VII The Alain Connes' Radon-Nikodym Theorem.

A natural problem is: given two n.s.f. weights φ, ψ on a von Neumann algebra M, what is the connection between them. The problem is natural because for faithful measures there is the answer given by the Radon-Nikodym theorem. To guess what kind of connection could exists look at two automorphism groups of the form

$$\sigma_t = A^{it} \cdot A^{-it} \quad , \quad \tau_t = B^{it} \cdot B^{-it}.$$

The „connection" is the function

$$u : t \longmapsto u(t) = B^{it} A^{it}.$$

This function is a „cocycle":

$$u(t+s) = u(t)\, \sigma_t(u(s))$$

and connects the two groups:

$$\tau_t(x) = u_t\, \sigma_t(x)\, u_t^*.$$

In case that function is a unitary group

$$u(t+s) = u(t)\, u(s),$$

the analytic generator A: $A^{it} = u_t$, could be considered as a Radon-Nikodym derivative of a weight ψ with $\sigma_t^\psi = \tau_t$ with respect to a weight φ with $\sigma_t^\varphi = \sigma_t$ and we can expect also a relation of the form

$$\psi(\cdot) = \varphi(A\cdot)$$

which justifies the „Radon-Nikodym" terminology.

1°. Consider two different n.s.f. weights φ, ψ on the von Neumann algebra M. Consider the von Neumann algebra $P = Mat_2(M)$ and the n.s.f. weight $\theta = \theta(\varphi, \psi)$ on P defined by

$$\theta(X) = \varphi(x_{11}) + \psi(x_{22}) \quad \text{for } X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \geq 0$$

(Here, and in what follows, we give the complete arguments only for normal positive forms φ and ψ. In the general case the arguments are completely similar - Lectures, 10.28).

Consider also the projections

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in P.$$

We have

$$\theta(E_{11}X) = \varphi(x_{11}) = \theta(XE_{11}), \; (\forall)\, X \in P$$

$$\theta(E_{22}X) = \psi(x_{22}) = \theta(XE_{22}), \; (\forall)\, X \in P$$

hence $E_{11}, E_{22} \in \mathcal{P}^{\theta}$, i.e. $\sigma_t^{\theta}(E_{11}) = E_{11}$, $\sigma_t^{\theta}(E_{22}) = E_{22}$, $(t\in\mathbb{R})$, according to $\overline{\text{VI}}.4°$. For $x\in M$ we have

$$\sigma_t^{\theta}\begin{pmatrix} x & 0 \\ 0 & 0\end{pmatrix} = \sigma_t^{\theta}\left(E_{11}\begin{pmatrix} x & 0 \\ 0 & 0\end{pmatrix}E_{11}\right) = E_{11}\,\sigma_t^{\theta}\begin{pmatrix} x & 0 \\ 0 & 0\end{pmatrix}E_{11}, \quad (x\in M),$$

hence $\sigma_t^{\theta}\begin{pmatrix} x & 0 \\ 0 & 0\end{pmatrix}$ is of the form $\begin{pmatrix} \sigma_t(x) & 0 \\ 0 & 0\end{pmatrix}$ with $\sigma: \mathbb{R}\to\sigma_t\in \mathrm{Aut}(M)$ a continuous group of automorphisms. Using the KMS-conditions we infer that $\sigma_t(x) = \sigma_t^{\varphi}(x)$, $(x\in M, t\in\mathbb{R})$. Hence

$$\sigma_t^{\theta}\begin{pmatrix} x & 0 \\ 0 & 0\end{pmatrix} = \begin{pmatrix} \sigma_t^{\varphi}(x) & 0 \\ 0 & 0\end{pmatrix} \text{ and, similarly, } \sigma_t^{\theta}\begin{pmatrix} 0 & 0 \\ 0 & \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_t^{\psi}(x)\end{pmatrix}$$

for any $x\in M$. Then:

$$E_{11}\,\sigma_t^{\theta}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} = \sigma_t^{\theta}\left(E_{11}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\right) = \sigma_t^{\theta}(0) = 0$$

$$\sigma_t^{\theta}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}E_{22} = \sigma_t^{\theta}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}E_{22}\right) = \sigma_t^{\theta}(0) = 0$$

hence

$$\sigma_t^{\theta}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ u(t) & 0\end{pmatrix} \quad ; \quad t\in\mathbb{R}.$$

