VIII Radon-Nikodym type theorems.

1° We begin with a contluction and a general theorem due to Pedersen and Takesaki.

Let φ be an n.s.f weight. The caracterization of the centralizer M^φ in this general case is

(*) $$a \in M^\varphi := \{a \in M;\ \sigma_t^\varphi(a) = a,\ (\forall) t \in \mathbb{R}\} \iff$$
$$\iff a\,\mathfrak{M}_\varphi \subset \mathfrak{M}_\varphi,\ \mathfrak{M}_\varphi a \subset \mathfrak{M}_\varphi \text{ and } \varphi(ax) = \varphi(xa),\ (\forall) x \in \mathfrak{M}_\varphi$$

For $a \in M^\varphi$, $a \geq 0$, a new weight φ_a on M is defined by

(1) $$\varphi_a(x) = \varphi(a^{1/2} x a^{1/2}) \ ; \ x \in M^+.$$

It is clear that φ_a is normal. Since $a \in M^\varphi$ we have $\mathfrak{N}_\varphi \subset \mathfrak{N}_{\varphi_a}$, $\mathfrak{M}_\varphi \subset \mathfrak{M}_{\varphi_a}$ and

(2) $$\varphi_a(x) = \varphi(ax) = \varphi(xa) \ , \quad (x \in \mathfrak{M}_\varphi)$$

In particular φ_a is semifinite. Also $s(\varphi_a) = s(a)$ for supports. If a is invertible, then

(3) $$\sigma_t^{\varphi_a}(x) = a^{it}\sigma_t^\varphi(x)\,a^{-it} \quad (x \in M,\ t \in \mathbb{R})$$

This follows using the KMS-conditions or even by a direct argument using the constuction of the modular automorphisms group (MOD.TH. 4.2 pp. 62-63).

2° If $a, b \in M^\varphi$, $a \geq 0$, $b \geq 0$ then

(1) $$\varphi_{a+b} = \varphi_a + \varphi_b .$$

Indeed, if a and b are invertible then
$$\mathfrak{M}_{\varphi_a} = \mathfrak{M}_{\varphi_b} = \mathfrak{M}_{\varphi_{a+b}} = \mathfrak{M}_\varphi$$
and for every $x \in \mathfrak{M}_\varphi$ we have
$$\varphi_{a+b}(x) = \varphi((a+b)x) = \varphi(ax) + \varphi(bx) = \varphi_a(x) + \varphi_b(x)$$

In the general case consider the elements
$$u = w\text{-}\lim_{\varepsilon \to 0} a^{1/2}(a+b+\varepsilon)^{-1/2}, \quad v = w\text{-}\lim_{\varepsilon \to 0} b^{1/2}(a+b+\varepsilon)^{-1/2}$$

Then: $u, v \in M^\varphi$ and
$$a^{1/2} = u(a+b)^{1/2},\ b^{1/2} = v(a+b)^{1/2} \text{ and } u^*u + v^*v = s(a+b)$$

If $x \in \mathfrak{M}_{\varphi_{a+b}} \cap M^+$, then $y := (a+b)^{1/2} x (a+b)^{1/2} \in \mathfrak{M}_\varphi$ and $uyu^*, vyv^* \in \mathfrak{M}_\varphi$, that is

$$+\infty > \varphi(uyu^*) + \varphi(vyv^*) = \varphi(a^{1/2} x a^{1/2}) + \varphi(b^{1/2} x b^{1/2})$$

and hence, using once more $1^\circ(*)$,

$$\varphi_a(x) + \varphi_b(x) = \varphi(y(u^*u + v^*v)) = \varphi(y) = $$
$$= \varphi_{a+b}(x)$$

Conversely, let $x \in \mathfrak{M}_{\varphi_a + \varphi_b} \cap M^+$. Then $a^{1/2} x a^{1/2} \in \mathfrak{M}_\varphi \cap M^+$, $b^{1/2} x b^{1/2} \in \mathfrak{M}_\varphi \cap M^+$ and with a further application of $1^\circ(*)$ we infer that

$$u^* a^{1/2} x a^{1/2} u \in \mathfrak{M}_\varphi \cap M^+, \quad v^* b^{1/2} x b^{1/2} v \in \mathfrak{M}_\varphi \cap M^+$$

As:

$$(a+b)^{1/2} x (a+b)^{1/2} = w\text{-}\lim_{\varepsilon \to 0} (a+b+\varepsilon)^{-1} (a+b) x (a+b) (a+b+\varepsilon)^{-1}$$
$$\leq w\text{-}\lim_{\varepsilon \to 0} 2 (a+b+\varepsilon)^{-1} (axa + bxb)(a+b+\varepsilon)^{-1}$$
$$= 2 [u^* a^{1/2} x a^{1/2} u + v^* b^{1/2} x b^{1/2} v)$$

it follows that $x \in \mathfrak{M}_{\varphi_{a+b}} \cap M^+$. This completes the proof of $2^\circ(1)$.

