \{(x, v) \in X \times \mathbb{C}^n : p(x)v = v\}$$

Then E is a vector bundle over X.

Definition 1.7 *Let $p : E \to X$ be a vector bundle. A section is a map $s : X \to E$ such that $s(x) \in E_x$ for every $x \in X$.*

Exercise 1.1 *Let E be a vector bundle over X of rank n. Prove that E is trivial if and only if there exists n-linearly independent sections.*

Thus choosing a trivialisation is the same as choosing local sections which form a basis at each fibre.

Exercise 1.2 *Prove that the bundle described in 1.6 is indeed a vector bundle.*

Pullback: Let $f : Y \to X$ be continuous and $p : E \to X$ be a vector bundel. Define

$$f^*(E) := \{(y, e) : f(y) = p(e)\} \subset Y \times E$$

Check that $f^*(E)$ is a vector bundle over Y.

Whitney sum: Let $p : E \to X$ and $q : F \to X$ be vector bundles over X. Define

$$E \oplus F := \{(e, f) \in E \times F : p(e) = q(f)\}$$

Check that $E \oplus F$ is a vector bundle over X. Clearly $E \oplus F$ is isomorphic to $F \oplus E$. Also upto isomorphism $\oplus$ is associative.

Let us denote the set of isomorphism classes of vector bundles over X by $V(X)$. The Whitney sum of vector bundles makes $V(X)$ an abelian semigroup with an identity element. The abelian group $K(X)$ is defined to be the group obtained from $V(X)$ by the Grothendieck construction. The group $K(X)$ is called the K-group of X.

Let us recall the Grothendieck construction. Suppose $(R, +)$ is an abelian semigroup with identity. Define an equivalence relation $\sim$ on $R \times R$ as follows:

$$(a, b) \sim (c, d) \text{ if there exists } e \in R \text{ such that } a + d + e = b + c + e.$$

We think of the equivalence class $[(a, b)]$ as representing the difference $a-b$. The addition $+$ on $R \times R/\sim$ is defined as

$$[(a, b)] + [(c, d)] = [(a + c, b + d)].$$

Then $+$ is well defined on $R \times R/\sim$ and $(R \times R/\sim, +)$ is an abelian group with $[(a, a)]$ as the identity element for any $a \in R$ and the inverse of $[(a, b)]$ is $[(b, a)]$.

The map $X \to K(X)$ is a contravariant functor from the category of compact Hausdorff spaces to the category of abelian groups. It is homotopy invariant and the K-groups can be computed for a large family of topological spaces.

Exercise 1.3 *Let (U_α, h_α) be a trivialising cover for a vector bundle E over X. Then the map $h_\alpha h_\beta^{-1} : U_\alpha \cap U_\beta \times \mathbb{C}^n \to U_\alpha \cap U_\beta \times \mathbb{C}^n$ has the form*

$$h_\alpha h_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x)v)$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{C})$ is continuous. Prove that

$$g_{\alpha\alpha} = 1$$
$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$$

The above relations is expressed by saying $\{g_{\alpha\beta}\}$ is a co-cycle. Also $g'_{\alpha\beta}s$ are also called transition maps.

Exercise 1.4 *Write down the transition functions for the pull-back and the Whitney sum.*

Exercise 1.5 *Let $\{g_{\alpha\beta}\}$ be a co-cycle. Consider the disjoint union $\bigsqcup_\alpha U_\alpha \times \mathbb{C}^n$. Define a equivalence relation on $\bigsqcup_\alpha U_\alpha \times \mathbb{C}^n$ by declaring*

$$U_\alpha \times \mathbb{C}^n \ni (x, v) \sim (y, w) \in U_\beta \times \mathbb{C}^n \quad \text{if and only if} \quad x = y \quad \text{and} \quad g_{\alpha\beta}(x)w = v$$

Let

$$E := \frac{\bigsqcup_\alpha U_\alpha \times \mathbb{C}^n}{\sim}.$$

Prove that E is a vector bundle over X with the obvious projection map.

2 Serre-Swan theorem

If $p : E \to X$ is a vector bundle, let $\Gamma(E)$ denote the space of sections. Then $\Gamma(E)$ is a $C(X)$ module.

Exercise 2.1 *Prove that* $\Gamma(E_1 \oplus E_2) = \Gamma(E_1) \oplus \Gamma(E_2)$ *as* $C(X)$*-modules.*

The main aim of this section is to prove the following theorem.

