X More Radon-Nikodym type theorems.

1° Let $M \subset B(H)$ be a von Neumann algebra with a bicyclic vector $\xi_0 \in H$. As in III·1° we define for each $\xi \in H$ an operator L^0_ξ with $D(L^0_\xi) = M'\xi_0$ and

$$L^0_\xi(x'\xi_0) = x'\xi \ , \quad (x' \in M')$$

which is affiliated to M and $L^0_\xi(\xi_0) = \xi$.

This operator is positive if and only if for all $x' \in M'$

$$(x'\xi \mid x'\xi_0) \geq 0 \ \text{ i.e. } (x'^* x'\xi \mid \xi_0) \geq 0 \ \text{ i.e. } \ \omega'_{\xi,\xi_0} \geq 0 .$$

Also, as we have seen in III·1°, if $\xi \in D_S$ the operator L^0_ξ is preclosed. The operator L^0_ξ is preclosed also if it is symmetric, in particular if it is positive. In these cases we denote by L_ξ the closure of L^0_ξ.

If L^0_ξ is positive, then its Friedrichs extension A is a positive selfadjoint operator, affiliated to M and $\xi = A\xi_0$, $L_{A\xi_0} \subset A$. We denote

$$P_S = \{\xi \in H ; L^0_\xi \text{ is positive}\} =$$

$$= \{\xi \in H ; \omega'_{\xi,\xi_0} \geq 0\}$$

$$= \{A\xi_0 ; A \text{ positive self-adjoint affiliated to } M, \xi_0 \in D_A\}$$

2° With the above notation we have

LEMMA. For any normal positive form φ on M there exists a unique vector $\xi \in P_S$ such that

$$\varphi = \omega_\xi .$$

In particular, there exists a positive selfadjoint operator A, affiliated to M, such that $\xi_0 \in D(A)$ and

$$\varphi = \omega_{A\xi_0}$$

PROOF. <u>Uniqueness</u> Let $\xi_1, \xi_2 \in P_S$ such that $\omega_{\xi_1} = \omega_{\xi_2}$. Then

$$\| x\xi_1 \| = \| x\xi_2 \| , \quad (\forall)\, x \in M$$

so that there is a partial isometry $v' \in M'$ such that

$$V'x\xi_1 = x\xi_2,\ (\forall)\, x \in M \text{ and } V'([M\xi_1]^{\perp}) = 0.$$

In particular, $V'\xi_1 = \xi_2$, $V'^*\xi_2 = \xi_1$ and hence

$$\omega'_{\xi_2,\xi_0} = \omega'_{V'\xi_1,\xi_0} = R_{V'}\, \omega'_{\xi_1,\xi_0}$$

$$\omega'_{\xi_1,\xi_0} = \omega'_{V'^*\xi_2,\xi_0} = R_{V'^*}\, \omega'_{\xi_2,\xi_0}.$$

Since $\xi_1, \xi_2 \in P_S$ we have $\omega'_{\xi_1,\xi_0} \geq 0$, $\omega'_{\xi_2,\xi_0} \geq 0$ and, by the uniqueness of the polar decomposition it follows that

$$\omega'_{\xi_1,\xi_0} = \omega'_{\xi_2,\xi_0}$$

and, since $\overline{M'\xi_0} = H$, this entails $\xi_1 = \xi_2$.

<u>Existence</u> By the Spatial Theorem of von Neumann there exists a vector $\zeta \in H$ such that

$$\varphi = \omega_\zeta$$

Let

$$\omega'_{\zeta,\xi_0} = R_{V'}\psi'$$

be the polar decomposition. We define

$$\xi = V'^*\zeta$$

and then

$$\omega'_{\xi,\xi_0} = \omega'_{V'^*\zeta,\xi_0} = R_{V'^*}\, \omega'_{\zeta,\xi_0} = \psi' \geq 0$$

hence $\xi \in P_S$.

On the other hand

$$\omega'_{\zeta,\xi_0} = R_{V'}\psi' = R_{V'}\,\omega'_{\xi,\xi_0} = \omega'_{V'\xi,\xi_0}$$

hence $\zeta = V'\xi$. Thus for any $x \in M$ we have

$$\varphi(x) = \omega_\zeta(x) = (x\zeta|\zeta) = (x\zeta|V'\xi) = (V'^*x\zeta|\xi)$$
$$= (xV'^*\zeta|\xi) = (x\xi|\xi) = \omega_\xi(x)$$

that is $\varphi = \omega_\xi$.

3° Let φ be a normal positive form on M and A a positive self-adjoint operator in H, affiliated to M. Let also

$$e_n = \chi_{(0,n)}(A) \quad ; \quad n \in \mathbb{N}.$$

We say that A is of summable square with respect to φ if

$$c = \sup_n \varphi(A^2 e_n) < \infty \quad ; \quad \text{then } \varphi(A^2 e_n) \uparrow c.$$

In this case, for any $x \in M$ and any $m > n$ we have

$$|\varphi(Ae_m x Ae_m) - \varphi(Ae_n x Ae_n)| \leq$$

$$\leq |\varphi(Ae_m x A(e_m - e_n))| + |\varphi(A(e_m - e_n) x Ae_n)|$$

$$\leq \varphi(Ae_m x x^* Ae_m)^{1/2} \varphi(A^2(e_m - e_n))^{1/2} +$$

$$+ \varphi(A^2(e_m - e_n))^{1/2} \varphi(Ae_n x^* x Ae_n)^{1/2} \leq$$

$$\leq 2 \|x\| c^{1/2} |\varphi(A^2 e_m) - \varphi(A^2 e_n)|^{1/2}.$$

Consequently $\varphi(Ae_n x Ae_n)$ is a Cauchy sequence. We write

$$(L_A R_A \varphi)(x) = \lim_{n \to \infty} \varphi(Ae_n x Ae_n) \quad ; \quad x \in M.$$

Therefore we get a linear form $L_A R_A \varphi$ on M. Tending to the limit for $m \to \infty$ in the preceding inequality we get

$$|(L_A R_A \varphi)(x) - (L_{Ae_n} R_{Ae_n} \varphi)(x)| \leq$$

$$\leq 2 \|x\| c^{1/2} [c - \varphi(A^2 e_n)]^{1/2} \xrightarrow[n \to \infty]{} 0$$

hence $L_A R_A \varphi \in M_*^+$ since $L_{Ae_n} R_{Ae_n} \varphi \in M_*^+$, $(\forall) n \in \mathbb{N}$.

By taking into account the functional calculus (LECTURES 9.11.(iii)) it is obvious that if

$$\varphi = \omega_\xi \; , \; (\xi \in H),$$

then the operator A is of summable square with respect to φ if and only if

$$\xi \in D_A$$

and, in this case

$$L_A R_A \varphi = \omega_{A\xi}$$

4°. THEOREM. (Takesaki) Let φ and ψ be normal positive forms on the von Neumann algebra $M \subset B(H)$ such that

$$s(\varphi) \leq s(\psi)$$

Then there exists a positive self-adjoint operator A in H, affiliated to M such that

$$s(A) \leq s(\psi) \text{ and } \varphi = L_A R_A \psi.$$

Moreover if $s(\psi)$ is a finite projection in M, then the operator A is uniquely determined with the required properties.

