V The Tomita algebra

Consider again a von Neumann algebra $M \subset B(H)$ with a bicyclic vector $\xi_0 \in H$.

Although $M \cap M'$ can be very small ($M \cap M' = \mathbb{C}I$ when M is a factor) we will see that $M\xi_0 \cap M'\xi_0$ is dense in H.

Consider the associated objects

$$\varphi(x) = (x\xi_0 | \xi_0), \quad (x \in M), \quad s(\varphi) = 1$$

and S, $F = S^*$, J, Δ, $\sigma = \sigma^\varphi : \mathbb{R} \longrightarrow Aut(M)$, $j : M \longrightarrow M'$.

1° An element $a \in M$ is called <u>entire analytic</u> if the mapping $i\mathbb{R} \ni it \longmapsto \sigma_t(a) \in M$ has an entire analytic extension

$$F : \mathbb{C} \ni \alpha \longmapsto F_a(\alpha) \in M \quad (\text{automatically} \in M\ !)$$

and in this case we write

$$\sigma_\alpha(a) = F_a(-i\alpha) \quad \text{i.e. } F_a(\alpha) = \sigma_{-i\alpha}(a), \quad (\alpha \in \mathbb{C}).$$

The set of analytic elements will be denoted by A.

Recall that the operator Δ^α is recovered from the so-continuous unitary group $\{\Delta^{it};\ t \in \mathbb{R}\}$ in a similar manner (LECTURES, 9.20):

$(\xi, \eta) \in Graph(\Delta^\alpha) \iff$ the mapping $i\mathbb{R} \ni it \mapsto \Delta^{it}\xi \in H$ has a continuous bounded extension

$$f : \{\beta \in \mathbb{C};\ 0 \leq Re\beta \leq Re\alpha\} \longrightarrow H$$

analytic in $\{\beta \in \mathbb{C};\ 0 < Re\beta < Re\alpha\}$ and $\eta = f(\alpha)$

(We assumed $Re\alpha > 0, \dots$)

From the equality

$$\sigma_t(a)\Delta^{it}\xi = \Delta^{it} a\xi,$$

by analytic continuation, we infer that

$$a \in A,\ \xi \in D(\Delta^\alpha) \Rightarrow a\xi \in D(\Delta^\alpha) \text{ and } \Delta^\alpha a\xi = \sigma_{i\alpha}(a)\Delta^\alpha\xi$$

and then (by Lectures, 9.24):

$$\sigma_{-i\alpha}(a) = \overline{\Delta^\alpha a \Delta^{-\alpha} | D(\Delta^{-\alpha})}\ ; \quad (a \in A, \alpha \in \mathbb{C})$$

2° PROPOSITION: A is a w-dense $*$-subalgebra of M and

$$A\xi_0 \subset M\xi_0 \cap M'\xi_0 \cap \bigcap_{\alpha \in \mathbb{C}} D(\Delta^\alpha)$$

$$\overline{\Delta^\alpha | A\xi_0} = \Delta^\alpha, \quad \overline{S | A\xi_0} = S, \quad \overline{S^* | A\xi_0} = S^*$$

$A\xi_0$ is called the Tomita algebra associated to φ.

PROOF. A is a $*$-subalgebra of M: it is clearly a vector subspace of M and it is easy to check that

$$\sigma_{\bar\alpha}(a^*) = \sigma_\alpha(a)^* \ , \quad (a\in A)$$
$$\sigma_\alpha(ab) = \sigma_\alpha(a)\sigma_\alpha(b) \ , \ (a,b\in A)$$

by analytic continuation from the case $\alpha = it$, $t\in\mathbb{R}$.