Then one sees that:

1) the map $t\mapsto u(t)$ is so-continuous (since $t\mapsto \sigma_t^{\theta}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$ is continuous

2) $u(t)$ are unitaries:

$$\begin{pmatrix} u(t)^*u(t) & 0 \\ 0 & 0\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ u(t) & 0\end{pmatrix}^*\begin{pmatrix} 0 & 0 \\ u(t) & 0\end{pmatrix} = \sigma_t^{\theta}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}^*\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\right) =$$

$$= \sigma_t^{\theta}(E_{11}) = \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}$$

hence $u(t)^*u(t) = 1$ and, similarly, $u(t)u(t)^* = 1$

3) $u(t+s) = u(t)\,\sigma_t^{\varphi}(u(s))$, $(t,s\in\mathbb{R})$:

$$\begin{pmatrix} 0 & 0 \\ u(t+s) & 0\end{pmatrix} = \sigma_{t+s}^{\theta}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} = \sigma_t^{\theta}\left(\sigma_s^{\theta}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\right) =$$

$$= \sigma_t^{\theta}\begin{pmatrix} 0 & 0 \\ u(s) & 0\end{pmatrix} = \sigma_t^{\theta}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\begin{pmatrix} u(s) & 0 \\ 0 & 0\end{pmatrix}\right) =$$

$$= \sigma_t^{\theta}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\cdot\sigma_t^{\theta}\begin{pmatrix} u(s) & 0 \\ 0 & 0\end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 \\ u(t) & 0\end{pmatrix}\begin{pmatrix} \sigma_t^{\varphi}(u(s)) & 0 \\ 0 & 0\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ u(t)\,\sigma_t(u(s)) & 0\end{pmatrix}$$

4) $\sigma_t^{\psi}(x) = u(t)\,\sigma_t^{\varphi}(x)\,u(t)^*$, $(x\in M, t\in\mathbb{R})$:

$$\begin{pmatrix} 0 & 0 \\ 0 & \sigma_t^{\psi}(x) \end{pmatrix} = \sigma_t^{\theta}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = \sigma_t^{\theta}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) =$$

$$= \left(\begin{pmatrix} 0 & 0 \\ u(t) & 0 \end{pmatrix}\begin{pmatrix} \sigma_t^{\varphi}(x) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & u(t)^* \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & u(t)\sigma_t^{\varphi}(x)u(t)^* \end{pmatrix}.$$

Moreover, the KMS condition for θ gives

5) a certain KMS-condition for $u(t)$ (MOD. TH. 3.1)

With the properties 1)–5), the mapping $t \mapsto u(t)$ is uniquely determined (MOD. TH. 3.1), is called the Connes cocycle associated to φ, ψ and is denoted by:

$$u_t = \left[\frac{D\psi}{D\varphi}\right]_t \ ; \ t \in \mathbb{R}.$$

In particular, for any two nsf weights φ, ψ on M we have

$$\sigma_t^{\varphi}(\mathrm{mod}\, Int(M)) = \sigma_t^{\psi}(\mathrm{mod}\, Int\, M) \ ; \ t \in \mathbb{R},$$

where $Int(M) \subset Aut(M)$ is the normal subgroup of inner automorphisms. This defines an invariant of M called the modular homomorphism

$$\delta_M : \mathbb{R} \ni t \longmapsto \sigma_t^{\varphi}(\mathrm{mod}\, Int(M)) \in Out(M) := \frac{Aut(M)}{Int(M)}$$