PROOF. We may assume $s(\psi) = 1$. By the GNS construction there is a $*$-isomorphism π of M onto a von Neumann algebra $N \subset B(H_0)$, $\pi(1) = 1$, and a bicyclic vector $\xi_0 \in H_0$ for N such that

$$\psi = \omega_{\xi_0} \circ \pi$$

Then

$$\varphi = \theta \circ \pi$$

where θ is a normal positive form on $N \subset B(H_0)$ uniquely determined by this equality.

From Lemma 2° we infer that there exists a positive self-adjoint operator B in H_0, affiliated to N such that

$$\theta = \omega_{B\xi_0}$$

Let $A = \pi^{-1}(B)$ be the positive self-adjoint operator in H, affiliated to M, obtained by transporting the operator B by the $*$-isomorphism π^{-1} (LECTURES, 9.25). If we denote

$$e_n = \chi_{[0,n]}(A) \quad , \quad f_n = \chi_{[0,n]}(B),$$

then we have:

$$\psi(A^2 e_n) = \psi(\pi^{-1}(B^2 f_n)) = \omega_{\xi_0}(B^2 f_n) = \| f_n B \xi_0 \|^2 \leq \| B \xi_0 \|^2$$

hence A is of summable square with respect to ψ. Then

$$(L_A R_A \psi)(x) = \lim_{n \to \infty} \psi(Ae_n x Ae_n) = \lim_{n \to \infty} \psi(\pi^{-1}(Be_n) x \pi^{-1}(Be_n))$$

$$= \lim_{n \to \infty} \omega_{\xi_0}(Be_n \pi(x) Be_n) = \omega_{B\xi_0}(\pi(x)) = \theta(\pi(x)) = \varphi(x)$$

hence $L_A R_A \psi = \varphi$.

Assume now that $s(\psi)$ is finite, that is that M is finite. Moreover, by considering the $*$-isomorphism π, we may assume $M \subset B(H)$ with a bicyclic vector $\xi_0 \in H$, and $\psi = \omega_{\xi_0}$.

Let A be a positive self-adjoint operator in H, affiliated to M and such that $\xi_0 \in D(A)$ and

$$\varphi = L_A R_A \psi = \omega_{A\xi_0}$$

By Lemma 2°, the vector $A\xi_0$ is uniquely determined by φ. On the other hand, A is an extension of the positive operator $L_{A\xi_0}$. Since M is finite, A is indeed determined uniquely by φ. (see LECTURES, Cor. 9.8).

XI Hyperstandard Forms and the Spatial Derivative.

1° In X 1° we considered a von Neumann algebra $M \subset B(H)$ with a bicyclic vector $\xi_0 \in H$ and for each $\xi \in H$ we defined an operator L^0_ξ in H, affiliated to M:

$$D(L^0_\xi) = M'\xi_0 \,, \quad L^0_\xi x'\xi_0 = x'\xi \,, \quad (x' \in M')$$

The operator L^0_ξ is preclosed if $\xi \in D(S)$ or if L^0_ξ is symmetric and we saw that, with $L_\xi = \overline{L^0_\xi}$,

$$P_S := \{\xi \in H ; L_\xi \text{ is positive}\} = \{\xi \in H ; \omega'_{\xi,\xi_0} \geq 0\}$$
$$= \{A\xi_0 ; A \text{ positive self-adjoint affiliated to } M, \xi_0 \in D(A)\}.$$

Similarly, for each $\eta \in H$ we define an operator R^0_η in H, affiliated to M':

$$D(R^0_\eta) = M\xi_0 \,, \quad R^0_\eta x\xi_0 = x\eta \,, \quad (x \in M)$$

which is preclosed if $\eta \in D(S^*)$ or if R^0_η is symetric, $R_\eta = \overline{R^0_\eta}$ and

$$P_{S^*} := \{\eta \in H ; R_\eta \text{ is positive}\} = \{\eta \in H ; \omega_{\eta,\xi_0} \geq 0\}$$
$$= \{B\xi_0 ; B \text{ positive self-adjoint affiliated to } M', \xi_0 \in D(B)\}$$

2° It is an easy matter to show that (SEE LECTURES p. 260-261)

PROPOSITION: For $\xi \in H$ are equivalent

(i) $\xi \in P_S$

(ii) $\xi \in D_S$, $S\xi = \xi$ and L_ξ is positive

(iii) $\xi \in \overline{M^+\xi_0}$

(iv) $(\xi | \eta) \geq 0 \quad (\forall)\, \eta \in P_{S^*}$

and, similarly, for $\eta \in H$ are equivalent

(j) $\eta \in P_{S^*}$

(jj) $\eta \in D_{S^*}$, $S^*\eta = \eta$ and R_η is positive

(jjj) $\eta \in \overline{M'^+\xi_0}$

(jw) $(\xi | \eta) \geq 0 \quad (\forall)\, \xi \in P_S$

Thus $P_S \subset D_S$; $P_{S^*} \subset D_{S^*}$ are closed convex cones dual to each other. Moreover:

$$D_S = \operatorname{lin} P_S \qquad , \qquad D_{S^*} = \operatorname{lin} P_{S^*}$$

This is also elementary (<u>LECTURES</u>, pg. 262).

3° With the same kind of approximation arguments as used for the Tomita algebra $\mathcal{T}$ associated to the left Hilbert algebra $\mathcal{Q} = M\xi_0 = \mathcal{Q}''$ we see that

for any $\xi \in \mathcal{Q}''$ there is a sequence $\{\xi_n\} \subset \mathcal{T}$ such that

(1) $\xi_n \to \xi$, $S\xi_n \to S\xi$

(2) $L_{\xi_n} \xrightarrow{so} L_\xi$, $(L_{\xi_n})^* \xrightarrow{so} (L_\xi)^*$

(3) $\|L_{\xi_n}\| \leq \|L_\xi\|$, (4) $n \in \mathbb{N}$

and a similar statement for $\eta \in \mathcal{Q}' (= M'\xi_0)$.