It is also clear that $A\xi_0 \subset M\xi_0 \cap \bigcap_{\alpha\in\mathbb{C}} D(\Delta^\alpha)$, namely

$$\Delta^\alpha(a\xi_0) = \Delta^\alpha a \Delta^{-\alpha}\xi_0 = \sigma_{-i\alpha}(a)\xi_0 \ , \quad (a\in A, \alpha\in\mathbb{C})$$

Then $A\xi_0 \subset M'\xi_0$ since for $a\in A$ we have

$$Ja\xi_0 = JSa^*\xi_0 = JJ\Delta^{1/2}a^*\xi_0 = \Delta^{1/2}a^*\xi_0 =$$
$$= \sigma_{-i/2}(a^*)\xi_0 = \sigma_{i/2}(a)^*\xi_0$$

i.e. $a\xi_0 = J\sigma_{i/2}(a)^*\xi_0 = JM\xi_0 = M'\xi_0$ by the Tomita's Theorem.

It remains to prove the density assertions, that is to produce „enough many" elements in A. These elements are of the form

$$\sigma_f(x) := \int_{-\infty}^{\infty} f(t)\sigma_t(x)\,dt \quad \text{with } x\in M \text{ and „good" } f\in L^1(\mathbb{R});$$

notice that:

$$\sigma_f(x)\xi_0 = \int_{-\infty}^{\infty} f(t)\Delta^{it}x\xi_0\,dt \quad \left(\equiv \hat f(\ln\Delta)\,x\xi_0\right).$$

„Good" f means that f has an entire analytic continuation. But here we need just some concrete examples, namely

$$f_n(t) = \frac{1}{\sqrt{n\pi}}\, e^{-t^2/n}$$

Thus, for $x\in M$ we consider

$$x_n = \frac{1}{\sqrt{n\pi}}\int_{-\infty}^{\infty} e^{-t^2/n}\sigma_t(x)\,dt$$

$$x_n\xi_0 = \frac{1}{\sqrt{n\pi}}\int_{-\infty}^{\infty} e^{-t^2/n}\Delta^{it}x\xi_0\,dt$$

The mapping

$$i\mathbb{R} \ni is \longmapsto \sigma_s(x_n) = \frac{1}{\sqrt{n\pi}}\int_{-\infty}^{\infty} e^{-t^2/n}\sigma_{s+t}(x)\,dt =$$

$$= \frac{1}{\sqrt{n\pi}} \int_{-\infty}^{\infty} e^{-(t-s)^2/n}\, \sigma_t(x)\, dt$$

$$= \frac{1}{\sqrt{n\pi}} \int_{-\infty}^{\infty} e^{-(t+i(is))^2/n}\, \sigma_t(x)\, dt$$

has the analytic continuation

$$\mathbb{C} \ni \alpha \longmapsto \sigma_{-i\alpha}(x_n) = \frac{1}{\sqrt{n\pi}} \int_{-\infty}^{\infty} e^{-(t+i\alpha)^2/n}\, \sigma_t(x)\, dt$$

and

$$\Delta^{\alpha} x_n \xi_0 = \sigma_{-i\alpha}(x_n)\xi_0 = \frac{1}{\sqrt{n\pi}} \int_{-\infty}^{\infty} e^{-(t+i\alpha)^2/n} \Delta^{it} x\xi_0\, dt$$

On the other hand, using the Lebesgue dominated convergence theorem and the continuity at 0 of the mappings $t \longmapsto \sigma_t(x)$, it follows that

$$A \ni x_n \xrightarrow{w/s_0} x \in M,$$

$$A\xi_0 \ni x_n\xi_0 \longrightarrow x\xi_0 \in M\xi_0 .$$

Thus, A is w-dense in M.

Also, let $(\xi, \Delta^{\alpha}\xi) \in \mathrm{Graph}(\Delta^{\alpha})$ be orthogonal to $\mathrm{Graph}(\Delta^{\alpha}|A\xi_0)$, i.e.