Theorem 2.1 (Serre-Swan theorem) *The map* $[E] \to [\Gamma(E)]$ *is a bijection from the set of isomorphism classes of vector bundles over* X *and the set of isomorphism classes of finitely generated projective modules over* $C(X)$*.*

Lemma 2.2 *Let* F *be a subbundle of* E*. Consider a point* $x \in F$*. Then there exists an open set* U *containing* x *and linearly independent sections* $s_1, s_2, \cdots, s_m, s_{m+1}, \cdots, s_n$ *on* U *such that*

1. *For* $y \in U$*,* $F_y = span\{s_1(y), s_2(y), \cdots, s_m(y)\}$*.*
2. *For* $y \in U$*,* $E_y = span\{s_1(y), s_2(y), \cdots, s_m(y), s_{m+1}(y), \cdots, s_n(y)\}$*.*

Proof. Let V be a nbd around x on which both F and E are trivial. Assume that rank of F over V is m and that of E over V is n.

Identify $E|_V \cong V \times \mathbb{C}^n$. Choose m linearly independent sections for F over V. Name them $s_1, s_2, \cdots, s_m$. Choose $v_{m+1}, v_{m+2}, \cdots, v_n$ such that the vectors

$$\{s_1(x), s_2(x), \cdots, s_m(x), v_{m+1}, v_{m+2}, \cdots, v_n\}$$

forms a basis for $\mathbb{C}^n$.

By continuity (of what ?), it follows that there exists a nbd U around x such that $\{s_1(y), s_2(y), \cdots, s_m(y), v_{m+1}, v_{m+2}, \cdots, v_n\}$ is a basis for every $y \in U$. Now complete the proof. □

Definition 2.3 *Let* E *be a vector bundle over* X*. An inner product on* E *is a collection of inner products* $\{<,>_x: x \in X\}$*, one for each fibre* E_x*, such that if* $s, t \in \Gamma(E)$ *then the map* $X \ni x \to< s(x), t(x) >_x$ *is continuous. A vector bundle equipped with an inner product is called a Hermitian vector bundle.*

It is clear that trivial bundles admit an inner product. The proof of the following proposition is a partition of unity type argument.

Proposition 2.4 *Let E be a vector bundle over X. Then E admits an inner product.*

Proposition 2.5 *Let E be a Hermitian vector bundle over X and $F \subset E$ be a subbundle. Then $F^\perp$ is a vector bundle over X and $F \oplus F^\perp$ is isomorphic to E.*

Proof. It is enough to show that $F^\perp$ is a vector bundle (Justify). Choose locally independent sections $s_1, s_2, \cdots, s_m, s_{m+1}, \cdots s_n$ which form a basis for E and the first m sections form a basis for F.

Apply Gram-Schmidt process to replace $\{s_i\}$ by $\widetilde{s}_i$. Then

1. $\{\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_m\}$ form a local basis for F,
2. $\{\widetilde{s}_{m+1}, \widetilde{s}_{m+2}, \cdots, \widetilde{s}_n\}$ form a local basis for $F^\perp$, and
3. $\{\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_n\}$ form a local basis for E.

The local sections $\{\widetilde{s}_{m+1}, \widetilde{s}_{m+2}, \cdots, \widetilde{s}_n\}$ trivialises $F^\perp$.

Proposition 2.6 *Let E be a vector bundle over X. Then E is a subbundle of $X \times \mathbb{C}^N$ for some N.*

Proof. Choose finitely many trivialisations $(U_i, h_i)_{i=1}^n$. Let $\{\phi_i\}_{i=1}^n$ be a partition of unity such that $\text{supp}(\phi_i) \subset U_i$.

Let $i \in \{1, 2, \cdots, n\}$ be given. Consider the trivialisation $h_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^{m_i}$. We denote the projection from $U_i \times \mathbb{C}^{m_i} \to \mathbb{C}^{m_i}$ by π_2. Define $g_i : E \to \mathbb{C}^m$ by

$$g_i(e) \;:=\; \begin{cases} \phi_i(p(e))\pi_2 h_i(e) & \text{if } e \in p^{-1}(U_i), \\ 0 & \text{otherwise.} \end{cases}$$

Check that g_i is continuous.

Define $g : E \to X \times \mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \cdots \times \mathbb{C}^{m_n}$ by

$$g(e) := (p(e), g_1(e), g_2(e), \cdots, g_n(e))$$

Prove that

- g is injective,
- g is fibre-wise linear, and
- g is a topological embedding.

The proof is now complete. □

Using the last two propositions, do the following exercise.