2°. If φ, ψ are n.s.f. weights on M and $\sigma \in Aut(M)$, then, using the uniqueness of the modular automorphisms group and of the Connes cocycle with the KMS condition, we get that, for any $*$-automorphism $\sigma \in Aut(M)$,

(1) $\sigma_t^{\varphi\circ\sigma} = \sigma^{-1} \cdot \sigma_t^{\varphi} \cdot \sigma$, $M^{\varphi\circ\sigma} = \sigma^{-1}(M^{\varphi})$ and $\left[\frac{D(\psi\circ\sigma)}{D(\varphi\circ\sigma)}\right]_t = \sigma^{-1}\left(\left[\frac{D\psi}{D\varphi}\right]_t\right)$.

3° If φ is a normal semifinite (but not necessarily faithful) weight on the von Neumann algebra M, then we will denote by $\{\sigma_t^{\varphi}\}_{t\in\mathbb{R}}$ the modular automorphisms group associated to the nsf weight φ on the reduced algebra $s(\varphi) M s(\varphi)$. Also, if φ is an n.s.f. weight on M and ψ is a normal semifinite (not necessarily) faithful weight on M, then the Connes cocycle theorem takes the form:

There is an s^*-continuous mapping $\mathbb{R} \ni t \mapsto u_t \in M$ uniquely determined such that

1) $u_t u_t^* = s(\psi) = u_0$, $u_t^* u_t = \sigma_t^{\varphi}(s(\psi))$

2) $u_{t+s} = u_t \sigma_t^{\varphi}(u_s)$

3) $u_{-t} = \sigma_{-t}^{\varphi}(u_t^*)$

4) $\sigma_t^{\psi}(x) = u_t \sigma_t^{\varphi}(x) u_t^*$, $(x \in s(\psi) M s(\psi))$

5) a certain KMS condition

The proof is similar to the case when also ψ in n.s.f. (MOD.TH. Thm 3.1). In this case $\begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix} = \sigma_t^{\theta}\begin{pmatrix} 0 & 0 \\ s(\psi) & 0 \end{pmatrix}$.

4° If φ is an n.s.f. weight on M and $v \in M$ is a partial isometry with $vv^* \in M^{\varphi}$, then we get a new normal semifinite weight φ_v on M with $s(\varphi_v) = v^*v$.

$\varphi_v(x) = \varphi(vxv^*)$ and, using the KMS condition we get

$$\sigma_t^{\varphi_v}(x) = v^* \sigma_t^{\varphi}(vxv^*) v, \; (x \in v^*vMvv^*, t \in \mathbb{R})$$

$$M^{\varphi_v} = v^* M^{\varphi} v.$$

and

(2) $$\left[\frac{D\varphi_v}{D\varphi}\right]_t = v^* \sigma_t^{\varphi}(v)$$

In particular, if $e \in M^{\varphi}$ is a projection, then

$$\sigma_t^{\varphi_e}(x) = \sigma_t^{\varphi}(x), \; (x \in eMe, t \in \mathbb{R}) .$$

$$M^{\varphi_e} = eM^{\varphi}e$$

$$[D\varphi_e : D\varphi]_t = e, \; (\forall) t \in \mathbb{R}.$$

To prove (2) consider the „balanced weights" $\theta(\varphi,\varphi)$ and $\theta(\varphi,\varphi_v)$ on $Mat_2(M)$ and $u = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \in Mat_2(M)$. Then uu^* belongs to the centralizer of $\theta(\varphi,\varphi)$ and

$$\theta(\varphi,\varphi_v) = \theta(\varphi,\varphi)_u$$

Thus:

$$\begin{pmatrix} 0 & 0 \\ [D\varphi_v : D\varphi] & 0 \end{pmatrix} = \sigma_t^{\theta(\varphi,\varphi_v)}\begin{pmatrix} 0 & 0 \\ s(\varphi_v) & 0 \end{pmatrix} = \sigma_t^{\theta(\varphi,\varphi)_u}\begin{pmatrix} 0 & 0 \\ s(\varphi_v) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & v^* \end{pmatrix} \sigma_t^{\theta(\varphi,\varphi)}\left(\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} 0 & 0 \\ v^*v & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & v^* \end{pmatrix}\right)\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & v^* \end{pmatrix} \sigma_t^{\theta(\varphi,\varphi)}\left(\begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}\right)\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & v^* \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sigma_t^{\varphi}(v) & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ v^*\sigma_t^{\varphi}(v) & 0 \end{pmatrix}.$$