Instead of $a\xi_0$ take $\exp(-\Delta)\exp(-\Delta^{-1})a\xi_0$ in order to ensure that the operator $(1+\Delta^{2\mathrm{Re}\alpha})\exp(-\Delta)\exp(-\Delta^{-1})$ is bounded

$$(\xi \,|\, a\xi_0) + (\Delta^{\alpha}\xi \,|\, \Delta^{\alpha} a\xi_0) = 0, \quad (\forall)\, a \in A,$$

or

$$(\xi \,|\, (1+\Delta^{2\mathrm{Re}\alpha}) a\xi_0) = 0, \quad (\forall)\, a \in A .$$

Since $(1+\Delta^{2\mathrm{Re}\alpha})$ is injective and self-adjoint, it has dense range and, since $A\xi_0$ is dense in H, it follows that $\xi = 0$. Thus:

$$\overline{\Delta^{\alpha}|A\xi_0} = \Delta^{\alpha}.$$

Finally $\overline{S|A\xi_0} = S$ since $S = J\Delta^{1/2}$, and $\overline{S^*|A\xi_0} = S^*$ since $S^* = J\Delta^{-1/2}$.

3°. NOTE. The $*$-algebra of analytic elements $A \subset M$ and the Tomita algebra $A\xi_0$ are „the same" in the case of a bicyclic vector. However, in the general case, the Tomita algebra has to be constructed in a different manner namely (Lectures, 10.20) the Tomita algebra $\mathcal{T} \subset H$ is

$\mathcal{A}$ is the initial left Hilbert algebra and L_ξ, R_ξ are the left and right multiplication operators.

the set of all $\xi \in \bigcap_{\alpha \in \mathbb{C}} D(\Delta^\alpha)$ such that

$$\Delta^\alpha \xi \in \mathcal{A}'' \cap \mathcal{A}' \quad (\forall)\, \alpha \in \mathbb{C}$$

$$D(\Delta^\alpha L_\xi \Delta^{-\alpha}) = D(\Delta^{-\alpha}) \text{ and } \Delta^\alpha L_\xi \Delta^{-\alpha} \subset L_{\Delta^\alpha \xi}$$

$$D(\Delta^\alpha R_\xi \Delta^{-\alpha}) = D(\Delta^{-\alpha}) \text{ and } \Delta^\alpha R_\xi \Delta^{-\alpha} \subset R_{\Delta^\alpha \xi}$$

Then (Lectures, Theorem 10.20) $\mathcal{T}$ is a left Hilbert subalgebra of $\mathcal{A}''$, and

$$\mathcal{T}' = \mathcal{A}', \quad \mathcal{T}'' = \mathcal{A}''$$

$$\mathcal{T} \subset D(\Delta^\alpha), \quad \Delta^\alpha \mathcal{T} = \mathcal{T}, \quad \overline{\Delta^\alpha | \mathcal{T}} = \Delta^\alpha, \quad (\forall)\, \alpha \in \mathbb{C}$$

$$J\mathcal{T} = \mathcal{T}$$

$$\Delta^\alpha J \xi = J \Delta^{-\alpha} \xi, \quad (\forall)\, \xi \in \mathcal{T}, \ \alpha \in \mathbb{C}$$

$$\Delta^\alpha (\xi \eta) = (\Delta^\alpha \xi)(\Delta^\alpha \eta), \quad (\forall)\, \xi, \eta \in \mathcal{T}, \ \alpha \in \mathbb{C}$$

$$J(\xi \eta) = J(\eta) J(\xi), \quad (\forall)\, \xi, \eta \in \mathcal{T}$$

An algebra similar to the Tomita algebra is described in (Lectures, 10.22)

4° Let $Z = M \cap M'$ be the center of M and consider also the <u>centralizer</u> of the modular automorphism group σ, i.e. the fixed point algebra of σ:

$$C = \{x \in M;\ \sigma_t(x) = x,\ (\forall)\, t \in \mathbb{R}\}, \text{ denoted also } M^\varphi.$$

We have

$$Z \subset C \subset A$$

since σ_t acts identically on the center and, for $a \in C$, the constant mapping $it \longmapsto \sigma_t(a) = a$ has obviously the analytic extension $\mathbb{C} \ni \alpha \longmapsto a$, so

$$\sigma_\alpha(a) = a \quad (a \in C,\ \alpha \in \mathbb{C})$$

Note that for $a \in M$ we have

$$a \in C \iff a \text{ commutes with } \Delta$$

Indeed, $a \in C \iff \Delta^{it} a \Delta^{-it} = a,\ (\forall)\, t \in \mathbb{R} \iff \Delta^{it} a = a \Delta^{it}$, $(\forall)\, t \in \mathbb{R}$.

