Solutions to Home-work 11

1. (a) Notice that if $\phi \in \mathcal{B}^2(\mathbb{R}^3)$ and $x, y \in \mathbb{R}^3$ are arbitrary, and if we define $\phi_{ij} = \phi(e_i, e_j)$, then

$$\begin{aligned} \phi(x,y) &= \phi(x, \sum_{j=1}^{3} e^j(y) e_j) \\ &= \sum_{j=1}^{3} e^j(y)\phi(x, e_j) \\ &= \sum_{j=1}^{3} e^j(y)\phi(\sum_{i=1}^{3} e^i(x) e_i, e_j) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} e^i(x)e^j(y)\phi(e_i, e_j) \\ &= \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \phi(e_i,e_j) e^i \otimes e^j\right)(x,y) \ , \end{aligned}$$

thereby showing that $\{e^i \otimes e^j : 1 \leq i, j \leq 3\}$ is a spanning set for $\mathcal{B}^2(\mathbb{R}^3)$.

The above string of equations can also be used to see that the only linear combination of the $(e^i \otimes e^j)$'s which can vanish identically os the trivial linear combination, ie. that they are linearly independent.

(b) Notice, to start with, that if $\alpha_{ij}, 1 \leq i < j \leq 3$ are arbitrary scalars, then

$$\begin{aligned} \sum_{i<j} \alpha_{ij} e^i \wedge e^j = 0 \quad &\Rightarrow \quad \sum_{i<j} \alpha_{ij}(e^i \otimes e^j - e^j \otimes e^i) = 0 \\ &\Rightarrow \quad \alpha_{ij} = 0 \ \forall i, j \ , \end{aligned}$$

since $\{e^i \otimes e^j : 1 \leq i < j \leq 3\}$ is linearly independent, by (a) above. Hence, $\{e^i \wedge e^j : 1 \leq i < j \leq 3\}$ is linearly independent.

Next, the string of equations in 1(a) above show that if $\phi \in \mathcal{B}^2(\mathbb{R}^3)$, then $\phi = \sum_{i,j=1}^{3} \phi(e_i, e_j) e^i \otimes e^j$. If further

$\phi \in \bigwedge^2 \mathbb{R}^3$, then, $\phi(e_i, e_j) = -\phi(e_j, e_i)$, and we find that

$$\begin{aligned}\phi &= \sum_{i,j=1}^{3} \phi(e_i, e_j) e^i \otimes e^j \\ &= \sum_{1 \leq i < j \leq 3} \phi(e_i, e_j) e^i \wedge e^j\end{aligned}$$

and hence, $\{e^i \wedge e^j : 1 \leq i < j \leq 3\}$ spans $\bigwedge^2 \mathbb{R}^3$.

2. (a) If A and B are skew-symmetric matrices, and if $\alpha, \beta \in \mathbb{R}$, and if $C = \alpha A + \beta B$, then

$$\begin{aligned}c_{ij} &= \alpha a_{ij} + \beta b_{ij} \\ &= -\alpha a_{ji} - \beta b_{ji} \\ &- -c_{ij} \ ,\end{aligned}$$

thus demonstrating that the set of skew-symmetric matrices is (closed under formation of linear combinations, and is hence) a vector subspace of $M_n(\mathbb{R})$.

Notice that a skew-symmetric matrix must have zeroes on the main diagonal, and that the entries in the triangle below the diagonal are the negative of the corresponding (transposed) entries above the diagonal. Hence if $\{E(i,j)$ denotes the matrix with 1 in the $9i,j)$-th entry and 0 everywhere else - so the $\{E(i,j) : 1 \leq i,j \leq n\}$ are the 'standard basis' for $M_n(\mathbb{R})$, usually called a system of matrix units - then $\{E(i,j) - E(j,i) : 1 \leq i < j \leq n\}$ is a basis for the space of skew-symmetric matrices. Since this set has exactly $(n-1) + (n-2) + \cdots + 2 + 1 = \frac{n(-1)}{2}$, we are done.

(b) This is an easy verification, which we omit, since this fact can also be deduced from 2(c) below.

(c) With the foregoing notation, we see that the equation

$$T(E(i,j)) = e^i \otimes e^j$$

extends to a unique isomorphism of $M_n(\mathbb{R})$ onto $\mathcal{B}^2(\mathbb{R}^n)$ which maps $E(i,j) - E(j,i)$ to $e^i \wedge e^j$.

3. (a) This is almost the same as verifyting that the mapping $\epsilon : \mathcal{Z} \to \{\pm 1\}$ defined by

$$\epsilon(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

is a group homomorphism.

(b) If $\sigma \in A_n$ and $\pi \in S_n$ are arbitrary, then

$$\epsilon(\pi\sigma\pi^{-1}) = \epsilon(\pi)\epsilon(\pi)^{-1} = 1$$

and so indeed $\pi\sigma\pi^{-1} \in A_n$. The same argument shows, more generally that if $\phi : G_1 \to G_2$ is a homomorphism between groups, then $\{g \in G_1 : \phi(g) = 1\}$ is a normal subgroup of G_1. (We have written 1 above for the identity element of the group G_2.)

(c) Suppose that, for $i = 1, 2$, the transposition interchanges the distinct elemens p_i and q_i. Let σ be any permutation such that $\sigma(p_1) = p_2$ and $\sigma(q_1) = q_2$. Then, direct computation shows that $\sigma\tau_1\sigma^{-1} = \tau_2$.

(d) If even one transposition, say τ_1 were to belong to A_n, it would follow from (c) above and (b) that every transposition τ_2 should also belong to A_n. Since transpositions generate all of S_n, it would then follow that $S_n = A_n$, contrary to our assumption.

Next, if some $\sigma \in A_n$ were expressible as a product $\sigma = \tau_1 \cdots \tau_{2k+1}$ of an odd number of transpositions, that would imply that $\tau_{2k+1} = \sigma\tau_{2k} \cdots \tau_1 \in A_n$, which we have shown to not be possible.