III The Commutation Theorem for Tensor Products.

1°. Assume $M \subset B(H)$ is a von Neumann algebra with a bicyclic vector $\xi_0 \in H$. The following result is specific for this case.

We saw that $S \subset F^*$. Here we prove that

$$F^* = S,$$

and then we will get an important application. It is even a priori clear that this equality should be important, because it express a strong connection between M and M' in standard form.

So, what have we to prove? We have to show that $F^* \subset S$, i.e. that every $\xi \in D(F^*)$ belongs to $D(S)$ and $S\xi = F^*\xi$. Since $S = \overline{S|M\xi_0}$ we have to show that there exists a sequence $x_n \in M$ such that

(1) $$x_n\xi_0 \longrightarrow \xi \;,\; x_n^*\xi_0 = Sx_n\xi_0 \longrightarrow F^*\xi =: \eta$$

a) We can define two linear operators X_0, Y_0 in H as follows:

$$X_0 : M'\xi_0 \ni x'\xi_0 \longrightarrow x'\xi \in H \quad ; \quad Y_0 : M'\xi_0 \ni x'\xi_0 \longmapsto x'\eta \in H$$

We check that $X_0 \subset Y_0^*$, that is

$$(X_0\alpha \,|\, \beta) = (\alpha \,|\, Y_0\beta) \quad (\forall)\, \alpha \in D(X_0), \beta \in D(Y_0)$$

i.e.:

$$(X_0 a'\xi_0 \,|\, b'\xi_0) = (a'\xi_0 \,|\, Y_0 b'\xi_0), \; (\forall)\, a', b' \in M'$$

and indeed, since $\eta = F^*\xi$ we have:

$$(X_0 a'\xi_0 | b'\xi_0) = (a'\xi | b'\xi_0) = (\xi | a'^* b'\xi_0) = (\xi | F(b'^* a'\xi_0)) =$$
$$= (b'^* a'\xi_0 | F^*\xi) = (a'\xi_0 | b'\eta) = (a'\xi_0 | Y_0 b'\xi_0).$$

Thus X_0 is preclosed and we may consider its closure

$$X = \overline{X_0}.$$

On the other hand, X_0 commutes with each $x' \in M'$:

$$\alpha \in D(X_0) \Rightarrow x'\alpha \in D(X_0) \text{ and } X_0 x'\alpha = x' X_0\alpha$$

since for every $a' \in M'$ we have $x'a'\xi_0 \in M'\xi_0$ and

$$X_0 x'a'\xi_0 = x'a'\xi = x' X_0 a'\xi_0 .$$

It follows that also the closure X of X_0 commutes with each $x' \in M'$, that is, X is affiliated to $(M')' = M$. Clearly,

(2) $$X\xi_0 = \xi.$$

Also $\xi_0 \in D(Y_0)$ and $Y_0 \subset X_0^* = X^*$, so that

(3) $$X^* \xi_0 = \eta$$

Thus (1) is „almost proved", the only „bad" thing being that X can be unbounded.

b) Consider the polar decomposition of the closed linear operator X:

$$X = uA \;;\; A = (X^*X)^{1/2},\; u^*u = s(A) = \text{Range } X^* \ni \eta$$

$$X = Bu \;;\; B = (XX^*)^{1/2},\; uu^* = s(B) = \text{Range } X \ni \xi$$

Of course, $B = uAu^*$, hence, for any Borel function $\varphi : [0,\infty) \to \mathbb{C}$,

$$u\,\varphi(A) = \varphi(B)\,u .$$

Since X is affiliated to M, we have $u \in M$ and $\varphi(A), \varphi(B) \in M$ whenever φ is bounded. Note also that $X^* = u^*B$.