From Proposition 2° and the Tomita Theorem we infer that

$$J P_S = P_{S^*}$$

and since $S\xi = \xi$ for any $\xi \in P_S$ we get

$$\Delta^{1/2} P_S = P_{S^*}$$

and consequently

$$\Delta^{1/4} P_S = \Delta^{-1/4} P_{S^*}$$

We are thus led to consider the closed convex cone

(4) $$P = \overline{\Delta^{1/4} P_S} = \overline{\Delta^{-1/4} P_{S^*}}$$

In order to make notation as simple and as expressive as possible we shall write

$$\xi\eta = L_\xi(\eta) = xy\xi_0 \quad \text{for } \xi = x\xi_0,\ \eta = y\xi_0 \in \mathcal{Q}'' = M\xi_0$$

$$\xi\eta = R_\eta(\xi) = x'y'\xi_0 \quad \text{for } \xi = x'\xi_0,\ \eta = y'\xi_0 \in \mathcal{Q}' = M'\xi_0$$

If $\xi, \eta \in \mathcal{Q}'' \cap \mathcal{Q}'$, then $L_\xi(\eta) = R_\eta(\xi)$. We have seen that

$$P_S = \overline{M^+\xi_0} = \overline{\{\xi(S\xi)\,;\, \xi \in \mathcal{Q}''\}} \ , \ P_{S^*} = \overline{M'^+\xi_0} = \overline{\{\eta(S^*\eta)\,;\,\eta \in \mathcal{Q}'\}}$$

Using the approximations (1)-(3) we get

$$P_S = \overline{P_S^\circ} \quad \text{where} \quad P_S^\circ = \{\xi(S\xi)\,;\,\xi \in \mathcal{T}\}$$

$$P_{S^*} = \overline{P_{S^*}^\circ} \quad \text{where} \quad P_{S^*}^\circ = \{\eta(S^*\eta)\,;\,\eta \in \mathcal{T}\}$$

It is easy to see that for any $\xi, \eta \in \mathcal{T}$ and any $\alpha \in \mathbb{C}$:

$$\Delta^{\alpha}(\xi\eta) = (\Delta^{\alpha}\xi)(\Delta^{\alpha}\eta) \ , \ J(\xi\eta) = (J\eta)(J\xi)$$

so that for $\xi \in \mathcal{T}$:

$$\Delta^{1/4}(\xi(S\xi)) = \Delta^{1/4}(\xi)\,\Delta^{1/4}(S\xi) = (\Delta^{1/4}\xi)\,J(\Delta^{1/4}\xi)$$

Since $\Delta^{1/4}\mathcal{T} = \mathcal{T}$ we hence infer that, with the notation

$$P^{o} = \{\xi(J\xi)\,;\, \xi \in \mathcal{T}\}$$

we have

$$\Delta^{1/4}P_S^{o} = P^{o} \quad \text{and, similarly,} \quad \Delta^{-1/4}P_{S^*}^{o} = P^{o}$$

It follows that:

(5) $$P = \overline{P_o} = \overline{\{\xi(J\xi)\,;\,\xi \in \mathcal{T}\}}$$

Indeed, if $\xi \in P_S$, then there is a sequence $\{\xi_n\} \subset P_S^{o}$ with $\xi_n \to \xi$. Since $S\xi_n = \xi_n$, $S\xi = \xi$ and $\Delta^{1/2} = JS$, we get $\Delta^{1/2}\xi_n \to \Delta^{1/2}\xi$ and therefore

$$\|\Delta^{1/4}\xi - \Delta^{1/4}\xi_n\|^2 = (\Delta^{1/2}(\xi-\xi_n)\,|\,(\xi-\xi_n)) \longrightarrow 0$$

whence $\Delta^{1/4}\xi \in \overline{\Delta^{1/4}P_S^{o}} = \overline{P^{o}}$. Consequently $P \subset \overline{P_o}$ by the definition (4) of P. Conversely, it is obvious that

$$P^{o} = \Delta^{1/4}P_S^{o} \subset \Delta^{1/4}P_S \subset P$$

Again by approximation ((1)-(3)) it is now easy to check that

(6) $$P = \overline{\{\xi(J\xi)\,;\,\xi \in \mathcal{A}''\}} = \overline{\{(J\eta)\eta\,;\,\eta \in \mathcal{A}'\}}$$

4°. From 3°(5) it is clear that

(1) $$\xi \in P \Rightarrow J\xi = \xi$$

For any $t \in \mathbb{R}$ and any $\xi \in \mathcal{T}$ we have

$$\Delta^{it}(\xi(J\xi)) = (\Delta^{it}\xi)(\Delta^{it}J\xi) = (\Delta^{it}\xi)\,J(\Delta^{it}\xi),$$

thus $\Delta^{it}P_o \subset P_o$ and, again by 3°(5), we get

(2) $$\Delta^{it}P = P \ , \quad (t \in \mathbb{R})$$ and then

(3) $$\xi \in P,\ f \in L^1(\mathbb{R}),\ f \geq 0 \Rightarrow \int_{-\infty}^{\infty} f(t)\,\Delta^{it}\xi\,dt \in P$$

since P is a closed convex cone.

We show now that

$$x \in M \Rightarrow x(JxJ)P \subset P.$$

Indeed, if $\xi \in \mathfrak{A}'' = M\xi_0$ and $\zeta \in \mathfrak{I}$, then

DO THIS WITH $\xi = x\xi_0$, $\zeta = a\xi_0$

$$L_\xi(JL_\xi J)(\zeta(J\zeta)) = L_\xi J L_\xi (JL_\zeta J)(\zeta) = \text{(commutation)}$$

$$= L_\xi J(JL_\zeta J)L_\xi(\zeta) = L_\xi L_\zeta J L_\xi(\zeta) = (\xi\zeta)J(\xi\zeta)$$

hence $L_\xi(JL_\xi J)P^\circ \subset P$ and therefore $L_\xi(JL_\xi J)P \subset P$.

5° For $\xi \in P_S$ and $\eta \in P_{S^*}$ we have

$$(\Delta^{1/4}\xi \mid \Delta^{-1/4}\eta) = (\xi \mid \eta) \geq 0 \quad \text{by Proposition } 2^\circ,$$

and using the definition of P we see that

$$\xi, \eta \in P \Rightarrow (\xi \mid \eta) \geq 0.$$

On the other hand, let $\zeta \in H$ be such that $(\zeta \mid \theta) \geq 0$, $(\forall)\,\theta \in P$. and denote

$$\zeta_n = \sqrt{n/\pi}\int_{-\infty}^{\infty} e^{-nt^2}\Delta^{it}\zeta\,dt, \quad \theta_n = \sqrt{n/\pi}\int_{-\infty}^{\infty} e^{-nt^2}\Delta^{it}\theta\,dt.$$

If $\theta \in P$, then (by 4° (3)) $\theta_n \in P$ and therefore

$$(\zeta_n \mid \theta) = (\zeta \mid \theta_n) \geq 0 \;, \quad (n \in \mathbb{N}).$$

In particular, if $\xi \in P_S$ then $\Delta^{1/4}\xi \in P$ and

$$(\Delta^{1/4}\zeta_n \mid \xi) = (\zeta_n \mid \Delta^{1/4}\xi) \geq 0 \;, \quad (n \in \mathbb{N})$$

so that $\Delta^{1/4}\zeta_n \in P_{S^*}$ (by the duality P_S, P_{S^*}). Thus $\zeta_n \in \Delta^{-1/4}P_{S^*} \subset P$ and, since $\zeta_n \to \zeta$ and P is closed, $\zeta \in P$.

