Solutions to Home-work 8

1. (a) If

$$A = \begin{bmatrix} A^1 \\ A^2 \\ A^3 \\ A^4 \end{bmatrix},$$

where of course A^i denotes the i-th row of A, then

$$CA = \begin{bmatrix} A^1 \\ A^2 + cA^4 \\ A^3 \\ A^4 \end{bmatrix}.$$

(b) If A^i denotes the i-th row of A as in (a) above, then the i-th row of $R^{(m)}_{k,l}A$ is seen to be given by

$$\begin{array}{cl} A^i & \text{if } i \neq k \\ A^k + cA^j & \text{if } i = k \end{array}$$

2. (a) If A is as in 1(a), then

$$FA = \begin{bmatrix} A^3 \\ A^2 \\ A^1 \\ A^4 \end{bmatrix}.$$

(b) If A^i denotes the i-th row of A as in (a) above, then the i-th row of $F^{(m)}_{kl}A$ is seen to be given by

$$\begin{array}{cl} A^l & \text{if } i = k \\ A^k & \text{if } i = l \\ A^i & \text{if } i \notin \{k, l\} \end{array}$$

3. (a) It is clear that $v_1, v_2 \in \Pi$ and in particular, $Tv_i = v_i$ for $i = 1, 2$ (since in fact $Tv = v \forall v \in V$); also v_1 and v_2 are not multiples of one another, and so $\{v_1, v_2\}$ is a linearly independent set. As Π is a proper subspace of $\mathbb{R}^3$, it follos that Π must be two-dimensional. Hence $\{v_1, v_2\}$ is a basis for Π as asserted. As v_3 is perpendicular to Π, it follows from the definition of T that $Tv_3 = -v_3$. The preceding observations show that $\mathcal{B}$ is indeed a basis for $\mathbb{R}^3$ and that $[T]_{\mathcal{B}}$ is as shown.

(b) Routine matrix multilication shows that U is invertible and that U' is the inverse of U.

(c) If $S \in L(\mathbb{R}^3)$ is given by $[S]_{\mathcal{E}} = U$, then it is clear that $Se_j = v_j$ for $1 \leq j \leq 3$.

(d) The rule for 'change of basis' (see lecture 8) shows that we must have

$$
\begin{aligned}
[T]_{\mathcal{E}} &= U[T]_{\mathcal{B}}U^{-1} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} + \frac{1}{6} - \frac{1}{3} & \frac{-1}{2} + \frac{1}{6} - \frac{1}{3} & \frac{-1}{6} - \frac{1}{3} \\ \frac{-1}{2} + \frac{1}{6} - \frac{1}{3} & \frac{1}{2} + \frac{1}{6} - \frac{1}{3} & \frac{-1}{6} - \frac{1}{3} \\ \frac{1}{6} - \frac{1}{3} & \frac{1}{6} - \frac{1}{3} & \frac{-1}{6} - \frac{1}{3} \end{bmatrix} \\
&= \frac{1}{6} \begin{bmatrix} 3+1-2 & -3+1-2 & -1-2 \\ -3+1-2 & 3+1-2 & -1-2 \\ 1-2 & 1-2 & -1-2 \end{bmatrix} \\
&= \frac{1}{6} \begin{bmatrix} 2 & -4 & -3 \\ -4 & 2 & -3 \\ -1 & -1 & -3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{-1}{2} \\ \frac{-2}{3} & \frac{1}{3} & \frac{-1}{2} \\ \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{2} \end{bmatrix}
\end{aligned}
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