Let $e_n = \chi_{(\frac{1}{n}, n)}(A)$, $f_n = \chi_{(\frac{1}{n}, n)}(B)$ and $\varphi_n(t) = t\chi_{(\frac{1}{n}, n)}(t)$, $(t \geq 0)$. Since $\xi_0 \in D(X)$, $X\xi_0 = \xi$ and $X = uA$, we have:

$$u\,\varphi_n(A)\xi_0 = uAe_n\xi_0 \longrightarrow uA\xi_0 = X\xi_0 = \xi .$$

Since $\xi_0 \in D(X^*)$, $X^*\xi_0 = \eta$ and $X^* = u^*B$, we have

$$(u\,\varphi_n(A))^*\xi = \varphi_n(A)u^*\xi_0 = u^*\varphi_n(B)\xi_0 = u^*Bf_n\xi_0 \to u^*B\xi_0 = X^*\xi_0 = \eta .$$

Hence (1) is satisfied with $x_n = u\,\varphi_n(A) \in M$.

2° a) Let $M_1 \subset B(H_1)$ (resp. $M_2 \subset B(H_2)$) be a von Neumann algebra with a bicyclic vector $\xi_1 \in H_1$ (resp $\xi_2 \in H_2$).

Consider the tensor product Hilbert space $H = H_1 \bar{\otimes} H_2$, the tensor product von Neumann algebra $M = M_1 \bar{\otimes} M_2 \subset B(H)$ and the vector $\xi_0 = \xi_1 \otimes \xi_2 \in H$.

Then $M' \supset M_1' \bar{\otimes} M_2'$, hence ξ_0 is cyclic and separating for M.

A long-standing conjecture was that

$$M' = M_1' \bar{\otimes} M_2',$$

and this conjecture was first proved by M. Tomita.

For the proof here consider the corresponding operators S_1, S_2, S and F_1, F_2, F. We know that $F_1 = S_1^*$, $F_2 = S_2^*$, $F = S^*$, and it is fairly easy to check that

$$S = S_1 \bar{\otimes} S_2 .$$

By a classical theorem in operator theory we then have $S^* = S_1^* \bar{\otimes} S_2^*$, that is:

$$F = F_1 \bar{\otimes} F_2 .$$

However, it is clear that F is the S-operator $S^{M'}$ associated to $M' \subset B(H)$ and the bicyclic vector $\xi_0 \in H$. Also, $F_1 \bar{\otimes} F_2$ is the S-operator $S^{M_1' \bar{\otimes} M_2'}$ associated to $M_1' \bar{\otimes} M_2' \subset B(H)$ and $\xi_0 \in H$, by the same easy argument as above for $S = S_1 \bar{\otimes} S_2$

b) Thus we are led to the situation described in the following

LEMMA. Let $N \subset M \subset B(H)$ be von Neumann algebras having a common bycicl̈ic vector $\xi_0 \in H$.

If $S^N = S^M$, then $N = M$.

PROOF. Let $x \in M$. Then $\xi = x\xi_0 \in D(S^M) = D(S^N)$. It follows (see 1°, a)) that there is a closed linear operator X, affiliated with N, such that

$$X y'\xi_0 = y'\xi = y' x \xi_0 \quad (\forall)\ y' \in N'.$$

But for $x' \in M' \subset N'$ we have

$$X x'\xi_0 = x' x \xi_0 = x x' \xi_0$$

Hence $X = x$ and $x \in N$. Indeed, let $\eta \in H$ and $x_n' \in M'$ a sequence such that $x_n'\xi_0 \to \eta$. Then

$$X x_n'\xi_0 = x_n' x \xi_0 = x x_n'\xi_0 \longrightarrow x\eta$$

hence $\eta \in D(X)$ and $X\eta = x\eta$, since X is closed.

Thus

$$(M_1 \bar{\otimes} M_2)' = M_1' \bar{\otimes} M_2'$$

in the case of existence of bicyclic vectors.

3°. We can now treat the general case (no bycicl̈ic vectors). If we show that

$$(M_1 \bar{\otimes} M_2)' \text{ and } (M_1' \bar{\otimes} M_2')' \text{ commute},$$

then it follows that $(M_1 \bar{\otimes} M_2)' \subset (M_1' \bar{\otimes} M_2')'' = M_1' \bar{\otimes} M_2'$ by the von Neumann bicommutant theorem, which is the desired conclusion since the other inclusion is obvious.

To this end consider:

$T' \in (M_1\bar{\otimes} M_2)'$, $T \in (M_1'\bar{\otimes} M_2')'$ and $\xi\in H_1$, $\eta\in H_2$.