Consequently P is a self-polar cone: for $\zeta \in H$,

(1) $$\zeta \in P \iff (\zeta \mid \theta) \geq 0 \;, (\forall)\,\theta \in P.$$

If $\zeta \in P \cap (-P)$ then we infer that $(\zeta \mid -\zeta) \geq 0$, hence $\zeta = 0$.

(2) $$P \cap (-P) = \{0\}$$

We shall denote, for $\zeta = J\zeta$ and $\theta = J\theta$

$$\zeta \leq \theta \iff \theta - \zeta \in P$$

6° We now prove that for any $\zeta \in H$, $J\zeta = \zeta$, there exist $\zeta^+, \zeta^- \in P$ uniquely determined such that

(1)
$$\zeta = \zeta^+ - \zeta^- \quad ; \quad \zeta^+ \perp \zeta^-$$

Indeed let $\zeta \in H$, $J\zeta = \zeta$. Since P is a closed convex set there is a unique element $\zeta^+ \in P$ such that
$$\|\zeta^+ - \zeta\| = \inf\{\|\theta - \zeta\| ; \theta \in P\}$$
and we denote $\zeta^- = \zeta - \zeta^+$. For any $\theta \in P$ and any $t \geq 0$:
$$0 \leq \|(\zeta^+ + t\theta) - \zeta\|^2 - \|\zeta^+ - \zeta\|^2 = t^2\|\theta\|^2 + 2t\,\mathrm{Re}(\zeta^- | \theta),$$
whence $\mathrm{Re}(\zeta^- | \theta) \geq 0$. Since $J\zeta^- = \zeta^-$ and $J\theta = \theta$, we have
$$(\zeta^- | \theta) = \mathrm{Re}(\zeta^- | \theta) \geq 0$$
Therefore (by 5°,(1)), $\zeta^- \in P$.

On the other hand for any $t \in (0,1)$ we have
$$0 \leq \|(1-t)\zeta^+ - \zeta\|^2 - \|\zeta^+ - \zeta\|^2 = t^2\|\zeta^+\|^2 - 2t\,\mathrm{Re}(\zeta^+ | \zeta^-)$$
which entails $\mathrm{Re}(\zeta^+ | \zeta^-) \leq 0$ and, again by 5°(1) it follows that
$$(\zeta^+ | \zeta^-) = 0.$$

For the uniqueness, assume $\zeta = \xi - \eta$, $\xi, \eta \in P$, $\xi \perp \eta$. Then $\zeta^+ - \xi = \zeta^- - \eta$ and therefore
$$\|\zeta^+ - \xi\|^2 = (\zeta^+ - \xi | \zeta^- - \eta) = -(\zeta^+ | \eta) - (\xi | \zeta^-) \leq 0$$
hence $\xi = \zeta^+$, $\eta = \zeta^-$.

In particular we proved that $H = \mathrm{lin}\, P$:
$$H = (P - P) + i(P - P).$$

7° Notice that any self-polar convex cone P in a Hilbert space H determines a unique conjugation J of H such that $J\zeta = \zeta$ $(\forall)\, \zeta \in P$.

Indeed, if $\zeta \in H$ and $\zeta \perp P$ then $\zeta \in P$ since P is self-polar and therefore $(\zeta | \zeta) = 0$, i.e. $\zeta = 0$. Thus P is a total subset of H, i.e. $(P-P) + i(P-P)$ is dense in H. Consequently, the conjugation J is uniquely determined by

$$J(\xi+i\eta)=\xi-i\eta\ ,\quad (\xi,\eta\in P-P).$$

The decomposition $6^\circ(1)$ is valid in this more general setting, with the same proof

We shall say that a von Neumann algebra $M\subset B(H)$ is <u>hyperstandard</u> if there exist a conjugation $J: H\to H$ and a self-polar convex cone $P\subset H$ such that

1°. $M\ni x\longmapsto Jx^*J\in M'$ is a $*$-antiisomorphism acting identically on $Z(M)$

2° $\xi\in P\Rightarrow J\xi=\xi$

3° $x\in M\Rightarrow[x(JxJ)]P\subset P$

From the above we know that any von Neumann algebra is $*$-isomorphic to a hyperstandard von Neumann algebra.

Let $M\subset B(H)$ be a hyperstandard von Neumann algebra as above with conjugation $J:H\to H$ and self-polar convex cone $P\subset H$.

Let e be a projection in M. Then JeJ is a projection in M' with equal central supports: $z(JeJ)=z(e)$. Of course

$$e\neq 0\Rightarrow e(JeJ)\neq 0$$

Let $q=e(JeJ)=(JeJ)e\in B(H)$. Then $M_q\subset B(qH)$ is a von Neumann algebra with commutant M'_q and center Z_q, where $Z=Z(M)$ is the center of M. The mapping

$$x_e\longmapsto x_q$$

is a $*$-isomorphism of the reduced von Neumann algebra M_e onto the von Neumann algebra M_q. Since $Jq=qJ$ and $qP=[e(JeJ)]P\subset P$, it is easy to see that

$M_q\subset B(qH)$ is hyperstandard with conjugation qJq and self-polar convex cone qP.

This remark will help to reduce some problems to the case of hyperstandard von Neumann algebras of countable type.

LEMMA 1. For any projection $e \in M$ there is a family $\{\xi_n\}_{n\in I} \subset P$ such that $p_{\xi_n}^{\xi}\neq 0$ are mutually orthogonal and

$$e = \sum_{n\in I} p_{\xi_n}^{\xi}$$

If e is of countable type, then there is $\xi_0 \in P$ such that

$$e = p_{\xi_0}^{\xi}$$

In particular, if M is of countable type, then M has a bicyclic vector $\xi_0 \in P$.

PROOF. If $e \neq 0$, then $e(JeJ) \neq 0$ and, since $H = \overline{\text{lin}}\, P$, there is a non zero vector $\xi \in [e(JeJ)]P \subset P$. Clearly $p_{\xi}^{\xi} \leq e$, so that the first assertion follows by a maximality argument.

If e is of countable type, then the set I is at most countable and we may assume $\|\xi_n\| = 1$. Then

$$\xi_0 = \sum_n 2^{-n}\xi_n \in P \text{ and } e = p_{\xi_0}^{\xi}$$

[Notice that $p_{\xi}^{\prime} = J p_{J\xi} J, \ldots$]

In the next Lemma we suppose that the given hyperstandard von Neumann algebra $M \subset B(H)$ with the given J and P has a bicyclic vector $\xi_0 \in H$ and consequently a canonical conjugation J_0 and a self-polar convex cone

$$P_0 = \overline{\{[x(J_0 x J_0)]\xi_0\, ;\ x \in M\}}$$

which makes it again hyperstandard. Then

LEMMA 2. $J_0 = J$ and $P_0 = P$.