From (1) it follows that

(2) $$0 \leq a \leq b \Rightarrow \varphi_a \leq \varphi_b. \quad (\text{for } a, b \in \mathfrak{M}_\varphi)$$

and from this we get, for $a \in M^\varphi$, $a_i \in M^\varphi$,

(3) $$0 \leq a_i \uparrow a \Rightarrow \varphi_{a_i} \uparrow \varphi_a$$

Indeed, let $x \in M^+$. From (2) we get the inequality

$$\sup_i \varphi_{a_i}(x) \leq \varphi_a(x).$$

On the other hand, since φ is normal and $a_i^{1/2} x a_i^{1/2} \xrightarrow{w} a^{1/2} x a^{1/2}$ we get

$$\varphi_a(x) = \varphi(a^{1/2} x a^{1/2}) \leq \liminf_i \varphi(a_i^{1/2} x a_i^{1/2})$$
$$\leq \sup_i \varphi_{a_i}(x)$$

3°. Let φ be an nsf weight on the von Neumann algebra $M \subset B(H)$ and A be a positive selfadjoint operator in H affiliated to M^{φ}. Consider the bounded positive operators

$$A_{\varepsilon} = A(1+\varepsilon A)^{-1} \in M^{\varphi}, \quad (\varepsilon > 0).$$

and notice that $A_{\varepsilon} \uparrow A$ for $\varepsilon \downarrow 0$. Also, let

$$e_n = \chi_{(\frac{1}{n}, n)}(A) \in M^{\varphi}$$

and notice that Ae_n is a bounded positive operator, invertible in $e_n M^{\varphi} e_n$, and $e_n \uparrow s(A)$. By 2°(2), a normal weight φ_A on M is defined by

(1) $$\varphi_A(x) = \sup_{\varepsilon > 0} \varphi_{A_{\varepsilon}}(x) = \lim_{\varepsilon \to 0} \varphi_{A_{\varepsilon}}(x), \quad (x \in M^{+}).$$

If A is bounded we get the same thing as above by 2°(3).

If $x \in M^{+}$ and $s(x)s(A) = 0$, then clearly $\varphi_A(x) = 0$. On the other hand, if $x \in e_n(\mathfrak{M}_{\varphi} \cap M^{+})e_n \subset \mathfrak{M}_{\varphi} \cap M^{+} \subset \mathfrak{M}_{\varphi_{Ae_n}} \cap M^{+}$ then

$$\varphi_A(x) = \lim_{\varepsilon \to 0} \varphi(A_{\varepsilon}^{1/2} e_n x e_n A_{\varepsilon}^{1/2}) = \lim_{\varepsilon \to 0} \varphi((Ae_n)_{\varepsilon}^{1/2} x (Ae_n)_{\varepsilon}^{1/2}) =$$
$$= \varphi_{Ae_n}(x) < \infty.$$

It follows that φ_A is semifinite and $s(\varphi_A) \leq s(A)$. Actually:

(2) $$s(\varphi_A) = s(A)$$

Indeed, if $x \in M^{+}$ and $\varphi_A(x) = 0$, then for every $\varepsilon > 0$ we have $\varphi(A_{\varepsilon}^{1/2} x A_{\varepsilon}^{1/2}) = 0$ i.e. $A_{\varepsilon}^{1/2} x A_{\varepsilon}^{1/2} = 0$ (since we assumed φ faithful), hence $xA_{\varepsilon} = 0$, $x\, s(A_{\varepsilon}) = 0$ and finally $x\, s(A) = 0$.