It is sufficient to show that

(2) $$(TT'(\xi\otimes\eta)\,|\,\xi\otimes\eta) = (T'T(\xi\otimes\eta)\,|\,\xi\otimes\eta)$$

Consider the cyclic projections:

$$e = \overline{M_1'\xi}\in M_1,\quad e' = \overline{M_1\xi},\quad f = \overline{M_2'\eta}\in M_2,\quad f' = \overline{M_2\eta}\in M_2'$$

Since $\xi = e\xi = e'\xi$, $\eta = f\eta = f'\eta$ and $e\bar{\otimes} f \in M_1\bar{\otimes} M_2$, $e'\bar{\otimes} f' \in M_1'\bar{\otimes} M_2'$, while $T'\in(M_1\bar{\otimes} M_2)'$, $T\in (M_1'\bar{\otimes} M_2')'$, we are leed to the following transformations

$$(e\bar{\otimes} f)(e'\bar{\otimes} f')\,T\,T'\,(e\bar{\otimes} f)(e'\bar{\otimes} f') =$$

$$(ee'\bar{\otimes} ff')\,T\,(ee'\bar{\otimes} ff')\cdot(ee'\bar{\otimes} ff')\,T'\,(ee'\bar{\otimes} ff')$$

so that (2) will follow if we show that these two operators commute. They belong respectively to the following algebras

$$(ee'\otimes ff')(M_1'\otimes M_2')'(ee'\otimes ff') = (ee'M_1'e'e\bar{\otimes} ff'M_2'f'f)'$$
$$(ee'\otimes ff')(M_1\otimes M_2)'(ee'\otimes ff') = (ee'M_1e'e\otimes ff'M_2f'f)'$$

These equalities refer to the elementary operations of reduction, induction, commutant and tensor product.

Now $ee'M_1e'e\subset B(ee'H_1)$ and $ff'M_2f'f\subset B(ff'H_2)$ are von Neumann algebras in standard form with bicyclic vectors ξ and η respectively. In this case we know that (1) holds, so we get (2), and hence

$$(M_1\bar{\otimes} M_2)' = M_1'\bar{\otimes} M_2'$$

for any von Neumann algebras $M_1\subset B(H_1)$, $M_2\subset B(H_2)$.

$\underline{\text{IV}}$ Uniqueness of the standard form.

1° PROPOSITION. Let M be a von Neumann algebra, $e, f \in \mathcal{P}(M)$ projections in M. Assume that for the central supports we have $z(f) \leq z(e)$ and that

the reduced algebra $eMe \subset B(eH)$ has a cyclic vector $\xi \in eH$
the reduced algebra $fMf \subset B(fH)$ has a separating vector $\eta \in fH$

Then $f \prec e$.

PROOF. By the Comparison Theorem we may assume that $e \prec f$, hence that $e \leq f$ and finally that $f = 1$, without any loss of generality. Then

$$P_\eta = 1 \quad \text{and} \quad z(e) = 1$$

As ξ is eMe-cyclic, ξ is M'_e-separating and hence ξ is M' separating since the induction $M' \ni x \longmapsto xe \in M'e$ is a $*$-isomorphism because $z(e) = 1$. Thus ξ is M-cyclic:

$$P'_\xi = 1.$$

Finally, by the Spatial Theorem of von Neumann,

$$P'_\eta \leq 1 = P'_\xi \Rightarrow f = 1 = P_\eta \prec P_\xi = e, \text{ q.e.d.}$$

2° PROPOSITION. Let $\pi : M_1 \to M_2$ be a $*$-isomorphism between von Neumann algebras $M_1 \subset B(H_1)$, $M_2 \subset B(H_2)$. There exist:

a von Neumann algebra $M \subset B(H)$
two projections $e_1', e_2' \in M' \subset B(H)$
two spatial isomorphisms $\pi_1 : M_1 \to Me_1'$, $\pi_2 : M_2 \to Me_2'$

such that $z(e_1') = 1 = z(e_2')$ and

$$(\pi_2 \circ \pi \circ \pi_1^{-1})(xe_1') = xe_2', \quad (\forall) x \in M$$

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ Me_1' \arrow[rr, "\hat{\pi}"] & & Me_2' \\ M_1 \arrow[u, "\pi_1"] \arrow[uur, bend left, "\hat{i}_1"] \arrow[rr, "\pi"] & & M_2 \arrow[u, "\pi_2"'] \end{tikzcd}$$

