Solutions to Home-work 9

1.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 5 & 0 & 1 & 0 \\ 2 & 5 & 14 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 2 & 5 & 14 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 3 & 10 & -2 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 3 & 10 & -2 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 1 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 1 \\ 0 & 1 & 0 & -4 & 10 & -3 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right];$$

and hence, we see that

$$\left(\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 5 & 14 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 3 & -4 & 1 \\ -4 & 10 & -3 \\ 1 & -3 & 1 \end{bmatrix}.$$

2. (a) We start by row-reducing the augmented matrix

$$\left[\begin{array}{cccccc|c} 1 & 0 & 2 & 0 & 3 & 0 & 4 \\ 1 & 1 & 2 & 2 & 3 & 3 & 5 \\ 1 & 1 & 1 & 2 & 2 & 2 & 6 \end{array}\right]$$

thus:

$$\left[\begin{array}{cccccc:c} 1 & 0 & 2 & 0 & 3 & 0 & 4 \\ 1 & 1 & 2 & 2 & 3 & 3 & 5 \\ 1 & 1 & 1 & 2 & 2 & 2 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cccccc:c} 1 & 0 & 2 & 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 & 0 & 3 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 & 6 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccccc:c} 1 & 0 & 2 & 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 & -1 & 2 & 2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccccc:c} 1 & 0 & 2 & 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 & 0 & 3 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccccc:c} 1 & 0 & 2 & 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccccc:c} 1 & 0 & 0 & 0 & 1 & -2 & 6 \\ 0 & 1 & 0 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 \end{array}\right];$$

Hence the initial system of equations

$$\begin{aligned} x_1 + 2x_3 + 3x_5 &= 4 \\ x_1 + x_2 + 2x_3 + 2x_4 + 3x_5 + 3x_6 &= 5 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 + 2x_6 &= 6 \end{aligned}$$

are satisfied if only if the following system of equations is satisfied:

$$\begin{aligned} x_1 + x_5 - 2x_6 &= 6 \\ x_2 + 2x_4 + 3x_6 &= 1 \\ x_3 + x_5 + x_6 &= -1 \end{aligned}$$

i.e., if and only if

$$
\begin{aligned}
x_1 &= 6 - x_5 - 2x_6 && (0.1)\\
x_2 &= 1 - 2x_4 - 3x_6 && (0.2)\\
x_3 &= -1 - x_5 - x_6 && (0.3)
\end{aligned}
$$

So the answer to 2(a) is 'yes; in fact there are infinitely many distinct solutions (for varying choices of x_5)', as is seen from (b) below.

(b) It follows from the reasoning of (a) above and equations (0.1)-(0.3) that the most general solution is given by arbitrarily specifying

$$x_4 = a, x_5 = b, x_6 = c$$

and then requiring that

$$x_1 = 6 - b - 2c, x_2 = 1 - 2a - 3c, x_3 = -1 - b - c.$$

3. (a) Row-reduce thus:

$$
\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 0 & 3 \\ 1 & 3 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 2 & 0 & 3 \\ 1 & 3 & 0 & 5 \end{bmatrix}
$$

$$
\rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -2 & 2 \\ 1 & 3 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix}
$$

$$
\rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}
$$

$$
\rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\rightarrow \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

.

(b) It follows from (a) above that there exists a 4×4 matrix E which is (a product of elementary, and hence) invertible, such that $B = EA$. The fact that E is 1-1 implies that

$$Ax = 0 \Rightarrow Bx = 0$$

Thus,

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 0 \\ x_1 + 2x_2 + 2x_3 + 3x_4 &= 0 \\ x_1 + 2x_2 + 3x_4 &= 0 \\ x_1 + 3x_2 + 4x_3 + 5x_4 &= 0 \end{aligned}$$

if and only if

$$\begin{aligned} x_1 - x_4 &= 0 \\ x_2 + 2x_4 &= 0 \\ x_3 &= 0 \end{aligned}$$

if and only if $x_1 = a, x_2 = -2a, x_3 = 0, x_4 = a$ for some $a \in \mathbb{R}$

(c) It is seen by reading the steps of the row-reduction in (a) above, but in reverse order, that

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

Thus, we have

$$E_9 = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -0 & 1 \end{bmatrix} \quad \& \quad E_8 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Similarly, we find that

$$E_9 = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_8 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \qquad E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\& \quad E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d) The E that we seek is seen to be given (after some matrix multiplication) by

$$\begin{aligned} E &= (E_9 \cdots E_1)^{-1} \\ &= E_1^{-1} \cdots E_9^{-1} \\ &= \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix} . \end{aligned}$$

Since clearly $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \notin ran\ B$, and since E is 1-1, it follows that $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = E\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) \notin ran\ A.$

(Note: since the row-reduced achelon form a square matrix is the identity matrix if and only if it is invertible,

this argument shows that a square matrix is invertible if and only if it is expressible as a product of elementary matrices!)