Quantum Symmetries in Free Probability Theory

Roland Speicher
Queen's University
Kingston, Canada

Quantum Groups

- are generalizations of groups G (actually, of $C(G)$)

- are supposed to describe non-classical symmetries

- are Hopf algebras, with some additional structure ...

Quantum Groups: Deformation of Classical Symmetries

$$G \rightsquigarrow G_q$$

- quantum groups are often deformations G_q of classical groups, depending on some parameter q, such that for $q \to 1$, they go to the classical group $G = G_1$

- G_q and G_1 are incomparable, none is stronger than the other
  - G_1 is supposed to act on commuting variables
  - G_q is the right replacement to act on q-commuting variables

Quantum Groups: Strengthening of Classical Symmetries

$$G \rightsquigarrow G^+$$

- there are situations where a classical group G has a genuine non-commutative analogue G^+ (no interpolations)

- G^+ is "stronger" than G: $\qquad G \subset G^+$
  - G acts on commuting variables
  - G^+ is the right replacement for acting on non-commuting variables

We are interested in quantum versions of

real compact matrix groups

Think of

- orthogonal matrices
- permutation matrices

Such quantum versions are captured by the notion of

orthogonal Hopf algebra

Orthogonal Hopf Algebra

is a C^*-algebra A, given with a system of n^2 self-adjoint generators $u_{ij} \in A$ $(i, j = 1, \ldots, n)$, subject to the following conditions:

- The inverse of $u = (u_{ij})$ is the transpose matrix $u^t = (u_{ji})$.

- $\Delta(u_{ij}) = \Sigma_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$.

- $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.

- $S(u_{ij}) = u_{ji}$ defines a morphism $S : A \to A^{op}$.

These are compact quantum groups in the sense of Woronowicz.

In the spirit of non-commutative geometry, we are thinking of

$$A = C(G^+)$$

as the continuous functions, generated by the coordinate functions u_{ij}, on some (non-existing) quantum group G^+, replacing a classical group G.

Quantum Orthogonal Group O_n^+ (Wang 1995)

The quantum orthogonal group $A_o(n) = C(O_n^+)$ is the universal unital C^*-algebra generated by u_{ij} $(i, j = 1, \dots, n)$ subject to the relation

- $u = (u_{ij})_{i,j=1}^n$ is an orthogonal matrix

This means: for all i, j we have

$$\sum_{k=1}^n u_{ik} u_{jk} = \delta_{ij} \qquad \text{and} \qquad \sum_{k=1}^n u_{ki} u_{kj} = \delta_{ij}$$

Quantum Permutation Group S_n^+ (Wang 1998)

The quantum permutation group $A_s(n) = C(S_n^+)$ is the universal unital C^*-algebra generated by u_{ij} $(i, j = 1, \ldots, n)$ subject to the relations

- $u_{ij}^2 = u_{ij} = u_{ij}^*$ for all $i, j = 1, \ldots, n$

- each row and column of $u = (u_{ij})_{i,j=1}^n$ is a partition of unity:

$$\sum_{j=1}^n u_{ij} = 1 \qquad \sum_{i=1}^n u_{ij} = 1$$

Are there more of those?

$$\begin{array}{ccc} S_n^+ & \subset & O_n^+ \\ \cup & & \cup \\ S_n & \subset & O_n \end{array}$$

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$$\begin{array}{ccccc} S_n^+ & \subset & G_n^+ & \subset & O_n^+ \\ \cup & & \cup & & \cup \\ S_n & \subset & G_n & \subset & O_n \end{array}$$

- Are there more non-commutative versions G_n^+ of classical groups G_n?

Are there more of those?

$$
\begin{array}{ccccc}
S_n^+ & & \subset & & O_n^+ \\
 & & & / & \\
\cup & & G_n^* & & \cup \\
 & / & & & \\
S_n & & \subset & & O_n
\end{array}
$$

- Are there more non-commutative versions G_n^+ of classical groups G_n?

- Actually, are there more nice non-commutative quantum groups G_n^*, stronger than S_n?

How can we describe and understand intermediate quantum groups:

$$S_n \subset \mathbf{G_n^*} \subset O_n^+$$

$$C(S_n) \leftarrow \mathbf{C(G_n^*)} \leftarrow C(O_n^+)$$

How can we describe and understand intermediate quantum groups:

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Deal with quantum groups by looking on their representations!!!