Exercise 2.2 *If E is a vector bundle over X then $\Gamma(E)$ is a finitely generated projective $C(X)$-module.*

Exercise 2.3 *Let E be a vector bundle over X. Let $x \in X$ and $s : X \to E$ be a section such that $s(x) = 0$. Then s can be written as $s = \sum_{i=1}^{n} g_i s_i$ where $g_i \in C(X)$ and $s_i \in \Gamma(E)$ with g_i vanishing at x.*

Idea: Choose n-locally independent sections $s_1, s_2, \cdots, s_n$ around x and write $s := \sum_{i=1}^{n} f_i s_i$. Let ϕ be a continuous function such that $\phi \geq 0$, $\phi(x) = 1$ and supp(ϕ) concentrated around x. Now $s = (1-\phi)s + \sum_{i=1}^{n} \phi^{\frac{1}{2}} f_i \phi^{\frac{1}{2}} s_i$. Note that $\phi^{\frac{1}{2}} s_i$ are globally defined. □

Exercise 2.4 *Let $g : E_1 \to E_2$ be a bundle map. Then $g_* : \Gamma(E_1) \to \Gamma(E_2)$ defined by $g_*(s) = g \circ s$ is a $C(X)$-module map.*

Proposition 2.7 *Let $T : \Gamma(E_1) \to \Gamma(E_2)$ be a $C(X)$-module map. Then there exists a bundle map $g : E_1 \to E_2$ such that $g_* = T$.*

Idea of the proof: Let $v \in E_1$ be such that v lies over $x \in X$. Choose any section s such that $s(x) = v$. Define $g(v) := (Ts)(x)$. Now Exercise 2.3 implies that g is well-defined.

Proposition 2.8 (Surjectivity part) *Let $\mathcal{E}$ be a f.g. projective $C(X)$-module. Then there exists a vector bundle over X such that $\mathcal{E} \cong \Gamma(E)$.*

Proof. Let $p \in M_n(C(X))$ be the idempotent which corresponds to $\mathcal{E}$. Define

$$E := \{(x, v) : p(x)v = v\}$$

Then E is a vector bundle over X and $\Gamma(E) \cong \mathcal{E}$. □

Exercise 2.5 *Now convince yourself that we have proved Serre-Swan theorem.*

3 Basic K-theory

Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$. We consider only right $\mathcal{A}$ modules. For $n \geq 1$, we write elements of $\mathcal{A}^n$ as column vectors. The matrix algebra $M_n(\mathcal{A})$ acts on $\mathcal{A}^n$ by left multiplication as module maps.

Exercise 3.1 *Prove that* $End_{\mathcal{A}}(\mathcal{A}^n) = M_n(\mathcal{A})$.

Definition 3.1 *Let* $\mathcal{E}$ *be a right* $\mathcal{A}$ *module.*

1. *The module* $\mathcal{E}$ *is said to be finitely generated if there exists* $\xi_1, \xi_2, \cdots, \xi_n \in \mathcal{E}$ *such that the* $\mathcal{A}$*-module generated by* $\{\xi_1, \xi_2, \cdots, \xi_n\}$ *is* $\mathcal{E}$.

2. *The module* $\mathcal{E}$ *is said to be projective if it is a direct summand of a free* $\mathcal{A}$*-module.*

Exercise 3.2 *Prove that* $\mathcal{E}$ *is finitely generated if and only if there exists a* $\mathcal{A}$*-module surjection* $p : \mathcal{A}^n \to \mathcal{E}$.

Exercise 3.3 *Prove that the following are equivalent.*

1. *The module* $\mathcal{E}$ *is projective.*

2. *If* $p : M \to N$ *is a surjection and* $f : \mathcal{E} \to N$ *is any map then* f *admits a lift* $\tilde{f} : \mathcal{E} \to M$.

Exercise 3.4 *Let* $\mathcal{E}$ *be a finitely generated projective* $\mathcal{A}$*-module. Prove that* $\mathcal{E}$ *is a direct summand of* $\mathcal{A}^n$ *for some* $n \geq 1$.

For the rest of this section $\mathcal{E}, \mathcal{E}'$ will denote f.g. projective modules.

Exercise 3.5 *Let* $p \in M_n(\mathcal{A})$ *be an idempotent i.e.* $p^2 = p$. *Consider* p *as a* $\mathcal{A}$*-module map on* $\mathcal{A}^n$.