PROOF. Let $H = H_1 \oplus H_2$. Then $\hat{\pi}_1 : M_1 \ni x_1 \longmapsto x_1 \oplus \pi(x_1) \in B(H)$ is an injective w-continuous $*$-morphism and hence

$$M = \hat{\pi}_1(M_1) = \{ x_1 \oplus \pi(x_1) \,;\, x_1 \in M_1 \} \subset B(H)$$

is a von Neumann algebra. Consider the canonical isometries

$$u_1 : H_1 \ni \xi_1 \longmapsto \xi_1 \oplus 0 \in H, \quad u_2 : H_2 \ni \xi_2 \longmapsto 0 \oplus \xi_2 \in H,$$

the projections

$$e_1' = u_1 u_1^* = P_{H_1} \in M', \quad e_2 = u_2 u_2^* = P_{H_2} \in M'$$

and the spatial isomorphisms

$$\pi_1 : M_1 \ni x_1 \mapsto u_1 x_1 u_1^* \in Me_1' \ , \ \pi_2 : M_2 \ni x_2 \mapsto u_2 x_2 u_2^* \in Me_2' .$$

The canonical induction $M \ni x \mapsto xe_1' \in Me_1'$ coincides with the $*$-isomorphism $\pi_1 \circ \hat{\pi}_1^{-1}$, hence $z(e_1') = 1$. Then the map

$$\hat{\pi} : Me_1' \ni xe_1' \longmapsto xe_2' \in Me_2'$$

is correctly defined and one checks immediately that

$$\hat{\pi} = \pi_2 \circ \pi \circ \pi_1^{-1}$$

hence $\hat{\pi}$ is injective and $z(e_2') = 1$. QED.

3° THEOREM. Any $*$-isomorphism $\pi : M_1 \to M_2$ between von Neumann algebras $M_1 \subset B(H_1)$, $M_2 \subset B(H_2)$ with bicyclic vectors $\xi_1 \in H_1$, $\xi_2 \in H_2$ respectively, is spatial.

PROOF. By Proposition 2° we may assume

$$\pi : M_1 = Me_1' \ni xe_1' \longmapsto xe_2' \in Me_2' = M_2$$

where $M \subset B(H)$ is some von Neumann algebra $e_1', e_2' \in \mathcal{P}(M')$, $z(e_1') = 1 = z(e_2')$ and the reduced algebras $e_1' M' e_1'$, $e_2' M' e_2'$ have bicyclic vectors. By PROPOSITION 1° we get

$$e_1' \sim e_2' \text{ in } M'$$

which means that π is spatial.

4°. A countably decomposable von Neumann algebra $M \subset B(H)$ is in standard form if and only if it has a bicyclic vector $\xi_0 \in H$, by the Tomita's Theorem.

Therefore any $*$-isomorfism between two standard countably decomposable von Neumann algebras is spatial. Thus, the standard form of a countably decomposable von Neumann algebra is unique.

This is the main argument for the uniqueness of the standard form.

5°. Let $M \subset B(H)$ be a von Neumann algebra. A projection $e \in M$ is <u>piecewise</u> of <u>countable type</u> if there exists a family $\{q_k\}_{k \in K}$ of mutually orthogonal central projections with $\sum_{k \in K} q_k = 1$ and each eq_k of countable type

Any finite projection is piecewise of countable type (Lectures, 7.2)

Let M be a properly infinite von Neumann algebra and γ be an infinite cardinal. One says that M is uniform of type γ if there exists a family $\{e_i\}_{i\in I}$ of equivalent, mutually orthogonal projections in M, each piecewise of countable type such that

$$\sum_{i\in I} e_i = 1 \quad \text{and} \quad \operatorname{card} I = \gamma$$

(These conditions determine uniquely card I - Lectures 8.4).