5° Let $\varphi = \omega_{\xi_0} = (\cdot\, \xi_0 | \xi_0) \in M_*^+$. If φ were a trace, i.e. $\varphi(ax) = \varphi(xa)$ $(\forall)\, x, a \in M$, then S would be an isometry, hence Δ would be the identity operator and all the modular automorphisms σ_t would be the identity. When it is not so, the modular operator or the modular automorphism group σ should be able to express what change appears in the trace equality $\varphi(xa) = \varphi(ax)$. This is indeed so, as we shall see.

We want to change $xa\xi_0$ into $ax\xi_0$. This can be done as follows:

$$xa\xi_0 = S(a^* x^* \xi_0) = Sa^* S x\xi_0 = J\Delta^{1/2} a^* \Delta^{-1/2} J x\xi_0$$
$$= J\,\sigma_{-\frac{i}{2}}(a^*)\, J x\xi_0 = J\,\sigma_{\frac{i}{2}}(a)^* J\, x\xi_0$$

if we assume $a \in A$, and then:

$$\begin{aligned}
\varphi(xa) &= (xa\xi_0 \mid \xi_0) = (J\,\sigma_{\frac{i}{2}}(a)^* J x\xi_0 \mid \xi_0) \\
&= (J\xi_0 \mid \sigma_{\frac{i}{2}}(a)^* J x\xi_0) \\
&= (\sigma_{\frac{i}{2}}(a)\xi_0 \mid J x\xi_0) \\
&= (x\xi_0 \mid J\,\sigma_{\frac{i}{2}}(a)\xi_0) \\
&= (x\xi_0 \mid \Delta^{1/2} S\,\sigma_{\frac{i}{2}}(a)\xi_0) \\
&= (x\xi_0 \mid \Delta^{1/2}\sigma_{-\frac{i}{2}}(a^*)\Delta^{-1/2}\xi_0) \\
&= (x\xi_0 \mid \sigma_{-\frac{i}{2}}(\sigma_{-\frac{i}{2}}(a^*))\xi_0 \\
&= (x\xi_0 \mid \sigma_{-i}(a^*)\xi_0) \\
&= (x\xi_0 \mid \sigma_i(a)^*\xi_0) \\
&= (\sigma_i(a)x\xi_0 \mid \xi_0) \qquad = \varphi(\sigma_i(a)x)
\end{aligned}$$

Thus, for $a \in A$ and $x \in M$ we have obtained

(1) $$\varphi(xa) = \varphi(\sigma_i(a)x)$$

(2) $$xa\xi_0 = J\,\sigma_{-\frac{i}{2}}(a^*)\, J x\xi_0$$

In particular

(3) $$a \in C \implies \varphi(xa) = \varphi(ax), \ (\forall)\, x \in M$$

(4) $$a \in C \implies xa\xi_0 = Ja^*Jx\xi_0, \ (\forall)\, x \in M$$

We shall see that, in essence, the equality (1) determines uniquely the modular automorphism group, and that (3) is actually an equivalence.

VI The KMS-condition

The constuction of the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbb{R}} \subset Aut(M)$ associated to a normal state or to an n.s.f weight φ on the W*-algebra M was rather complicated. It gives however a fact that involves only the pair (M,φ), namely the equality

$$\varphi(xa) = \varphi(\sigma_i(a)x) \quad (\forall)\ x\in M, a\in A.$$

1°. The problem is to characterize $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ only in terms of M and φ and, preferably, to avoid any analytical extension.

In what follows we work with (M,ξ_0) but we think at $\varphi=\omega_{\xi_0}$.