Prove that $Ker(1-p)$ *is a finitely generated projective* $\mathcal{A}$*-module. Show that any f.g. projective module arises this way.*

Exercise 3.6 *Let* $p \in M_m(\mathcal{A})$ *and* $q \in M_n(\mathcal{A})$ *be idempotents.*

Prove that $Im(p)$ *and* $Im(q)$ *are isomorphic as* $\mathcal{A}$ *modules if and only if there exists* $x \in M_{m,n}(\mathcal{A})$ *and* $y \in M_{n,m}(\mathcal{A})$ *such that* $xy = p$ *and* $yx = q$.

In view of the above exercises, we will identify a f.g. projective module(its isomorphism class) with an idempotent in $\bigcup_{n=1}^{\infty} M_n(\mathcal{A})$ (its equivalence class) .

Exercise 3.7 *Let $\mathcal{E}$ and $\mathcal{E}'$ be f.g. projective $\mathcal{A}$-modules. Then $\mathcal{E} \oplus \mathcal{E}'$ is finitely generated and projective. Moreover if $\mathcal{E}$ and $\mathcal{E}'$ are given by the idempotents p and q respectively then $\mathcal{E} \oplus \mathcal{E}'$ is given by the idempotent $\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$.*

Let $E(\mathcal{A}) := \{e \in \mathcal{A} : \ e^2 = e\}$ and $E_\infty(\mathcal{A}) := \bigcup_{n=1}^{\infty} E(M_n(\mathcal{A}))$. Define an equivalence relation on $E_\infty(\mathcal{A})$ as follows: Let $p \in M_m(\mathcal{A})$ and $q \in M_n(\mathcal{A})$.

$$p \sim q \Leftrightarrow \text{ there exists } u \in M_{m\times n}(\mathcal{A}), v \in M_{n\times m}(\mathcal{A}) \text{ such that } uv = p \text{ and } vu = q.$$

We also denote the set of equivalence classes by $E_\infty(\mathcal{A})$. Then we have the following proposition.

Proposition 3.2 *The operation $\oplus$ defined as $[p] \oplus [q] := \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ is well defined on $E_\infty(\mathcal{A})$. Moreover, $(E_\infty(\mathcal{A}), \oplus)$ is a commutative semigroup with identity.*

Remark 3.3 *The abelian semigroup $E_\infty(\mathcal{A})$ is nothing but the semigroup of isomorphic classes of finitely generated projective $\mathcal{A}$-modules.*

Definition 3.4 *The K-group $\widehat{K}_0(\mathcal{A})$ is the the Grothendieck group of the abelian semigroup $(E_\infty(\mathcal{A}), \oplus)$.*

Elements of $\widehat{K}_0(\mathcal{A})$ are of the form $[e]-[f]$ where e and f are idempotents in $M_N(\mathcal{A})$ for some N. Also $[e] - [f] = [e'] - [f']$ if and only if there exists $g \in M_k(\mathcal{A})$ such that $e \oplus f' \oplus g \sim e' \oplus f \oplus g$.

$\widehat{K}_0$ is a functor from the category of unital algebras to abelian groups.

Non-unital case: Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. The algebra $\mathcal{A}$ is not assumed to be unital. Consider $\mathcal{A}^+ := \mathcal{A} \oplus \mathbb{C}$ with the multiplication defined by

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu).$$

Let $\epsilon : \mathcal{A}^+ \to \mathbb{C}$ be the map defined by $\epsilon(a, \lambda) = \lambda$. Then ϵ is an algebra homomorphism.

Define $K_0(\mathcal{A}) := Ker\widehat{K}_0(\epsilon)$.

Remark 3.5 *If $\mathcal{A}$ is unital then $K_0(\mathcal{A})$ and $\widehat{K}_0(\mathcal{A})$ are naturally isomorphic. Reason: If $\mathcal{A}$ is unital then $\mathcal{A}^+$ is isomorphic to $\mathcal{A} \oplus \mathbb{C}$ as algebras. $\widehat{K}_0$ preserves direct sums.*

Theorem 3.6 *If X is a compact smooth manifold then to define $K(X)$ it is enough to consider smooth vector bundles.*

We end our discussion by seeing a similar theorem for non-commutative algebras.

Throughout A will stand for a unital Banach algebra and $\mathcal{A} \subset A$ is a dense subalgebra which contains the unit of A.