PROOF. Let $x \in M$. Then $JxJ \in M'$ and, consequently,

$$[JS_0^*J]x\xi_0 = JS_0^*(JxJ)\xi_0 = J(JxJ)^*\xi_0 = x^*\xi_0$$

This entails $S_0 \subset JS_0^*J$ and, similarly $S_0^* \subset JS_0J$. Thus $JS_0 = S_0^*J$ which entails

$$(JS_0)^* = S_0^*J = JS_0$$

hence the linear operator JS_0 is self-adjoint. For any $x \in M$:

$$((JS_0)(x\xi_0)\,|\,x\xi_0) = (Jx^*\xi_0\,|\,x\xi_0) = (xJx\xi_0\,|\,\xi_0) =$$
$$= ([x(JxJ)]\xi_0\,|\,\xi_0) \geq 0 \text{ since both sides are in } P$$

Since $JS_0 = \overline{JS_0|M\xi_0}$ it follows that JS_0 is a positive operator. Since $S_0 = J(JS_0)$ we get $J = J_0$ by the uniqueness of the polar decomposition $S_0 = J_0\Delta_0^{1/2}$.

Then

$$P_0 = \overline{\{[x(J_0 x J_0)]\xi_0 ;\ x \in M\}} = \overline{\{[x(JxJ)]\xi_0 ;\ x \in M\}} \subset P$$

hence $P = P_0$ since both cones are self-polar.

Under the same assumptions we have

LEMMA 3 The mapping $\Phi : a \longmapsto \Delta^{1/4} a \xi_0$ is an order isomorphism between the set of all self-adjoint $a \in M$ onto the set of all vectors $\xi \in H$ such that

$$J\xi = \xi \ \text{ and } \ (\exists)\ \lambda > 0 \text{ such that } -\lambda\xi_0 \leq \xi \leq \lambda \xi_0$$

PROOF. Since $\Delta^{1/4}$ is injective and ξ_0 is M-separating, the map Φ is injective. For $a = a^* \in M$ we have

$$a \geq 0 \iff a\xi_0 \in P_S \iff \Delta^{1/4} a \xi_0 \in P \qquad (\text{cf. } 2^\circ)$$

hence Φ is an order isomorphism. Let show that Φ is surjective.

For $a \in M$, $a = a^*$, we have $(1+\Delta^{1/2})(a\xi_0) = a\xi_0 + Ja\xi_0$ whence

$$\Delta^{1/4}(a\xi_0) = (\Delta^{1/4} + \Delta^{-1/4})^{-1}(a\xi_0 + Ja\xi_0)$$

and, as the operator $(\Delta^{1/4} + \Delta^{-1/4})^{-1}$ is bounded,

$$a_i \xrightarrow{w} a \implies \Phi(a_i) \to \Phi(a) \text{ weakly.}$$

Since the set $\{a \in M ;\ 0 \leq a \leq 1\}$ is w-compact, it follows that $\Phi(\{a \in M ;\ 0 \leq a \leq 1\})$ is weakly compact in H.

Let $\xi \in H$, $J\xi = \xi$, $0 \leq \xi \leq \xi_0$. Then (see $4^\circ(3)$)

$\Delta^\alpha \xi_0 = \xi_0$

$$0 \leq \xi_n = \sqrt{n/\pi} \int_{-\infty}^{\infty} e^{-nt^2} \Delta^{it} \xi \, dt \leq \sqrt{n/\pi} \int_{-\infty}^{\infty} e^{-nt^2} \Delta^{it} \xi_0 \, dt = \xi_0$$

For any $\eta \in P_{S^*}$ we have $\Delta^{-1/4}\eta \in P$, hence

$$(\Delta^{-1/4}\xi_n \mid \eta) = (\xi_n \mid \Delta^{-1/4}\eta) \geq 0$$

Thus $\Delta^{-1/4}\xi_n \in P_S$ by the duality (P_S, P_{S^*}). Similarly

$$\xi_0 - \Delta^{-1/4}\xi_n = \Delta^{-1/4}(\xi_0 - \xi_n) \in P_S .$$

It follows that the operators $L^0_{\Delta^{-1/4}\xi_n}$ and $1 - L^0_{\Delta^{-1/4}\xi_n} = L^0_{\xi_0 - \Delta^{-1/4}\xi_n}$ are positive (see 2°).

Consequently the operator $a_n = L_{\Delta^{-1/4}\xi_n}$ is bounded and $0 \le a_n \le 1$. It is obvious that $\xi_n = \Phi(a_n)$. Since $\Phi(a_n) = \xi_n \to \xi$ and since the set

$$\Phi(\{a \in M;\, 0 \le a \le 1\}) \text{ is closed}$$

it follows that there exists an $a \in M$, $0 \le a \le 1$, such that

$$\xi = \Phi(a)$$

8° Let $M \subset B(H)$ be a hyperstandard von Neumann algebra with conjugation J and self-polar cone P.

<u>PROPOSITION</u>. For any $\xi, \eta \in P$ we have

$$\|\xi - \eta\|^2 \le \|\omega_\xi - \omega_\eta\| \le \|\xi - \eta\|\,\|\xi + \eta\|$$

<u>PROOF</u>. The second inequality follows from:

$$(\omega_\xi - \omega_\eta)(x) = \frac{1}{2}\left[(x(\xi+\eta)|\xi-\eta) + (x(\xi-\eta)|\xi+\eta)\right];\ x \in M,$$

(relation valid for any $\xi, \eta \in H$).

Let $e = s(\omega_\xi) \vee s(\omega_\eta)$ and $q = e(JeJ)$. Then e is of countable type and $\xi, \eta \in qP$. As we have seen at the beginning of section 7°, M_q is hyperstandard and $*$-isomorphic to M_e. So, we may assume that M is of countable type.

We first consider the case when the vector $\xi + \eta$ is separating. Since $J(\xi+\eta) = \xi+\eta$, the vector $\xi+\eta$ is also cyclic. Let Δ be the modular operator associated to $\xi+\eta$. By Lemma 2/7° P is the self-polar cone associated to $\xi+\eta$. Since

$$-(\xi+\eta) \le \xi - \eta \le \xi + \eta$$

from Lemma 3/7° we infer that there exists an $a \in M$, $-1 \le a \le 1$, such that

$$\xi - \eta = \Delta^{1/4} a(\xi+\eta)$$

Then:

$$\|\omega_\xi - \omega_\eta\| \ge (\omega_\xi - \omega_\eta)(a) = \mathrm{Re}\,(a(\xi+\eta)|(\xi-\eta)) =$$
$$= (\Delta^{-1/4}(\xi-\eta)|\xi-\eta)$$

Since $J\Delta^{1/4} = \Delta^{-1/4}J$, it follows that

$$(\Delta^{-1/4}(\xi-\eta)|(\xi-\eta)) = (\Delta^{1/4}(\xi-\eta)|\xi-\eta)$$

hence

$$\|\omega_\xi - \omega_\eta\| \geq \left(\tfrac{1}{2}(\Delta^{1/4}+\Delta^{-1/4})(\xi-\eta) \mid \xi-\eta\right) \geq \|\xi-\eta\|^2$$