Spaces of Intertwiners

Associated to an orthogonal Hopf algebra $(A = C(G_n^*), (u_{ij})_{i,j=1}^n)$ are the spaces of intertwiners:

$$\mathbf{I}_{G_n^*}(k,l) = \{T : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l} \mid Tu^{\otimes k} = u^{\otimes l}T\}$$

where $u^{\otimes k}$ is the $n^k \times n^k$ matrix $(u_{i_1 j_1} \dots u_{i_k j_k})_{i_1 \dots i_k, j_1 \dots j_k}$.

$$u \in M_n(A) \qquad u : \mathbb{C}^n \to \mathbb{C}^n \otimes A$$

$$u^{\otimes k} : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes k} \otimes A$$

$\mathrm{I}_{G_n^*}$ is Tensor Category with Duals

Collection of vector spaces $\mathbf{I}_{G_n^*}(k,l)$ has the following properties:

- $T, T' \in \mathbf{I}_{G_n^*}$ implies $T \otimes T' \in \mathbf{I}_{G_n^*}$.
- If $T, T' \in \mathbf{I}_{G_n^*}$ are composable, then $TT' \in \mathbf{I}_{G_n^*}$.
- $T \in \mathbf{I}_{G_n^*}$ implies $T^* \in \mathbf{I}_{G_n^*}$.
- $id(x) = x$ is in $\mathbf{I}_{G_n^*}(1,1)$.
- $\xi = \sum e_i \otimes e_i$ is in $\mathbf{I}_{G_n^*}(0,2)$.

Quantum Groups $\leftrightarrow$ Intertwiners

The compact quantum group G_n^* can actually be rediscovered from its space of intertwiners:

There is a one-to-one correspondence between:

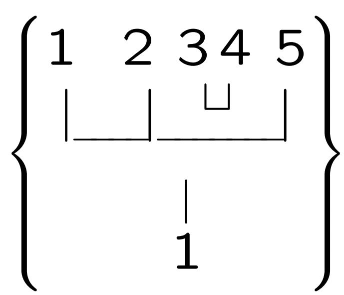
- orthogonal Hopf algebras $C(O_n^+) \to \mathbf{C(G_n^*)} \to C(S_n)$
- tensor categories with duals $\mathbf{I}_{O_n^+} \subset \mathbf{I}_{\mathbf{G_n^*}} \subset \mathbf{I}_{S_n}$.

We denote by $P(k,l)$ the set of partitions of the set with repetitions $\{1,\ldots,k,1,\ldots,l\}$. Such a partition will be pictured as

$$p = \begin{Bmatrix} 1\ldots k \\ \mathcal{P} \\ 1\ldots l \end{Bmatrix}$$

where $\mathcal{P}$ is a diagram joining the elements in the same block of the partition.

Example in $P(5,1)$:



Associated to any partition $p \in P(k,l)$ is the linear map

$$T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$$

given by

$$T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \cdots j_l} \delta_p(i,j)\, e_{j_1} \otimes \ldots \otimes e_{j_l}$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$.

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Examples:

$$T_{\left\{ \big| \;\; \big| \right\}}(e_a \otimes e_b) = e_a \otimes e_b$$

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Examples:

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$$T_{\left\{ \begin{smallmatrix} \sqcup \\ |\;\;| \end{smallmatrix} \right\}}(e_a \otimes e_b) = \delta_{ab} \sum_{cd} e_c \otimes e_d$$

Intertwiners of (Quantum) Permutation and of (Quantum) Orthogonal Group

$$
\begin{array}{ccccccc}
S_n^+ & \subset & O_n^+ & \qquad\qquad & \mathbf{I}_{S_n^+} & \supset & \mathbf{I}_{O_n^+} \\
\cup & & \cup & & \cap & & \cap \\
S_n & \subset & O_n & & \mathbf{I}_{S_n} & \supset & \mathbf{I}_{O_n}
\end{array}
$$

Intertwiners of Permutation Group

$$\text{span}(T_p | p \in P(k,l)) = \mathbf{I}_{S_n}(k,l)$$

Intertwiners of (Quantum) Permutation Group

Let $NC(k,l) \subset P(k,l)$ be the subset of noncrossing partitions.

$$\text{span}(T_p | p \in NC(k,l)) = \mathbf{I}_{S_n^+}(k,l)$$

$$\cap$$

$$\text{span}(T_p | p \in P(k,l)) = \mathbf{I}_{S_n}(k,l)$$

Intertwiners of (Quantum) Permutation and of (Quantum) Orthogonal Group

Let $NC(k,l) \subset P(k,l)$ be the subset of noncrossing partitions.