LEMMA. Let φ be an n.s.f. weight on the von Neumann algebra M, and $v \in M$ a partial isometry with $f = vv^* \in M^\varphi$ and $e = v^*v \in M^\varphi$. Then

$$\varphi_v = \varphi_e \iff v \in M^\varphi$$

Indeed, formaly we have (MOD. TH. Prop. 2.21):
If $v \in M^\varphi$, then:

$$\varphi(vxv^*) = \varphi(vexv^*) = \varphi(exv^*v) = \varphi(exe)$$

Conversely, if $\varphi_v = \varphi_e$, then

$$\varphi(vx) = \varphi(fvx) = \varphi(vxf) = \varphi(vxvv^*) = \varphi(exve) =$$
$$= \varphi(xve) = \varphi(xv)$$

5° Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be n.s.f. weights on the von Neumann algebra M and

$$N = Mat_n(M) \ni [x_{ij}] \equiv \sum_{i,j=1}^{n} x_{ij} \otimes e_{ij} \in M \otimes Mat_n(\mathbb{C})$$

where $\{e_{ij}\}$ is the canonical system of matrix units, in $Mat_n(\mathbb{C})$. The equation

$$\theta([x_{ij}]) = \sum_{k=1}^{n} \varphi_k(x_{kk}) \ ; \ [x_{ij}] \in N^+$$

defines an n.s.f. weight on N. Using the above Lemma we see that all elements of the form $\sum_k \pm 1 \otimes e_{kk}$ are in the centralizer of θ, hence

$$E_k = 1 \otimes e_{kk} \in N^\theta, \ (k = 1, \ldots, n).$$

Using the KMS conditions we see that

$$\sigma_t^\theta(x \otimes e_{kk}) = \sigma_t^{\varphi_k}(x) \otimes e_{kk}, \ (1 \le k \le n)$$

On the other hand:

$$\sigma_t^\theta(1 \otimes e_{ij}) = \left[\frac{D\varphi_i}{D\varphi_j}\right]_t \otimes e_{ij}, \ (1 \le i,j \le n) \tag{3}$$

Indeed, assume for instance $i = 2$ and $j = 1$. The projection $e = 1 \otimes (e_{11} + e_{22})$ belongs to the centralizer N^θ so that we may consider the n.s.f. weight θ_e on the reduced algebra eNe. It is clear that eNe can be identified with $Mat_2(M)$ in such a way that θ_e corresponds to the balanced weight $\theta(\varphi_1, \varphi_2)$. Using this identification we obtain

$$\sigma_t^{\theta}(1\otimes e_{21}) = \sigma_t^{\theta_e}(1\otimes e_{21}) = \sigma_t^{\theta(\varphi_1,\varphi_2)}\begin{pmatrix}0&0\\1&0\end{pmatrix}$$
$$= \begin{pmatrix}0&0\\ [D\varphi_2 : D\varphi_1]_t & 0\end{pmatrix} = [D\varphi_2 : D\varphi_1]_t \otimes e_{21}$$