If the vector $\xi+\eta$ is not separating, then $1-P_{\xi+\eta} \neq 0$. By Lemma 1/7° there is $\zeta \in P$ such that

$$P_\zeta = 1 - P_{\xi+\eta}.$$

Consider the vectors:

$$\xi_n = \xi + \tfrac{1}{n}\zeta \in P \quad , \quad \eta_n = \eta + \tfrac{1}{n}\zeta \in P.$$

It follows that

$$P'_{\xi_n+\eta_n} = P'_{(\xi+\eta+\frac{2}{n}\zeta)} = 1$$

since $P_{\xi+\eta}$ and $P_{\frac{2}{n}\zeta} = P_\zeta$ are orthogonal. Hence

$$P_{\xi_n+\eta_n} = 1.$$

By the case considered above we have

$$\|\omega_{\xi_n} - \omega_{\eta_n}\| \geq \|\xi_n - \eta_n\|^2 \quad , \ (n \in \mathbb{N})$$

and, with $n \to \infty$,

$$\|\omega_\xi - \omega_\eta\| \geq \|\xi - \eta\|^2.$$

<u>COROLLARY 1</u> For $\xi, \eta \in P$ we have:

(i) $\xi \perp \eta \iff P_\xi \perp P_\eta$

(ii) if $\zeta \in P$ and ($\zeta \perp \eta \Rightarrow \zeta \perp \xi$), then $P_\xi \leq P_\eta$

(iii) $\xi \leq \eta \Rightarrow P_\xi \leq P_\eta$

<u>PROOF</u>. (i) If $\xi \perp \eta$, then

$$\|\omega_\xi - \omega_\eta\| \geq \|\xi-\eta\|^2 = \|\xi\|^2 + \|\eta\|^2 = \|\omega_\xi\| + \|\omega_\eta\|;$$

It follows that (LECTURES E.5.15)

$$P_\xi = s(\omega_\xi) \perp s(\omega_\eta) = P_\eta$$

The converse is obvious.

(ii) follows easily from (i).
(iii) If $0 \le \xi \le \eta$, then, for any $\zeta \in P$, we have

$$0 \le (\xi|\zeta) \le (\eta|\zeta)$$

and (iii) follows from (ii)

<u>COROLLARY 2</u> The mapping $\xi \longmapsto \omega_\xi$ is a homeomorphism of P onto a closed subset of $M_*^+ := \{\varphi \in M_*; \varphi \ge 0\}$ with respect to the norm topologies.

<u>PROOF</u> By the PROPOSITION it is clear that the mapping $\xi \longmapsto \omega_\xi$ is a homeomorphism of P onto $\{\omega_\xi; \xi \in P\} \subset M_*^+$. If $\{\omega_{\xi_n}\}$ with $\xi_n \in P$ is a Cauchy sequence, then the same PROPOSITION shows that $\{\xi_n\}$ is a Cauchy sequence in P, hence there is $\xi \in P$ such that $\omega_{\xi_n} \to \omega_\xi$.

9°. <u>LEMMA 1</u> Let $\xi_0 \in P$. For any normal positive form φ on M with $\varphi \le \omega_{\xi_0}$, there is $\eta \in P$ such that

$$\varphi = \omega_{\xi_0,\eta} + \omega_{\eta,\xi_0} \quad \text{and} \quad \eta \le \frac{1}{2}\xi_0$$

<u>PROOF</u>. Let $e = s(\omega_{\xi_0})$ and $q = e(JeJ)$. Then $\xi_0 \in qP$ is a bicyclic vector for $M_q \subset B(qH)$ which is $*$-isomorphic to M_e. Consequently, we may assume that ξ_0 is bicyclic for M. In this case we denote by S and Δ the operators corresponding to ξ_0 and then also J and P are associated to ξ_0.

By the simplest Radon-Nikodym theorem (LECTURES, 5.19), there exist an element $a' \in M'$, $0 \le a' \le 1$, such that

$$\varphi(x) = (x\xi_0 | a'\xi_0), \quad (x \in M).$$

Then $a'\xi_0 \in P_{S^*}$, $(1-a')\xi_0 \in P_{S^*}$, thus

$$\zeta := \Delta^{-1/4}(a'\xi_0) \in P, \quad \xi_0 - \zeta = \Delta^{-1/4}((1-a')\xi_0) \in P.$$

Define:

$$\eta := (1+\Delta^{1/2})^{-1} a'\xi_0 = (1+\Delta^{1/2})^{-1}\Delta^{1/4}\zeta$$

If we denote by f the function $t \longmapsto \dfrac{2}{e^{2\pi t} + e^{-2\pi t}}$, then by applying the Corollary in 6° for $A = \Delta^{-1/2}$ we infer that see pg. 18

$$\eta = \int_{-\infty}^{\infty} f(t)\,\Delta^{it}\zeta$$

Since $0 \leq \zeta \leq \xi_0$, by $\overline{XI}.4°3$ we get

$$0 \leq \int_{-\infty}^{\infty} f(t)\Delta^{it}\zeta\, dt \leq \int_{-\infty}^{\infty} f(t)\Delta^{it}\xi_0\, dt$$

that is $0 \leq \eta \leq \frac{1}{2}\xi_0$.

On the other hand, since $J\eta = \eta$ we have

$$a'\xi_0 = (1+\Delta^{1/2})\eta = \eta + S^*J\eta = \eta + S^*\eta$$

hence, for any $x \in M$:

$$\begin{aligned}\varphi(x) = (x\xi_0 | a'\xi_0) &= (x\xi_0|\eta) + (x\xi_0|S^*\eta)\\ &= (x\xi_0|\eta) + (\eta|Sx\xi_0)\\ &= (x\xi_0|\eta) + (\eta|x^*\xi_0)\\ &= (\omega_{\xi_0,\eta} + \omega_{\eta,\xi_0})(x).\end{aligned}$$

<u>LEMMA 2</u> Let $\xi_0 \in P$. For any normal positive form $\varphi \leq \omega_{\xi_0}$ there is a $\xi \in P$ such that $\varphi = \omega_\xi$

<u>PROOF</u> Apply Lemma 1 to $\varphi_1 := \omega_{\xi_0} - \varphi$ to get $\eta_1 \in P$ such that

$$\varphi_1 = \omega_{\xi_0,\eta_1} + \omega_{\eta_1,\xi_0} \quad \text{and} \quad \eta_1 \leq \xi_0$$

Denote $\xi_1 = \xi_0 - \eta_1 \in P$. Then, as easily checked

$$\omega_{\xi_1} - \varphi = \omega_{\eta_1},$$

$$\|\varphi_1\| = \varphi_1(1) = 2\,\mathrm{Re}(\xi_0|\eta_1) = 2(\xi_0|\eta_1) \geq 0.$$

Since $P \ni \eta_1 \leq \xi_0$, we have

$$(\eta_1|\eta_1) \leq (\xi_0|\eta_1).$$

Consequently

$$\|\omega_{\xi_1} - \varphi\| = \|\omega_{\eta_1}\| = \|\eta_1\|^2 \leq \frac{1}{2}\|\varphi_1\| = \frac{1}{2}\|\omega_{\xi_0} - \varphi\|$$

By induction we get a sequence $\{\xi_n\} \subset P$ such that

$$\|\omega_{\xi_{n+1}} - \varphi\| \leq \frac{1}{2}\|\omega_{\xi_n} - \varphi\|.$$

Then $\|\omega_{\xi_n} - \varphi\| \to 0$ and, by COROLLARY 2 in 8°, it follows that there exists $\xi \in P$ such that $\varphi = \omega_\xi$.