$$
\begin{array}{ccc}
\text{span}(T_p | p \in NC(k,l)) = \mathbf{I}_{S_n^+}(k,l) & \supset & \mathbf{I}_{O_n^+}(k,l) = \text{span}(T_p | p \in NC_2(k,l)) \\
\cap & & \cap \\
\text{span}(T_p | p \in P(k,l)) = \mathbf{I}_{S_n}(k,l) & \supset & \mathbf{I}_{O_n}(k,l) = \text{span}(T_p | p \in P_2(k,l))
\end{array}
$$

Easy Quantum Groups

(Banica, Speicher 2009)

A quantum group $S_n \subset G_n^* \subset O_n^+$ is called **easy** when its associated tensor category is of the form

$$\mathbf{I}_{S_n} = \text{span}(T_p \mid p \in P)$$

$$\cup$$

$$\mathbf{I}_{\mathbf{G}_\mathbf{n}^*}$$

$$\cup$$

$$\mathbf{I}_{O_n} = \text{span}(T_p \mid p \in NC_2)$$

Easy Quantum Groups

(Banica, Speicher 2009)

A quantum group $S_n \subset G_n^* \subset O_n^+$ is called **easy** when its associated tensor category is of the form

$$\begin{array}{c} \mathbf{I}_{S_n} = \text{span}(T_p \mid p \in P) \\ \cup \\ \mathbf{I}_{\mathbf{G_n^*}} = \mathbf{span}(\mathbf{T_p} \mid \mathbf{p} \in \mathbf{P_{G^*}}), \\ \cup \\ \mathbf{I}_{O_n} = \text{span}(T_p \mid p \in NC_2) \end{array}$$

for a certain collection of subsets $P_{G^*} \subset P$.

What are we interested in?

- classification of easy (and more general) quantum groups (Banica&S, Banica&Vergnioux, Banica&Curran&S)

- understanding of meaning/implications of symmetry under such quantum groups; in particular, under quantum permutations S_n^+, or quantum rotations O_n^+ (Köstler&S, Curran, Banica&Curran&S)

- treating series of such quantum groups (like S_n^+ or O_n^+) as fundamental examples of non-commuting random matrices (Banica&Curran&S)

Classification Results

The **category of partitions** $P_{G^*} \subset P$ for an easy quantum group G_n^* must satisfy:

- P_{G^*} is stable by tensor product.
- P_{G^*} is stable by composition.
- P_{G^*} is stable by involution.
- P_{G^*} contains the "unit" partition $|$.
- P_{G^*} contains the "duality" partition $\sqcap$.

Classification Results

There are:

- 6 Categories of Noncrossing Partitions and
- 6 Categories of Partitions containing Basic Crossing:

$$
\begin{array}{ccccc}
\left\{\begin{matrix}\text{singletons and}\\ \text{pairings}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{singletons and}\\ \text{pairings (even part)}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all}\\ \text{pairings}\end{matrix}\right\} \\
\cap & & \cap & & \cap \\
\left\{\begin{matrix}\text{all}\\ \text{partitions}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all partitions}\\ \text{(even part)}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{with blocks of}\\ \text{even size}\end{matrix}\right\}
\end{array}
$$

Classification Results

and thus:

- 6 free easy quantum groups $S_n^+ \subset G_n^+ \subset O_n^+$ and

- 6 classical easy quantum groups $S_n \subset G_n \subset O_n$

$$
\begin{matrix}
\left\{\begin{matrix}\text{singletons and}\\ \text{pairings}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{singletons and}\\ \text{pairings (even part)}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all}\\ \text{pairings}\end{matrix}\right\} \\
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\end{matrix}
$$

Classification Results

- there are easy quantum groups which are neither classical nor free

- we have partial classification of them

- problematic are the ones of hyperoctahedral type (corresponding to partitions with blocks of even size)

- one can also ask whether there are any other (not necessarily easy) quantum groups of this sort, e.g.: can one classify all quantum rotations $O_n \subset G_n^* \subset O_n^+$

Quantum Symmetries

A vector

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is (quantum) symmetric (with respect to some property) if

$$y = ux = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad \text{i.e.} \quad y_i = \sum_{j=1}^{n} u_{ij} \otimes x_j$$

satisfies the same property as x.

