Solutions to Home-work 7

1.

$$\begin{aligned} T(0+0) &= T0 \\ &\Rightarrow T0+T0=T0 \\ &\Rightarrow T0=0 \end{aligned}$$

2.

$$Tu = Tv \Leftrightarrow T(u-v)=0$$

and

$$u=v \Leftrightarrow u-v=0$$

3. This was done in the lecture:

$$R_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} cos\ \theta \\ sin\ \theta \end{bmatrix}$$

by definition of the trigonometric (=circular) functions, while an in argument involving congruent triangles yields

$$R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -sin\ \theta \\ cos\ \theta \end{bmatrix}$$

Hence, the matrix representing R_θ is given by

$$\begin{bmatrix} cos\ \theta & sin\ \theta \\ -sin\ \theta & cos\ \theta \end{bmatrix}$$

4.

$$\begin{aligned} R_{\theta+\phi} &= R_\theta R_\phi \\ &= \begin{bmatrix} cos\ \theta & sin\ \theta \\ -sin\ \theta & cos\ \theta \end{bmatrix} \cdot \begin{bmatrix} cos\ \phi & sin\ \phi \\ -sin\ \phi & cos\ \phi \end{bmatrix} \\ &= \begin{bmatrix} cos\ \theta\ cos\ \phi - sin\ \theta\ sin\ \phi & sin\ \theta\ cos\ \phi + cos\ \theta\ sin\ \phi \\ -(sin\ \theta\ cos\ \phi + cos\ \theta\ sin\ \phi) & cos\ \theta\ cos\ \phi - sin\ \theta\ sin\ \phi \end{bmatrix} \end{aligned}$$

and hence, we see that

$$\begin{aligned} cos(\theta+\phi) &= cos\ \theta\ cos\ \phi - sin\ \theta\ sin\ \phi \\ sin(\theta+\phi) &= sin\ \theta\ cos\ \phi + cos\ \theta\ sin\ \phi \end{aligned}$$

5.
$$MR_\theta = \left[\begin{array}{cc} cos\ \theta & sin\ \theta \\ sin\ \theta & -cos\ \theta \end{array} \right]$$

while
$$R_\theta M = \left[\begin{array}{cc} cos\ \theta & -sin\ \theta \\ -sin\ \theta & -cos\ \theta \end{array} \right]$$

and hence

$$MR_\theta = R_\theta M \Leftrightarrow sin\ \theta = 0 \Leftrightarrow \theta = n\pi$$

6. It is clear that the equation
$$T(a+ib) = \left[\begin{array}{cc} a & -b \\ b & a \end{array} \right]$$

defines a 1-1 mapping T of $\mathbb{C}$ onto its image and that T is (real-) linear, meaning that, for any $\alpha_i, a_i, b_i \in \mathbb{R}, i = 1, 2,$

$$\begin{aligned} T(\alpha_1(a_1+ib_1)+\alpha_2(a_2+ib_2) &= T((\alpha_1 a_1 + i\alpha_1 b_1) + (\alpha_2 a_2 + i\alpha_2 b_2) \\ &= T((\alpha_1 a_1 + \alpha_2 a_2) + i(\alpha_1 b_1 + \alpha_2 b_2) \\ &= \left[\begin{array}{cc} \alpha_1 a_1 + \alpha_2 a_2 & -\alpha_1 b_1 - \alpha_2 b_2 \\ \alpha_1 b_1 + \alpha_2 b_2 & \alpha_1 a_1 + \alpha_2 a_2 \end{array} \right] \\ &= \alpha_1 \left[\begin{array}{cc} a_1 & -b_1 \\ b_1 & a_1 \end{array} \right] + \alpha_2 \left[\begin{array}{cc} a_2 & -b_2 \\ b_2 & a_2 \end{array} \right] \\ &= \alpha_1 T(a_1+ib_1) + \alpha_2 T(a_2+ib_2)\ ; \end{aligned}$$

while

$$\begin{aligned} T(a_1+ib_1) \times T(a_2+ib_2) &= \left[\begin{array}{cc} a_1 & -b_1 \\ b_1 & a_1 \end{array} \right] \times \left[\begin{array}{cc} a_2 & -b_2 \\ b_2 & a_2 \end{array} \right] \\ &= \left[\begin{array}{cc} a_1 a_2 - b_1 b_2 & -a_1 b_2 - a_2 b_1 \\ a_1 b_2 + a_2 b_1 & a_1 a_2 - b_1 b_2 \end{array} \right] \\ &= T((a_1 a_2 - b_1 b_2) + i(\overline{a}_1 b_2 + a_2 b_1)) \\ &= T((a_1+ib_1)(a_2+ib_2))\ , \end{aligned}$$

so T also preserves products.