From (3) with $n=2$ we get

(4) $$\left[\frac{D\varphi_1}{D\varphi_2}\right]_t = \left[\frac{D\varphi_2}{D\varphi_1}\right]_t^{-1}, \ (t\in\mathbb{R})$$

since $e_{12} = e_{21}^*$. From (3) with $n=3$ we get

(5) $$\left[\frac{D\varphi_1}{D\varphi_3}\right]_t = \left[\frac{D\varphi_1}{D\varphi_2}\right]_t \cdot \left[\frac{D\varphi_2}{D\varphi_3}\right]_t$$

since $e_{13} = e_{12}e_{23}$. For ψ_1, ψ_2, φ n.s.f weights we have

(6) $$\frac{D\psi_1}{D\varphi} = \frac{D\psi_2}{D\varphi} \iff \psi_1 = \psi_2 .$$

Indeed, using (5) this reduces to show that

(7) $$\left[\frac{D\psi}{D\varphi}\right]_t = 1, \ (\forall)\, t\in\mathbb{R} \implies \psi = \varphi .$$

Let $\theta = \theta(\varphi,\psi)$ be the balanced weight on $N = Mat_2(M)$. From the assumption in (7) it follows that $\sigma_t^{\theta}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $(t\in\mathbb{R})$, that is $\begin{pmatrix}0&1\\1&0\end{pmatrix}\in N^{\theta}$, so:

$$\varphi(x) = \theta\begin{pmatrix}x&0\\0&0\end{pmatrix} = \theta\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}x&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) = \theta\begin{pmatrix}0&0\\0&x\end{pmatrix} = \psi(x).$$

6°. Let φ, ψ be n.s.f. weights on M, $\theta = \theta(\varphi,\psi)$ the balanced weight an $u_t = [D\psi : D\varphi]_t$, $(t\in\mathbb{R})$, the Connes cocycle. For $x\in M$, $t\in\mathbb{R}$, put

$$\sigma_t^{\varphi,\psi}(x) = u_t\,\sigma_t^{\varphi}(x) = \sigma_t^{\psi}(x)\,u_t .$$

Then, for $[x_{ij}]\in Mat_2(M)$ we have

$$\sigma_t^{\theta}\left(\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}\right) = \begin{pmatrix}\sigma_t^{\varphi}(x_{11}) & \sigma_t^{\varphi,\psi}(x_{12})\\ \sigma_t^{\psi,\varphi}(x_{21}) & \sigma_t^{\psi}(x_{22})\end{pmatrix}$$

It is easy to see that $\{\sigma_t^{\psi,\varphi}\}_t$ is an s^*-continuous one parameter group of isometries of M onto M:

$$\sigma_s^{\psi,\varphi}(\sigma_t^{\psi,\varphi}(x)) = \sigma_{s+t}^{\psi,\varphi}(x)$$

and

$$\sigma_t^{\psi,\varphi}(x^*) = \left(\sigma_t^{\varphi,\psi}(x)\right)^*$$

and, for a third n.s.f. weight τ on M:

$$\sigma_t^{\psi,\varphi}(xy) = \sigma_t^{\psi,\tau}(x)\cdot\sigma_t^{\tau,\varphi}(y)$$

These properties together with a certain KMS-condition determine uniquely the group $\{\sigma_t^{\psi,\varphi}\}_{t\in\mathbb{R}}$ (MOD.TH-3.10)

7°. In the early 1960's S. Sakai obtained a powerful Radon-Nikodym theorem for normal positive forms φ, ψ on a von Neumann algebra M, namely (LECTURES 5.21)

$\varphi \le \psi \Rightarrow (\exists)\, a\in M$ uniquely determined such that
$$0\le a\le 1,\ s(a)\le s(\psi),\ \varphi(x)=\psi(axa),\ (\forall)\, x\in M$$

but the question: "for which $0\le a\le 1$ $\varphi(a\cdot a)\le\varphi$?" had not an answer at that time. The following result supplies the answer.

PROPOSITION. Let φ be an n.s.f. weight on the von Neumann algebra M, $a\in M$ and $\lambda\in(0,\infty)$. The following statements are equivalent:

(i) $\varphi(ax^*xa^*)\le\lambda^2\varphi(x^*x)$, $(\forall)\, x\in M$

(ii) $x\in\mathfrak{N}_\varphi \Rightarrow xa^*\in\mathfrak{N}_\varphi$ and $\|(xa^*)_\varphi\|_\varphi \le \lambda\|x_\varphi\|_\varphi$

(iii) $a\in D(\sigma^\varphi_{-i/2})$ and $\|\sigma^\varphi_{-i/2}(a)\|\le\lambda$.