First, it is clear that

(1) $$\varphi\circ\sigma_t = \varphi, \quad (\forall)\ t\in\mathbb{R}$$

since, with $x\in M$,

$$\varphi(\sigma_t(x)) = (\sigma_t(x)\xi_0|\xi_0) = (\Delta^{it}x\Delta^{-it}\xi_0|\xi_0) =$$
$$= (x\Delta^{-it}\xi_0|\Delta^{-it}\xi_0) = (x\xi_0|\xi_0) = \varphi(x).$$

Second, we want to rewrite the equality

$$\varphi(xa) = \varphi(\sigma_i(a)x), \quad (x\in M, a\in A),$$

without using any analytical extension. To this end let us write explicitely the equality

$$\varphi(x\sigma_{-i\alpha}(a)) = \varphi(\sigma_i(\sigma_{-i\alpha}(a))x) = \varphi(\sigma_{i(1-\alpha)}(a)x)$$

(We have replaced a by $\sigma_{-i\alpha}(a)$, which is the same, since $a\in A$).

Thus:

$$(x\sigma_{i\alpha}(a)\xi_0|\xi_0) = (\sigma_{i(1-\alpha)}(a)x\xi_0|\xi_0)$$
$$(x\Delta^{\alpha}a\Delta^{-\alpha}\xi_0|\xi_0) = (\Delta^{\alpha-1}a\Delta^{1-\alpha}x\xi_0|\xi_0)$$

(*) $$(x\Delta^{\alpha}a\xi_0|\xi_0) = (x\xi_0|\Delta^{1-\bar\alpha}a^*\xi_0)$$

But, in (*)

– the left hand side is meaningful even if $a\in M$ is an arbitrary element and $0\le \operatorname{Re}\alpha \le \frac{1}{2}$ since $a\xi_0 \in D(\Delta^{1/2}) = D(S)$;

– the right hand side is meaningful even if $a\in M$ is an arbitrary element and $0 \le \operatorname{Re}(1-\bar\alpha) \le \frac{1}{2}$, i.e. $\frac{1}{2}\le \operatorname{Re}\alpha \le 1$ since $a^*\xi_0 \in D(\Delta^{1/2}) = D(S)$.

Thus, due to the equality (*), we can define a function
$$f : \{\alpha \in \mathbb{C} ; 0 \leq \operatorname{Re}\alpha \leq 1\} \longrightarrow \mathbb{C}$$
continuous and bounded on this strip and analytic in its interior, by putting
$$f(\alpha) = \begin{cases} (x\,\Delta^{\alpha} a\xi_0 \mid \xi_0) & \text{if } 0 \leq \operatorname{Re}\alpha \leq \frac{1}{2} \\ (x\xi_0 \mid \Delta^{1-\bar{\alpha}} a^*\xi_0) & \text{if } \frac{1}{2} \leq \operatorname{Re}\alpha \leq 1 \end{cases}$$
and we have
$$f(it) = (x\,\Delta^{it} a\xi_0 \mid \xi_0) = \varphi(x\,\sigma_{i(it)}(a)) = \varphi(x\,\sigma_t(a))$$
$$f(1+it) = (x\xi_0 \mid \Delta^{1-(1+it)} a^*\xi_0) = \varphi(\sigma_{i(1-(1+it))}(a)\,x)$$
$$= \varphi(\sigma_t(a)\,x)$$
(for $a, x \in M$ arbitrary elements, f depending on a, x).

Therefore

(2) for every $x, a \in M$ there exists a function
$$f : \{\alpha \in \mathbb{C} ; 0 \leq \operatorname{Re}\alpha \leq 1\} \longrightarrow \mathbb{C}$$
continuous, bounded and analytic in the interior, s.t.
$$f(it) = \varphi(x\,\sigma_t(a)) \;,\; f(1+it) = \varphi(\sigma_t(a)\,x)$$

The conditions (1) (φ is σ_t-invariant) and (2) are called the <u>KMS conditions</u> (Kubo, Martin, Schwinger - physicists who have discovered it in physical considerations, before the Tomita theory).