Quantum Exchangeability

$x_1, \ldots, x_n \in (\mathcal{A}, \varphi)$ is

(quantum) exchangeable

if

$$y_1, \ldots, y_n \in (C(S_n^{(+)}) \otimes \mathcal{A}, \mathrm{id} \otimes \varphi)$$

has the same distribution as x. Concretely this means

$$\varphi(x_{i_1} \cdots x_{i_k}) \cdot 1_{C(S_n^{(+)})} = \sum_{j_1, \ldots, j_k = 1}^{n} u_{i_1 j_1} \cdots u_{i_k j_k} \varphi(x_{j_1} \cdots x_{j_k})$$

de Finetti Theorem

(de Finetti 1931, Hewitt, Savage 1955)

The following are equivalent for an infinite sequence of classical, commuting random variables:

- the sequence is exchangeable (i.e., invariant under all S_n)

- the sequence is independent and identically distributed with respect to the conditional expectation E onto the tail σ-algebra of the sequence

Non-commutative de Finetti Theorem

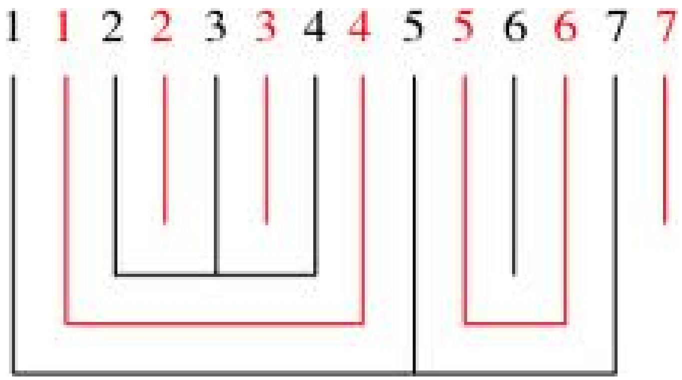
(Köstler, Speicher 2008)

The following are equivalent for an infinite sequence of non-commutative random variables:

- the sequence is quantum exchangeable (i.e., invariant under all S_n^+)

- the sequence is free and identically distributed with respect to the conditional expectation E onto the tail-algebra of the sequence

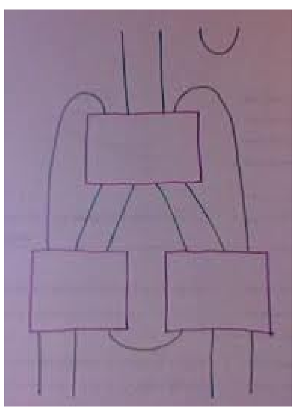
This "explains" occurrence of non-crossing pictures in free probability as emerging from the fact that free probability goes nicely with our quantum symmetries.

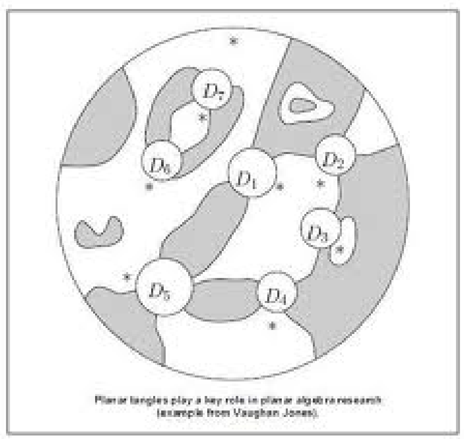


This "explains" occurrence of non-crossing pictures in free probability as emerging from the fact that free probability goes nicely with our quantum symmetries.

Question

Could it be that occurrence of planar pictures in subfactor theory emerges also somehow from the fact that subfactors have some nice relation with quantum permutations or alike symmetries?





Non-Commutative Random Matrices

- there exists, as for any compact quantum group, a unique Haar state on the easy quantum groups, thus one can integrate/average over the quantum groups

- actually: for the easy quantum groups, there exist nice and "concrete" formula for the calculation of this state:

$$\int_{G_n^*} u_{i_1 j_1} \cdots u_{i_k j_k} du = \sum_{\substack{p,q \in P_{G^*}(k) \\ p \leq \ker i \\ q \leq \ker j}} W_n(p,q),$$

where W_n is inverse of

$$G_n(p,q) = n^{|p \vee q|}.$$

Non-Commutative Random Matrices

- this allows the calculation of distributions of functions of our non-commutative random matrices G_n^*, in the limit $n \to \infty$

- in particular, in analogy to Diaconis&Shashahani, we have results about the asymptotic distribution of $\text{Tr}(u^k)$

- note: in the classical case, knowledge about traces of powers of the matrices is the same as knowledge about the eigenvalues of the matrices

Question

What are eigenvalues of a non-commutative (random) matrix?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