Lemma 3.7 *Let $e, f \in A$ be idempotents such that $||e - f|| < \frac{1}{||2e-1||}$. Then e and f are similar i.e. there exists $z \in A$ such that $zez^{-1} = f$.*

Proof. Let $z := (2e-1)(2f-1)+1$. Then $zf = ez$. Note that $z - 2 = 2(f-e)(2e-1)$. Hence $||z-2|| < 2$. Thus z is invertible and $z^{-1}ez = f$. This completes the proof. □

Lemma 3.8 *Let $e, f \in M_n(A)$ be idempotents such that $e \sim f$. Then in $M_{2n}(A)$, the idempotents $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$ are similar.*

Hint: Split

$$A^n = eA^n \oplus (1-e)A^n = fA^n \oplus (1-f)A^n.$$

Definition 3.9 *Let $\mathcal{A} \subset A$ be a dense subalgebra and assume that the inclusion $\mathcal{A} \subset A$ is unital. We call $\mathcal{A}$ smooth if*

1. *$\mathcal{A}$ admits a Frechet algebra structure,*
2. *the inclusion $\mathcal{A} \subset A$ is continuous, and*
3. *$\mathcal{A}$ is spectrally invariant i.e. if $a \in \mathcal{A}$ is invertible in A then $a^{-1} \in \mathcal{A}$.*

Exercise 3.8 *Let $\mathcal{A} \subset A$ be smooth and let $a \in \mathcal{A}$ be given. Show that $\sigma_{\mathcal{A}}(a) = \sigma_A(a)$.*

Exercise 3.9 *Let $a \in A$ be such that $||a^2 - a|| < \frac{1}{4}$. Prove that the spectrum $\sigma(a)$ does not intersect the line $\{z \in \mathbb{C} : Re(z) = \frac{1}{2}\}$.*

Exercise 3.10 *Let* $g : \{z : Re(z) \neq \frac{1}{2}\} \to \mathbb{C}$ *be holomorphic such that* $g^2 = g$ *and* $g(z) = z$ *if* $z \in \{0, 1\}$.

Let K *be a compact set. Let*

$$X := \{a \in A : \sigma(a) \subset K, \sigma(a) \text{ does not intersect } Re(z) = \frac{1}{2}\}.$$

Show that $X \ni a \to g(a) \in A$ *is norm continuous.*

If $\mathcal{A}$ is smooth in A then $\mathcal{A}$ is closed under holomorphic functional calculus i.e. for $a \in \mathcal{A}$ and f a holomorphic function in the nbd of $\sigma(a)$, $f(a) \in \mathcal{A}$. It is also true that $M_n(\mathcal{A})$ is closed under holomorphic functional calculus.

Theorem 3.10 *Let* $\mathcal{A} \subset A$ *be a smooth subalgebra. The inclusion* $\mathcal{A} \subset A$ *induces isomorphism between* $K_0(\mathcal{A})$ *and* $K_0(A)$.

Proof. Let us denote the inclusion map by i. We need to prove that $i_* : K_0(\mathcal{A}) \to K_0(A)$ is an isomorphism. Use Exercises 3.9 and 3.10 to make the following proof precise.

Surjectivity of i_***.** Let e be an idempotent in $M_n(A)$. Choose a in $M_n(\mathcal{A})$ close to e. Then a^2 is close to a. Thus $\sigma(a)$ does not intersect the line $Re(z) = \frac{1}{2}$. Then $g(a)$ is an idempotent and $g(a) \in M_n(\mathcal{A})$. By 3.10, it follows that $g(a)$ is close to $g(e) = e$. By Lemma 3.7, it follows that $[g(a)] = [e]$ in $K_0(A)$. Hence i_* is surjective.

Injectivity of i_***.** Suppose $[e] - [f] = [0]$ in $K_0(A)$ with $e, f \in M_n(\mathcal{A})$. Then there exists $g \in M_n(A)$ such that $e \oplus g \sim f \oplus g$. By the surjectivity part, we can assume that $g \in M_n(\mathcal{A})$. By Lemma 3.8, it follows that there exists $v \in M_n(A)$ such that $v(e \oplus g \oplus 0)v^{-1} = f \oplus g \oplus 0$. Choose $u \in M_n(\mathcal{A})$ close enough to v. Then $u(e \oplus g \oplus 0)u^{-1}$ is close to $f \oplus g \oplus 0$. Thus again by Lemma 3.8 and its proof, it follows that there exists z such that $zu(e \oplus g \oplus 0)u^{-1}z^{-1} = f \oplus g \oplus 0$. Hence $[e] - [f] = 0$ in $K_0(\mathcal{A})$.

This completes the proof. □

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