10°. THEOREM. Let $M \subset B(H)$ be a hyperstandard von Neumann algebra with conjugation J and self-polar cone P. The mapping

$$\xi \longmapsto \omega_\xi$$

is a homeomorphism of P onto M_*^+ wrt norm-topologies.

PROOF. Due to the last Lemma we have to prove only that the set $\{\omega_\xi ; \xi \in P\}$ is norm-dense in M_*^+.

We may assume that M is of countable type. Then, by Lemma 1 from 7° (pg. 85), there exists a bicyclic vector $\xi_0 \in P$. By Lemma 2 from 9°, the set $\{\omega_\xi ; \xi \in P\}$ contains the set

$$\{\varphi \in M_*^+ ; (\exists) \lambda > 0 ,\ \varphi \leq \lambda \omega_{\xi_0}\}$$

so it is enough to show that this last set is dense in M_*^+.

Let $\varphi \in M_*^+$. Then (... by the Spatial Theorem of von Neumann) there exists

$$\eta \in H = \overline{M'\xi_0} \quad \text{such that} \quad \varphi = \omega_\eta .$$

Consequently, there exists a sequence $\{x_n'\} \subset M'$, $x_n'\xi_0 \to \eta$. Then

$$\omega_{x_n'\xi_0} \longrightarrow \omega_\eta = \varphi \quad \text{and} \quad \omega_{x_n'\xi_0} \leq \|x'\|^2 \omega_{\xi_0} .$$

This completes the proof of the Theorem

11° For $\varphi \in M_*^+$ we denote by $\varphi^{1/2} \in P$ the unique vector $\xi \in P$ such that $\varphi = \omega_\xi$.

If $\varphi, \psi \in M_*^+$ and $\varphi \leq \psi$, then $\omega_{\varphi^{1/2}} \leq \omega_{\psi^{1/2}}$. Since $J\varphi^{1/2} = \varphi^{1/2}$, $J\psi^{1/2} = \psi^{1/2}$, we infer that also

$$\omega'_{\varphi^{1/2}} \leq \omega'_{\psi^{1/2}}$$

and then we see that there is a uniquely determined operator $x \in M$ such that

$$\varphi^{1/2} = x \psi^{1/2} \quad \text{and} \quad s(|x|) \leq s(\psi)$$

namely

$$x(x'\psi^{1/2}) = x'\varphi^{1/2}, \ (x' \in M') ; \quad x\eta = 0 \ \text{for} \ \eta \in \overline{M'\psi^{1/2}}^{\perp}$$

In particular:

$$\varphi = \psi(x^* \cdot x) , \quad s(|x|) \leq s(\psi)$$

This last assertion is of Radon-Nikodym type, similar to the Sakai RN type theorem, but weaker (in one respect) since <u>this</u> x is not necessarily positive.

The uniquely determined $x \in M$ as above is denoted

$$\frac{d\varphi}{d\psi}$$

and is called the <u>spatial</u> <u>derivative</u> of φ wrt ψ.

If θ, φ, ψ are normal positive forms and

$$\theta \leq \varphi \leq \psi$$

then it is easily proved that the following „chain rule" holds:

$$\frac{d\theta}{d\varphi} \cdot \frac{d\varphi}{d\psi} = \frac{d\theta}{d\psi}.$$

COROLLARY. Let $M_k \subset B(H_k)$ be hyperstandard von Neumann algebras with conjugation J_k and self-polar cone P_k. If

$$\Pi : M_1 \longrightarrow M_2$$

is a $*$-isomorphism, then there exists a unitary operator

$$u : H_1 \longrightarrow H_2$$

uniquely determined such that

(1) $\Pi(x_1) = u \circ x_1 \circ u^*$ $(\forall)$ $x_1 \in M_1$

(2) $J_2 = u \circ J \circ u^*$

(3) $P_2 = u(P_1)$

<u>PROOF</u>. <u>UNIQUENESS</u>: From (1)-(3) it follows that for any $\xi_1 \in P_1$ we have $\omega_{u\xi_1} = \omega_{\xi_1} \circ \Pi^{-1}$ and $u\xi_1 \in P_2$. By the previous Theorem, $u\xi_1$ is uniquely determined with this property. Since $H_1 = \operatorname{lin} P_1$, this shows the uniqueness of u.

a) <u>EXISTENCE</u> Assume first that M_1 is of countable type. Then M_1 has a bicyclic vector $\xi_1 \in P_1$ by Lemma 1/7° (pg. 85). By the previous Theorem, there exists a vector $\xi_2 \in P_2$ such that

$$\omega_{\xi_2} = \omega_{\xi_1} \circ \Pi^{-1}.$$

It follows that ξ_2 is separating for M_2 and, since $J\xi_2 = \xi_2$, is is also cyclic for M_2. The relations

$$u(x_1\xi_1) = \pi(x_1)\xi_2 \quad ; \quad (x_1 \in M_1)$$

determine a unitary operator $u: H_1 \to H_2$ satisfying (1).

It is also easy to prove that (S_k asso to $M_k, \xi_k ; k=1,2$)

$$S_2 = u \circ S_1 \circ u^* ;$$

it follows that u satisfies also (2). Finally

$$P_2 = \{[x_2(J_2 x_2 J_2)]\xi_2 ; x_2 \in M_2\}^- =$$

$$= \{[(ux_1u^*)(uJ_1u^*)(ux_1u^*)(uJ_1u^*)]\xi_2 ; x_1 \in M_1\}^-$$

$$= u(\{[x_1(J_1x_1J_1)]\xi_1 ; x_1 \in M_1\}^-) = u(P_1)$$

b) In the general case there exists an increasingly directed family $\{e_{1,i}\}_{i\in I}$ consisting of projections of countable type with supremum equal to 1. For each $i \in I$ denote

$$e_{2,i} = \pi(e_{1,i}), \quad q_{1,i} = e_{1,i}(J_1 e_{1,i} J_1), \quad q_{2,i} = e_{2,i}(J_2 e_{2,i} J_2).$$

Then (by the remark on pg. 84) for any $i \in I$ there exists a uniquely determined $*$-isomorphism

$$\pi_i : (M_1)_{q_{1,i}} \to (M_2)_{q_{2,i}}$$

such that $\pi_i((x_1)_{q_{1,i}}) = (\pi(x_1))_{q_{2,i}}$

Using a), for any $i \in I$ there is a unitary operator

$$u_i : q_{1,i}(H_1) \to q_{2,i}(H_2)$$

uniquely determined by properties similar to (1), (2), (3).