We first explain what is $\sigma^\varphi_{-i/2}$.

An element $a\in M$ is called <u>analytic</u> in the vertical strip $\{\alpha\in\mathbb{C};\ -\varepsilon_1\le \operatorname{Re}\alpha\le\varepsilon_2\}$, $(0\le\varepsilon_1,\varepsilon_2<\infty)$, if there is an M-valued function F, defined and w-continuous on the strip and analytic in the interior of the strip, such that

$$F(it)=\sigma_t^\varphi(a),\quad t\in\mathbb{R}$$

and in this case, for each α in the strip we write

$$\sigma^\varphi_{-i\alpha}(a)=F(\alpha).$$

For $\alpha\in\mathbb{C}$ we shall write $a\in D(\sigma^\varphi_\alpha)$ if the element a is analytic in some vertical strip containing $i\alpha$.

It is easy to check that:

$$a\in D(\sigma^\varphi_\alpha)\Rightarrow a^*\in D(\sigma^\varphi_{\bar\alpha}),\ \sigma^\varphi_{\bar\alpha}(a^*)=\sigma^\varphi_\alpha(a)^*$$

$$a,b\in D(\sigma^\varphi_\alpha)\Rightarrow ab\in D(\sigma^\varphi_\alpha),\ \sigma^\varphi_\alpha(ab)=\sigma^\varphi_\alpha(a)\cdot\sigma^\varphi_\alpha(b)$$

$$a\in D(\sigma^\varphi_\beta),\ \sigma^\varphi_\beta(a)\in D(\sigma^\varphi_\alpha)\Rightarrow a\in D(\sigma^\varphi_{\alpha+\beta}) \text{ and}$$
$$\sigma^\varphi_{\alpha+\beta}(a)=\sigma^\varphi_\alpha(\sigma^\varphi_\beta(a))$$

From the equalities $\pi_\varphi(\sigma_t^\varphi(a))\Delta_\varphi^{it}\xi = \Delta_\varphi^{it}\pi_\varphi(a)\xi$ we infer, by analytic continuation, that

$$a \in D(\sigma^\varphi_{-i\alpha}),\ \xi \in D(\Delta_\varphi^\alpha) \Rightarrow \pi_\varphi(a)\xi \in D(\Delta_\varphi^\alpha) \text{ and}$$

(*) $$\Delta_\varphi^\alpha \pi_\varphi(a)\xi = \pi_\varphi(\sigma^\varphi_{-i\alpha}(a))\Delta_\varphi^\alpha \xi$$

or replacing α by $-i\alpha$: (see LECTURES 9.24)

$$a \in D(\sigma_\alpha^\varphi) \Rightarrow \pi_\varphi(\sigma_\alpha^\varphi(a)) = \overline{\Delta_\varphi^{i\alpha}\pi_\varphi(a)\Delta_\varphi^{-i\alpha} \mid D(\Delta_\varphi^{-i\alpha})}$$

PROOF OF THE PROPOSITION.

(i) $\Leftrightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) From (ii) it follows that there exists $T \in B(H_\varphi)$, $\|T\| \leq \lambda$, such that $Tx_\varphi = (xa^*)_\varphi$, $(x \in \mathfrak{N}_\varphi)$. For x in $\mathfrak{A}_\varphi = \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$ we then have

$$Tx_\varphi = (xa^*)_\varphi = S_\varphi(ax^*)_\varphi = S_\varphi \pi_\varphi(a) S_\varphi x_\varphi =$$
$$= J_\varphi \Delta_\varphi^{1/2}\pi_\varphi(a)\Delta_\varphi^{-1/2} J_\varphi x_\varphi$$

so the operator $\Delta_\varphi^{1/2}\pi_\varphi(a)\Delta_\varphi^{-1/2} \mid \mathfrak{A}'_\varphi$ is bounded with norm $\leq \lambda$. Since

$$\overline{\Delta_\varphi^{-1/2} \mid \mathfrak{A}'_\varphi} = \Delta_\varphi^{-1/2}$$

we infer that (LECTURES, 9.24) $a \in D(\sigma^\varphi_{-i/2})$ and $\|\sigma^\varphi_{-i/2}(a)\| \leq \lambda$

