Solutions to Home-work 10

1. (a)

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 5 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 2 & 5 & 14 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 3 & 10 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 3 & 10 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

and hence, A is row-reducible to the identity matrix I and therefore invertible.

(b) Deduce from (a) that $I = E_6E_5E_4E_3E_2E_1A$, where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$
$$\rightarrow E_3 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix}$$
$$\rightarrow E_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

and we deuce that

$$\begin{aligned} A &= (E_6E_5E_4E_3E_2E_1)^{-1} \\ &= E_1^{-1}E_2^{-1}\cdots E_6^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\qquad \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

is a decomposition of the desired sort.

2. (a)

$$\begin{aligned} adj(A) &= \begin{bmatrix} \begin{vmatrix} 2 & 5 \\ 5 & 14 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 5 & 14 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} \\ -\begin{vmatrix} 1 & 5 \\ 2 & 14 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 14 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 3 & -4 & 1 \\ -4 & 10 & -3 \\ 1 & -3 & 1 \end{bmatrix} \end{aligned}$$

(b) In this case $det(A) = 1$ and it is readily verified that indeed the matrix $adj(A) = A^{-1}$.

(c) The displayed ratios are seen to be nothing but

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 3a - 4b + c \\ -4a + 10b - 3c \\ a - 3b + c \end{bmatrix} \\ &= adj(A) \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \end{aligned}$$

as desired.