It follows that $i \leq k \Rightarrow u_i \subset u_k$ so that glueing together the operators u_i we get the desired operator u.

The above Corollary implies that if $M \subset B(H)$ is a hyperstandard von Neumann algebra, then there exists a left Hilbert algebra $\mathcal{A} \subset H$ such that J and P are associated to $\mathcal{A}$.

Notice that there are standard von Neumann algebras which are NOT hyperstandard.

This presentation (the same as in LECTURES) follows an article by HAAGERUP (Math. Scand. 37 (1975), 271-283) —

The implementation given by the last Corollary is called the canonical implementation. In particular it gives an injective

$$\mathrm{Aut}(M) \ni \sigma \longmapsto u_\sigma \in U(H)$$

We thus get a topology on $\mathrm{Aut}(M)$ by transporting the so (= wo) topology from $U(H)$. This is the same as the topology of uniform convergence on the predual M_* which is defined by the seminorms

$$\mathrm{Aut}(M) \ni \sigma \longmapsto \|\varphi \circ \sigma\| \quad ; \quad \varphi \in M_*,$$

as we will see now.

<u>THEOREM</u>. The canonical implementation $\sigma \mapsto u_\sigma$ of $\mathrm{Aut}(M)$ is an isomorphism of topological groups between $\mathrm{Aut}(M)$ with the topology of uniform convergence on M_* and a closed subgroup of $U(M)$ with the so(=wo)-topology.

<u>PROOF.</u> Let $\sigma \in \mathrm{Aut}(M)$ and $\{\sigma_i\}_{i\in I} \subset \mathrm{Aut}(M)$ a net. Also let $u = u_\sigma \in \mathcal{U}(H)$ and $u_i = u_{\sigma_i} \in \mathcal{U}(H)$. We have

$$\begin{aligned}
&\sigma_i \to \sigma \text{ in } \mathrm{Aut}(M) \text{ uniformly on } M_* \iff \\
&\iff \|\varphi \circ \sigma_i - \varphi \circ \sigma\| \to 0, \ (\forall)\, \varphi \in M_*^+ \\
&\iff \|\omega_\xi \circ \sigma_i - \omega_\xi \circ \sigma\| \to 0, \ (\forall)\, \xi \in P \\
&\iff \|\omega_{u_i^*\xi} - \omega_{u^*\xi}\| \to 0, \ (\forall)\, \xi \in P \\
&\iff \|u_i^*\xi - u^*\xi\| \to 0, \ (\forall)\, \xi \in P \\
&\iff \|u_i^*\xi - u^*\xi\| \to 0, \ (\forall)\, \xi \in H = \mathrm{lin}(P) \\
&\iff u_i \xrightarrow{\text{wo}} u .
\end{aligned}$$

This topology on $\mathrm{Aut}(M)$ will be called the <u>natural topology</u> on $\mathrm{Aut}(M)$

If $\sigma : G \to \mathrm{Aut}(M)$ is a continuous action, then there is an so-continuous one parameter group of unitaries

$$G \ni g \longmapsto u(g) \in \mathcal{U}(H)$$

such that $\sigma_g = \mathrm{Ad}(u(g))$, $(\forall)\, g \in G$ –

Consider a von Neumann algebra $M \subset B(H)$ with a bicyclic vector $\xi_0 \in H$ and associated S, J, Δ. We have the cones

$$P_S = \overline{\{x\xi_0 ; x \in M^+\}} \quad , \quad P_{S^*} = \overline{\{x'\xi_0 ; x' \in M'^+\}}$$

and the self-polar cone

$$P = \Delta^{1/4} P_S = \Delta^{-1/4} P_{S^*} = \overline{\{xJxJ\xi_0 ; x \in M\}}$$

We show that

$$P = \overline{\{xJxJ\xi_0 ; x \in M^+\}}$$

PROOF. It is sufficient to show that for every $x \in M^+$ there exists a sequence $\{x_n\} \subset M^+$ such that

$$x_n J x_n J \xi_0 \longrightarrow \Delta^{1/4} x \xi_0$$

We may assume that x is invertible. In this case, both ξ_0 and $x^{1/2}\xi_0$ are bicyclic for $M' \subset B(H)$. By the uniqueness of the hyperstandard form, there exists a unitary $u \in (M')' = M$ such that $u x^{1/2}\xi_0 \in P$

Thus, $y = ux^{1/2} \in M$ and $y\xi_0 \in P$. With

$$y_n = \sqrt{n/\pi} \int_{-\infty}^{\infty} e^{-nt^2} \sigma_t^{\varphi}(y)\, dt \quad (\text{where } \varphi = \omega_{\xi_0})$$

we obtain that y_n are analytic elements wrt σ^φ and

$$y_n\xi_0 = \sqrt{n/\pi} \int_{-\infty}^{\infty} e^{-nt^2} \Delta^{it} y\xi_0 \in P \qquad (\underline{\overline{\text{XI}}}.\,4^\circ (3)\, p.81)$$

Let $x_n = \sigma^{\varphi}_{i/4}(y_n) = \overline{\Delta^{-1/4} y_n \Delta^{1/4}} \in M$, $(n \in \mathbb{N})$. Then

$\Delta^{1/4} x_n \xi_0 = y_n \xi_0 \in P$, i.e. $x_n\xi_0 \in \Delta^{-1/4} P \subset P_S$ and hence $x_n \geqslant 0$ $(\underline{\overline{\text{XI}}}.\,2^\circ)$

In particular,

$$x_n = x_n^* = \sigma^{\varphi}_{i/4}(y_n)^* = \sigma^{\varphi}_{-i/4}(y_n^*)$$

and

$$x_n J x_n J \xi_0 = \Delta^{1/4} y_n^* y_n \xi_0 \longrightarrow \Delta^{1/4} x \xi_0 .$$

Indeed

$$x_n J x_n J \xi_0 = \Delta^{1/4} y_n^* \Delta^{-1/4} J \Delta^{1/4} y_n^* \Delta^{-1/4} J \xi_0 =$$

$$= \Delta^{1/4} y_n^* J \Delta^{1/4} \Delta^{1/4} y_n^* \xi_0 = \Delta^{1/4} y_n^* J \Delta^{1/2} y_n^* \xi_0 =$$

$$= \Delta^{1/4} y_n^* y_n \xi_0 \longrightarrow \Delta^{1/4} y^* y \xi_0 = \Delta^{1/4} x \xi_0 .$$