(iii) $\Rightarrow$ (ii) Let $x \in \mathfrak{A}_\varphi = \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$. Then $(x^*)_\varphi \in D(\Delta_\varphi^{1/2})$ and, using (*) with $\alpha = 1/2$, we get

$$\Delta_\varphi^{1/2}\pi_\varphi(a)(x^*)_\varphi = \pi_\varphi(\sigma^\varphi_{-i/2}(a))\Delta_\varphi^{1/2}(x^*)_\varphi$$
$$S_\varphi(ax^*)_\varphi = J_\varphi \pi_\varphi(\sigma^\varphi_{-i/2}(a)) J_\varphi S_\varphi (x^*)_\varphi$$

hence:

$$(xa^*)_\varphi = J_\varphi \pi_\varphi(\sigma^\varphi_{-i/2}(a)) J_\varphi x_\varphi$$

It follows that

$$\varphi(ax^*xa^*) \leq \|\sigma^\varphi_{-i/2}(a)\|\,\varphi(x^*x), \quad x \in M$$

since if $\varphi(x^*x) = \infty$ the inequality is obvious, and if $\varphi(x^*x) < \infty$, then $(x^*x)^{1/2} \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^* = \mathfrak{A}_\varphi$. This proves (i).

8°. Consider now two n.s.f. weights φ, ψ on M. By applying the previous result to the balanced weight $\theta \overset{\text{d}}{=} \theta(\varphi,\psi)$ and to the element $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$ it follows that the following statements are equivalent

(i) $\psi(a x^* x a^*) \leq \lambda^2 \varphi(x^* x)$

(ii) $x \in \mathfrak{N}_\varphi \Rightarrow x a^* \in \mathfrak{N}_\psi$ and $\|(x a^*)_\psi\|_\psi \leq \lambda \|x_\varphi\|_\varphi$

(iii) $a \in D(\sigma^{\psi,\varphi}_{-i/2})$ and $\|\sigma^{\psi,\varphi}_{-i/2}(a)\| \leq \lambda$

(cf MOD. TH Prop. 3.12).

In particular, for $a = 1$ and $\lambda = 1$, we get a weak form of the Sakai's Radon-Nikodym theorem.

COROLLARY. Let φ, ψ be n.s.f. weights on the von Neumann algebra M. The following statements are equivalent

(i) $\psi \leq \varphi$

(ii) there is an M-valued function defined and w-continuous on the strip $\{\alpha \in \mathbb{C};\ 0 \leq \operatorname{Re}\alpha \leq 1/2\}$ and analytic in the interior, such that

$$F(it) = [D\psi : D\varphi]_t, \ (t \in \mathbb{R}) \text{ and } \|F(1/2)\| \leq 1$$

If $\psi \leq \varphi$ there exists an element a $(= F(1/2)^*)$ in M with $\|a\| \leq 1$ such that

$$\psi(x^* x) = \varphi(a x^* x a^*), \quad (x \in \mathfrak{N}_\psi).$$

Indeed, since $\sigma^{\psi,\varphi}_t(1) = [D\psi : D\varphi]_t$, $(t \in \mathbb{R})$, we can take

$$F(\alpha) = \sigma^{\psi,\varphi}_{-i\alpha}(1), \quad (\alpha \in \mathbb{C};\ 0 \leq \operatorname{Re}\alpha \leq 1/2).$$