Solutions to Home-work 3

1.

$$\begin{aligned} sp\ \{(1,1,0),(1,0,1)\} &= \{\alpha(1,1,0)+\beta(1,0,1) : \alpha,\beta \in \mathbb{R}\} \\ &= \{\alpha+\beta,\alpha,\beta) : \alpha,\beta \in \mathbb{R}\} \\ &= \{(x,y,z) \in \mathbb{R}^3 : x = y+z\} \end{aligned}$$

is the plane with equation $x - y - z = 0$ (which clearly passes through the origin).

2. (a) Clearly $V = \mathbb{R}^2$ is a vector space, but the displayed W is not a subspace for various reasons; for instance, $(1,2) \in W$ but $(2,4) = 2(1,2) \notin W$.

   (b) The displayed W is not a vector space for various reasons, the most important one being that $(0,0) \notin W$, and the only vector $x \in \mathbb{R}^2$ which can satisfy $x+x = x$ is $x = (0,0)$ so that W cannot have a vector playing the role of 0.

   (c) The set $V = \mathbb{R}^{\mathbb{N}}$ of real sequences is a vector space, with respect to component-wise addition and scalar multipica-tion. For instance, associativity of addition is verified thus (almost exactly as in the case of $\mathbb{R}^2$ or $\mathbb{R}^n$ for any $n \in \mathbb{N}$): Suppose $x, y, z \in \mathbb{R}^{\mathbb{N}}$ and that their n-th coordinates are x_n, y_n, z_n respectively. Then the n-th co-ordinate of $(x + y) + z$ as well as of $x + (y + z)$ are both seen to be equal to $x_n + y_n + z_n$ thanks to associativity of addition in $\mathbb{R}$. Since the two sequences have the same co-ordinate in the n-th place, and since this is true for every $n \in \mathbb{N}$, the two sequences should themselves be identical.

   That the displayed W is indeed a subspace is a consequence of W being closed under the vector operations: for if $x, y \in \mathbb{R}^{\mathbb{N}}$ are as above, and if $\alpha, \beta \in \mathbb{R}$, then we see that if n is a prime number, the assumption that x and y belong to W shows that

$$\begin{aligned} (\alpha x + \beta y)_n &= \alpha x_n + \beta y_n \\ &= 0 \end{aligned}$$

   and hence $\alpha x + \beta y \in W$.

   (d) $\mathbb{R}^5$ is clearly a vector space, but the displayed W is not a subspace since it is not closed under multiplication by negative scalars. (It is closed under vector addition, how-ever.)

(e) The set of $\mathbb{R}^X$ real-valued functions on any set X is a vector space, with respect to pointwise operations. The verificationn is no different for general X than for $\mathbb{N}$ or $\{1, 2, \cdots, n\}$. The displayed W is indeed a subspace because if $f, g \in W$ and $\alpha, \beta \in \mathbb{R}$, an easy verification shows that the defining property of W implies that also $\lim_{x \to 0}(\alpha f + \beta g)(x) = 0$ whence $\alpha f + \beta g \in W$.

3. (a) $\{(1,0,0),(0,1,0),(0,0,0)\}$ is not linearly independent, since

$$0(1,0,0) + 0(0,1,0) + 1(0,0,0) = (0,0,0)$$

is a non-trivial linear combination resulting in the zero vector.

(b) $\{(0,1,1),(1,0,1),(1,1,0)\}$ is linearly independent; for,

$$\begin{aligned} 0 &= \alpha(0,1,1) + \beta(1,0,1) + \gamma(1,1,0) \\ \Rightarrow \; 0 &= (\beta+\gamma, \alpha+\gamma, \alpha+\beta) \\ \Rightarrow \; 0 &= \beta+\gamma = \alpha+\gamma = \alpha+\beta \ , \end{aligned}$$

which is seen to imply that $\alpha = -\beta = \gamma = -\alpha$, and hence that $\alpha = \beta = \gamma = 0$

(c) $\{(1,2,3),(4,5,6),(7,8,9),(10,11,12)\}$ is not linearly dependent; in fact, even the subset $\{(1,2,3),(4,5,6),(7,8,9)\}$ is linearly dependent, since the equation $(4,5,6)-(1,2,3) = (3,3,3) = (7,8,9)-(4,5,6)$ shows that $1(1,2,3)-2(4,5,6)+(7,8,9) = (0,0,0)$

(d) $\{(1,2,3),(7,14,21)\}$ is linearly dependent, since $7(1,2,3)-1(7,14,21) = (0,0,0)$. More generally, it is not hard to show that a linear combination of two vectors can be the zero vector if and only if one of them is a scalar multie of the other (prove this!).

4. If $S_0 = \{v_1, v_2, \cdots, v_k\} \subset \{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots v_n\} = S$, and if $\sum_{i=1}^{k} \alpha_i v_i = 0$ where $\alpha_i \neq 0$ for some $1 \leq i \leq k$, define $\alpha_j = 0$ for $k < j \leq n$, and note that $\sum_{i=1}^{k} \alpha_i v_i = 0$ where $\alpha_i \neq 0$ for some $1 \leq i \leq n$. (In fact an infinite set is said to be linearly independent precisely when every finite subset is linearly independent.)

5. Suppose $S = \{v_1, v_2, \cdots, v_n\} \neq \{0\}$. Let $W = sp\ S$. We shall show, by induction on n that S contains a basis for W.

If $n = 1$, then $S = \{v_1\}$ and $v_1 \neq 0$ by assumption and hence s is a (necessarily) linearly independent spanning set, hence a basis, for W.

Suppose the assertion holds for every set of cardinality smaller than n and suppose $|S| = n$. If S is linearly independent, it is a linearly dependent spanning set, hence a basis for W. If S is not linearly dependent, there exists an element - which we may assume is v_n which is a linear combination of $S_0 = S \setminus \{v_n\}$. Then S_0 is a spanning set for W with less than n elements. It follows from the induction hypothesis that S_0, and hence S, contains a basis for W.