2° The KMS-conditions characterize the modular automorphism group, i.e. $\{\sigma_t^{\varphi}\}_{t\in\mathbb{R}}$ is the only continuous action $\sigma : \mathbb{R} \longrightarrow \mathrm{Aut}(M)$ satisfying (1) and (2).

PROOF. Due to condition (1), the mappings
$$M\xi_0 \ni x\xi_0 \longmapsto \sigma_t(x)\xi_0 \in M\xi_0 \subset H \;,\; (t \in \mathbb{R}),$$
define an so-continuous unitary representation
$$u : \mathbb{R} \ni t \longmapsto u_t \in B(H) \;,\; u_t x\xi_0 = \sigma_t(x)\xi_0 \;,\; (x \in M, t \in \mathbb{R}).$$
For $a \in M$ we have
$$u_t S a\xi_0 = u_t a^*\xi_0 = \sigma_t(a^*)\xi_0 = \sigma_t(a)^*\xi_0 = S u_t a\xi_0$$

Since $S = \overline{S|M\xi_0}$, it follows that

u_t commutes with S, u_t commutes with $\Delta = S^*S$.

Using condition (2) we see that for every $a, x \in M$ there exists a function $f : \{\alpha \in \mathbb{C} ; 0 \leq \mathrm{Re}\,\alpha \leq 1\} \to \mathbb{C}, \ldots$, such that

$$f(it) = \varphi(x^* \sigma_t(a)) = (\sigma_t(a)\xi_0 | x\xi_0) = (u_t a\xi_0 | x\xi_0)$$

$$f(1+it) = \varphi(\sigma_t(a)x^*) = (x^*\xi_0 | \sigma_t(a)^*\xi_0) = (Sx\xi_0 | u_t S a\xi_0)$$

that is:

(3) $$f(it) = (u_t \zeta | \xi), \ f(1+it) = (S\xi | u_t S\zeta) \text{ for } \zeta = a\xi_0 \in M\xi_0, \xi = x\xi_0 \in M\xi_0$$

Since $S = \overline{S|M\xi_0}$, using the Phragmen-Lindelöf principle we see that (3) holds for every $\zeta, \xi \in D(S)$ (i.e. the function f exists). If, moreover, $\zeta \in D(\Delta)$ then

(4) $$f(it) = (u_t\zeta|\xi), \ f(1+it) = (u_t\Delta\zeta|\xi), \ (\zeta \in D(\Delta), \xi \in D(S))$$

Since $D(S)$ is dense in H, these equalities remain valid (the Phragmen-Lindelöf principle gives again a function f) with $\zeta \in D(\Delta)$ and $\xi \in H$ arbitrary. Also:

(5) $$|f(\alpha)| \leq \max\{\|\zeta\|, \|\Delta\zeta\|\} \cdot \|\xi\|.$$

Let A be the unique positive selfadjoint operator given by the Stone's theorem such that

$$A^{it} = u_t, \ (t \in \mathbb{R}).$$

We want to show that $\Delta \subset A$. So, let $\zeta \in D(\Delta)$. From (4), (5) it follows that the map $i\mathbb{R} \ni it \longmapsto u_t\zeta$ has an analytic extension to the strip $\{\alpha \in \mathbb{C} ; 0 \leq \mathrm{Re}\,\alpha \leq 1\}$, which means that $\zeta \in D(A)$, and the value of this extension at 1 is $\Delta\zeta$, which means that $A\zeta = \Delta\zeta$. Hence $\Delta \subset A$ indeed. It follows that $A \subset \Delta$ by adjunction, hence $\Delta = A$, and so:

$$\sigma_t(x)\xi_0 = u_t x\xi_0 = A^{it}x\xi_0 = \Delta^{it}x\xi_0 = \sigma_t^\varphi(x)\xi_0$$

hence $\sigma_t(x) = \sigma_t^\varphi(x)$ $(\forall)\, x \in M, t \in \mathbb{R}$, i.e. $\sigma_t = \sigma_t^\varphi$, $(t \in \mathbb{R})$, which concludes the proof.