For a general properly infinite von Neumann algebra there exist a family Γ of distinct cardinals and a family $\{p_\gamma\}_{\gamma\in\Gamma}$ of non-zero, mutually orthogonal central projections, uniquely determined by the conditions

$$\sum_{\gamma\in\Gamma} p_\gamma = 1 \quad \text{and} \quad Mp_\gamma \text{ is uniform of type } \gamma. \quad \text{(Lectures 8.5)}$$

Now let $M\subset B(H)$ be a von Neumann algebra whose commutant $M'\subset B(H)$ is properly infinite. Let Γ' and $\{p'_\gamma\}_{\gamma\in\Gamma'}$ be associated to M' such that $p'_\gamma \in M\cap M'$,

$$\sum_{\gamma\in\Gamma'} p'_\gamma = 1 \quad \text{and} \quad M'p'_\gamma \text{ is uniform of type } \gamma.$$

One can then define the symbol

$$u_{M'} = \{\Gamma', \{p'_{\gamma'}\}_{\gamma'\in\Gamma'}\} \text{ called the uniformity of } M.$$

If $\pi : M_1 \to M_2$ is a $*$-isomorphism between two von Neumann algebras with properly infinite commutants and

$$u_{M'_j} = \{\Gamma'_j, \{p'_{j,\gamma'}\}_{\gamma'\in\Gamma'_j}\} \ ; \quad j=1,2$$

then we write

$$\tilde{\pi}(u_{M'_1}) = u_{M'_2}$$

if $\Gamma'_1 = \Gamma'_2$ and $\pi(p'_{1,\gamma'}) = p'_{2,\gamma'}$ for any $\gamma' \in \Gamma'_1 = \Gamma'_2$, and we say that π preserves the uniformity

THEOREM. Let $M_1\subset B(H_1)$, $M_2\subset B(H_2)$ be von Neumann algebras with properly infinite commutants. Then any $*$-isomorphism $\pi : M_1 \to M_2$ which preserves the uniformity is spatial.

The proof (Lectures 8.6) is easily reduced to:

$$e'_1, e'_2 \in \mathcal{P}(M') \text{ properly infinite}, z(e'_1)=z(e'_2) \Rightarrow e'_1 \sim e'_2 \text{ in } M'$$

6° If $M \subset B(H)$ is a von Neumann algebra in standard form, then $j : M \ni x \longmapsto Jx^*J \in M'$ is a $*$-antiautomorphism hence the uniformity of M is equal to the uniformity of the commutant M'.

Thus, if $\pi : M_1 \to M_2$ is a $*$-isomorphism of standard von Neumann algebras then π preserves the uniformity and hence is a spatial isomorphism.

This proves the uniqueness of the standard form in the general case.

We may consider separately the cases

1) M_1 is finite   2) M_1 is properly infinite

In case 1) the unit 1 is piecewise of countable type, so that the problem is reduced to countably decomposable algebras

In case 2) π preserves the uniformity, etc.

In a factor (to avoid the mess with central projections)

$0\neq$ $e, f \in M$ projections, e countably decomposable, f infinite

$$\Rightarrow e \prec f$$

<u>Proof</u> Since f is infinite, $f = \sum_{n=1}^{\infty} f_n$ with $f_n \sim f$, $(\forall) n$

Since $z(e) z(f) = 1 \neq 0$ there are projections $e_1 \leq e$, $g_1 \leq f$ with $e_1 \sim g_1 \leq f \sim f_1$, hence $e_1 \sim h_1 \leq f_1$.

If $e - e_1 \neq 0$ we similarly find $e_2 \leq e - e_1$, $e_2 \sim h_2 \leq f_2$.

And so on. ~~If that~~ Take $\{e_n\}_{n \in \mathbb{N}}$ a maximal family of orthogonal subprojections of e such that $e_n \sim h_n \leq f_n$

Then $\sum_{n=1}^{\infty} e_n = e$, hence

$$e = \sum_n e_n \sim \sum_n h_n \leq \sum_n f_n = f, \quad \text{q.e.d.}$$

<u>Consequence</u>. Two (properly) infinite countably decomposable projections with the same central support are equivalent