Solutions to Home-work 6

1. (a) Any $v \in V$ admits a unique decomposition $u = \sum_{i=1}^{n} \alpha_i v_i$; then define $Tv = \sum_{i=1}^{n} \alpha_i w_i$. If also $V \ni v = \sum_{i=1}^{n} \beta_i v_i$, then we find, for any $\alpha, \beta \in \mathbb{R}$, that

$$\begin{aligned} T(\alpha u + \beta v) &= T\left(\sum_{i=1}^{n} (\alpha\alpha_i + \beta\beta_i) v_i\right) \\ &= \left(\sum_{i=1}^{n} (\alpha\alpha_i + \beta\beta_i) w_i\right) \\ &= \alpha T u + \beta T v \ , \end{aligned}$$

and hence T is indeed linear; clearly T satisfies $Tv_i = w_i$. Conversely, if $S \in L(V, W)$ satisfies $Sv_i = w_i$ for all i, then the assumed lilnearity of S will force $S(\sum_{i=1}^{n} \alpha_i v_i)$ to be equal to $(\sum_{i=1}^{n} \alpha_i w_i$ and thus $S = T$.

(b) If T is 1-1, then note that

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i w_i = 0 &\Rightarrow T(\sum_{i=1}^{n} \alpha_i v_i) = 0 \\ &\Rightarrow \sum_{i=1}^{n} \alpha_i v_i = 0 \ \text{ by the assumed 1-1-ness of } T \\ &\Rightarrow \alpha_i = 0 \forall i \ \text{ by the independence of the } v_i\text{s} \end{aligned}$$

so the w_is are independent.

Conversely, if the w_is are independent, and if $V \ni v = \sum_{i=1}^{n} \alpha_i v_i$, then observe that

$$\begin{aligned} Tv = 0 &\Rightarrow \sum_{i=1}^{n} \alpha_i w_i = 0 \\ &\Rightarrow \alpha_i = 0 \forall i \ \text{ by the independence of the } w_i\text{s} \\ &\Rightarrow v = 0 \end{aligned}$$

so T is indeed 1-1.

(c) Since $V = \{\sum_{i=1}^{n} \alpha_i v_i : \alpha_i \in \mathbb{R}\}$, it follows that $ran(T) = \{\sum_{i=1}^{n} \alpha_i w_i : \alpha_i \in \mathbb{R}\}$; hence T is onto iff $W = ran(T) = sp(\{w_1, \cdots, w_n\})$.

2. If $u, v \in \ker\ T, \alpha, \beta \in \mathbb{R}$, then $Tu = Tv = 0$ and so $T(\alpha u + \beta v) = \alpha T u + \beta T v = 0$ so $ker\ T$ is closed under forming linear combinations and is indeed a subspace.

Similarly, if $w_1, w_2 \in ran\ T$, there exist $v_1, v_2 \in V$ such that $w_i = Tv_i, i = 1, 2$, and so $\alpha_1 w_1 + \alpha_2 w_2 = T(\alpha_1 v_1 + \alpha_2 v_2) \in ran\ T$ as desired.

3. If T is 1-1, then $Tv = 0 (= T0) \Rightarrow v = 0$, so $ker\ T = \{0\}$.

   Conversely. if $ker\ T = \{0\}$, then $Tu = Tv \Rightarrow u - v \in ker\ T \Rightarrow u = v$.

4. $T0 = T(0 + 0) = T0 + T0 \Rightarrow T0 = 0$.

5. The validity of all the vector space axioms in $L(V, W)$ is a consequence of their validity in V and W. Thus, for instance, associativity of addition in $L(V, W)$ is verified thus: if $S, T.U \in L(V, W)$, then for arbitrary $v \in V$, we have $((S + T) + U)v = (S + T)v + Uv = (Sv + Tv) + Uv = Sv + (Tv + Uv) = Sv + (T + U)v = (S + (T + U))v$ and since two functions are equal precisely when they agree throughout their common domain, this shows that $((S + T) + U) = (S + (T + U))$ as desired. The other verifications are entirely analogous.

6. Suppose $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_m\}$ are bases for V and W respectively. It follows from Problem 1(a) that for $1 \leq i \leq m, 1 \leq j \leq n$, there exists a unique $T_{ij} \in L(V, W)$ such that $T_{ij} v_k = \delta_{jk} w_i$, where the *Kronecker delta function* is defined by

   $$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

   Thus

   $$T_{ij} v_k = \begin{cases} w_i & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

   We assert that $\{T_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for $L(V, W)$; we shall verify that the above set is linearly independent as well as a spanning set.

   First note that if we define $T = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} T_{ij}$, then, by definition, we find that

$$\begin{aligned} Tv_k &= \sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}T_{ij}v_k \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}T_{ij}v_k \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}\delta_{jk}w_i \\ &= \sum_{i=1}^{m}\alpha_{ik}w_i \end{aligned} \tag{0.1}$$

So if $T = 0$, then $0 = Tv_k = \sum_{i=1}^{m}\alpha_{ik}w_i$ for each $1 \leq k \leq n$, but the linear independence of the w_i's, it is thus seen that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}T_{ij} = 0 \Rightarrow \alpha_{ij} = 0 \forall 1 \leq i \leq m, 1 \leq j \leq n$$

thus establishing linear independence of the T_{ij}'s.

Next, if $S \in L(V,W)$, and if, for each $1 \leq k \leq n$, $Sv_k = \sum_{i=1}^{m}\alpha_{ik}w_i$ is the representation of Sv_k as a linear combination of the w_i's, it follows from the uniqueness assertion of 1(a) and equation (0.1) that $S = \sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}T_{ij}$ so the T_{ij}'s do form a spanning set.

7. For any $v \in V$, observe that

$$\begin{aligned} ((ST)U)v &= (ST)Uv \\ &= S(T(U(v))) \\ &= S((TU)v) \\ &= (S(TU))v \\ (\alpha(ST))v &= \alpha(ST)v \\ &= \alpha S(Tv) \\ &= (\alpha S)Tv \\ &= ((\alpha S)T)v \\ ((S+T)U)v &= (S+T)Uv \\ &= S(Uv) + T(Uv) \\ &= SUv + TUv \\ &= (SU+TU)v \end{aligned}$$

The last verificartion is similar.