3° The same result is true if, instead of $\varphi = \omega_{\xi_0}$, we consider an n.s.f. weight φ on M. In this case the condition (2), i.e. the KMS-condition reads:

$$(2') \quad \left[\begin{array}{l} \text{for every } x, a \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^* \text{ there exists a function} \\ \qquad f : \{\alpha \in \mathbb{C} ; 0 \le Re\,\alpha \le 1\} \to \mathbb{C} \\ \text{continuous, bounded and analytic in the interior, s.t.} \\ \qquad f(it) = \varphi(x\sigma_t(a)), \; f(1+it) = \varphi(\sigma_t(a)x) \end{array}\right.$$

and the proof is similar to the proof in the bicyclic case (Lectures, 10.17).

4° When regarded abstractly as (M, φ) we denote by M^φ the centralizer of $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$:

$$M^\varphi = \{ a \in M, \; \sigma_t(a) = a, \; (\forall) t \in \mathbb{R} \}$$

PROPOSITION: $M^\varphi = \{ a \in M ; \varphi(xa) = \varphi(ax), (\forall) x \in M \}$

PROOF. If $a \in M^\varphi$, then $a \in A$ is an entire analytic element and $\sigma_\alpha(a) = a \; (\forall) \alpha \in \mathbb{C}$, in particular $\sigma_i(a) = a$, hence

$$\varphi(xa) = \varphi(\sigma_i(a)x) = \varphi(ax) \quad (\forall) x \in M$$

using $\overline{\text{V}}$.4°(1).

Conversely, if $\varphi(xa) = \varphi(ax)$, $(\forall) x \in M$ we get

$$\varphi(xa) = \varphi(ax) = \varphi(\sigma_i(x)a)$$

for any analytic element x, hence

$$\varphi(\sigma_\alpha(x)a) = \varphi(\sigma_{\alpha+i}(x)a), \quad (\alpha \in \mathbb{C}).$$

Thus, the entire analytic function $\mathbb{C} \ni \alpha \mapsto \varphi(\sigma_\alpha(x)a) \in \mathbb{C}$ is periodic, with period i. But, for $0 \le Im\,\alpha \le 1$ this function is bounded:

$$|\varphi(\sigma_\alpha(x)a)| \le \|\sigma_\alpha(x)\| \|a\| \|\varphi\| = \|\sigma_{i\,Im(\alpha)}(x)\| \|a\| \|\varphi\|$$

$$\le \sup_{0\le s\le 1} \|\sigma_{is}(x)\| \|a\| \|\varphi\|.$$

Thus it is globally bounded and hence constant by the Liouville's theorem. Therefore

$$\varphi(xa) = \varphi(\sigma_{-t}(x)a) = \varphi(x\sigma_t(a)).$$

The equality $\varphi(x(a - \sigma_t(a))) = 0$ can now be extended from analytic elements to arbitrary elements $x \in M$ and hence $a - \sigma_t(a) = 0$, $\sigma_t(a) = a$, $(t \in \mathbb{R})$, and $a \in M^\varphi$

For an n.s.f weight φ on M the corresponding statement is that, for an element $a \in M$, we have $a \in M^{\varphi}$ if and only if

$$a\,\mathfrak{M}_{\varphi} \subset \mathfrak{M}_{\varphi}\,,\ \mathfrak{M}_{\varphi}\,a \subset \mathfrak{M}_{\varphi} \text{ and } \varphi(ax) = \varphi(xa),\ (\forall)\, x \in \mathfrak{M}_{\varphi}$$

and the proof is similar (Lectures 10.27).

Thus, $\{\sigma_t^{\varphi}\}_{t\in\mathbb{R}}$ and M^{φ} have „finite" characterizations not involving unbounded operators or analytical extensions of operator functions.