Solutions to Home-work 1

1. (a) Given the two equations

$$\begin{aligned} x + y &= 7 \\ x - y &= 3 \end{aligned}$$

add and subtract, to find that $2x = 10$ and $2y = 4$, and hence that

$$x = 5 \text{ and } y = 2$$

(b) Given the two equations

$$\begin{aligned} x + y &= 7 \\ 2x + 3y &= 15 \end{aligned}$$

subtract twice (resp., thrice) the first equation from the second, to find, respectively, that $y = 1$ and $-x = -6$; hence

$$x = 6 \text{ and } y = 1$$

(b) Given the two equations

$$\begin{aligned} x + 2y &= 4 \\ 2x + 4y &= 7 \end{aligned}$$

subtract twice the first equation from the second, to get $0x + 0y = -1$ which can clearly not be satisfied by any values of x and y.

Thus the pairs of lines in (a) and (b) meet - at the points $(5, 2)$ and $(6, 1)$ respectively - while the pair of equations in (c) represent a pair of distinct parallel lines - they pass through $(4, 0)$ and $(\frac{7}{2}, 0)$ respectively - and are both perpendicular to the vector $(1, 2)$.

2. In order for a line to be tangent to a circle, it must be perpendicular to the diameter through the point of tangency; but the only line through the origin (the centre of our circle) which is pendicular to the line $x + y = c$ is the line $y = x$, which meets the circle at (x, x) where $2x^2 = 1$, i.e., when $x = \pm\frac{1}{\sqrt{2}}$. So the desired points of tangency are $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. The two tangents are $x + y = \pm\sqrt{2}$. Thus $x + y = c$ is tangent to $x^2 + y^2 = 1$ if and only if $c = \pm\sqrt{2}$.

3. Let $M = (m_1, m_2)$ be the foot of the perpendicular from the point $P = (1, 2)$ to the line $L = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1\}$. Since $\mathbf{v} = (1, 1)$ is a vector orthogonal to L, it is seen that the vector $PM = (m_1 - 1, m_2 - 2)$ must be equal to $t\mathbf{v}$ for a unique scalar t. Thus, we see that $(m_1, m_2) = (1, 2) + (t, t) = (t + 1, t + 2)$. Since M is on the line L, we must have

$$\begin{aligned} 1 &= m_1 + m_2 \\ &= (t+1) + (t+2) \\ &= 2t + 3 \ , \end{aligned}$$

so we must have $t = -1$. So the distance from P to L is $\|t\mathbf{v}\| = \|(-1, -1)\| = \sqrt{2}$.

The same reasoning shows, more generally that the distancwe from a point $P = (p_1, p_2)$ from the line $L = \{x = (x_1, x_2) \in \mathbb{R}^2 : ax_1 + bx_2 + c = 0\}$ will be given by $d = |t|\sqrt{a^2 + b^2}$, where t must satisfy the requirement that $(p_1, p_2) + t(a, b)$ lies on L, i.e., that $ap_1 + bp_2 + t(a^2 + b^2) + c = 0$; unravelling this, we find that $d = \frac{|ap_1 + bp_2 + c|}{\sqrt{a^2+b^2}}$.

In order to make sure you have understood this reasoning, try to prove, in an entirely analogous fashion, that the distance from point $P(p_1, p_2, p_3)$ in $\mathbb{R}^3$ from the plane $\Pi = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : ax_1 + bx_2 + cx_3 + d = 0\}$ is given by

$$d = \frac{|ap_1 + bp_2 + cp_3 + d|}{\sqrt{a^2 + b^2 + c^2}} \ .$$

4. The definitions show that there must exist unique (positive) scalars p, q such that $z - x = p(y - x)$ and $y - z = q(y - x)$. Adding these equations, we have $p + q = 1$. But we are given that $r/s = \frac{\|z-x\|}{\|y-z\|} = p/q$; deduce that $p = \frac{p}{p+q} = \frac{r}{r+s}$ and hence that

$$\begin{aligned} z &= x + (z - x) \\ &= x + p(y - x) \\ &= (1 - p)x + py \\ &= \frac{s}{r+s}x + \frac{r}{r+s}y \ . \end{aligned}$$

5. By definition, the points D, E and F are given by the vectors $\frac{1}{2}(b+c), \frac{1}{2}(c+a)$ and $\frac{1}{2}(a+b)$, respectively. Hence by the formula of problem 4 above, we see that the point on the line segment AD which divides it in the ratio $2:1$ must be given by $\frac{1}{3}(a+2\times]half(b+c))$, i.e., by the vector $g = \frac{1}{3}(a+b+c)$ or the point G. Assertions (ii) and (iii) are verified in identical fashion.