Notice that for every $x \in \mathfrak{N}_{\varphi}$ we have

(3) $$\varphi_A(x^*x) = \| \pi_{\varphi}(A)^{1/2} J_{\varphi} x_{\varphi} \|_{\varphi}^{1/2}$$

Indeed:

$$\varphi_A(x^*x) = \lim_{\varepsilon \to 0} \varphi(A_{\varepsilon}^{1/2} x^* x A_{\varepsilon}^{1/2}) = \lim_{\varepsilon \to 0} \varphi(A_{\varepsilon} x^* x) =$$
$$= \lim_{\varepsilon \to 0} (x_{\varphi} \mid (xA_{\varepsilon})_{\varphi})_{\varphi} = \lim_{\varepsilon \to 0} (x_{\varphi} \mid J_{\varphi} \pi_{\varphi}(A_{\varepsilon}) J_{\varphi} x_{\varphi}) =$$
$$= \| \pi_{\varphi}(A)^{1/2} J_{\varphi} x_{\varphi} \|_{\varphi}^{2}.$$

4° PROPOSITION. $A \le B \iff \varphi_A \le \varphi_B$ $(A, B \ldots)$.

PROOF. If $A \le B$, then for every $\varepsilon > 0$ we have

(MOD. TH. A. 4) $A_\varepsilon \le B_\varepsilon$ so $\varphi_{A_\varepsilon} \le \varphi_{B_\varepsilon}$ and $\varphi_A \le \varphi_B$ by $3^\circ.(1)$.

Conversely, assume that $\varphi_A \le \varphi_B$. Let $\varepsilon > 0$ and $f_n = \chi_{(0,n)}(B) \in M^\varphi$, $(n \in \mathbb{N})$. Then for each $x \in \mathfrak{N}_\varphi$ we have (see $\underline{V}.5^\circ.(4)$):

$$(\pi_\varphi(f_n A_\varepsilon f_n) J_\varphi x_\varphi \mid J_\varphi x_\varphi) = \varphi_{A_\varepsilon}(f_n x^* x f_n) \le$$

$$\le \varphi_B(f_n x^* x f_n) = (\pi_\varphi(B f_n) J_\varphi x_\varphi \mid J_\varphi x_\varphi)$$

so that $f_n A_\varepsilon f_n \le B f_n$. Since $f_n \uparrow s(B) = s(\varphi_B) \ge$ $\ge s(\varphi_A) = s(A)$, it follows that $A_\varepsilon \le B$ and, since $A_\varepsilon \uparrow A$, we conclude $A \le B$.

5° PROPOSITION. $A_i \uparrow A \iff \varphi_{A_i}(x) \uparrow \varphi_A(x)$, $(\forall)\, x \in M^+$ $(A_i, A \ldots)$

PROOF. Assume $A_i \uparrow A$ and let $x \in M^+$. From 4° it follows that $\sup_i \varphi_{A_i}(x) \le \varphi_A(x)$. On the other

(MOD. TH. A.5) hand, since $A_i \uparrow A$ we have also $(A_i)_\varepsilon \uparrow A_\varepsilon$ $(\forall)\, \varepsilon > 0$. hence

$$\sup_i \varphi_{(A_i)_\varepsilon}(x) = \varphi_{A_\varepsilon}(x), \quad (\forall)\, \varepsilon > 0$$

and consequently:

$$\varphi_A(x) = \sup_\varepsilon \varphi_{A_\varepsilon}(x) = \sup_\varepsilon \sup_i \varphi_{(A_i)_\varepsilon} \le \sup_i \varphi_{A_i}(x)$$

Conversely, if $\varphi_{A_i} \uparrow \varphi_A$ then, by 4°, it follows that $\{A_i\}_i$ is an increasing net bounded above by A. so there is a positive selfadjoint operator B such that $A_i \uparrow A$. It follows that B is affiliated to M^φ and, by the first part of the proof $\varphi_{A_i} \uparrow \varphi_B$. Consequently, $\varphi_A = \varphi_B$ and from (4) it follows that $A \le B$ and $B \le A$, hence $A = B$ and $A_i \uparrow A$.

In particular we have

(1) $$\varphi_A(x) = \sup_n \varphi_{Ae_n}(x) = \lim_n \varphi_{Ae_n}(x)$$

6° PROPOSITION. $\sigma_t^{\varphi_A}(x) = A^{it}\sigma_t^{\varphi}(x)A^{-it}$ $(x \in s(A)Ms(A),\ t\in\mathbb{R})$

PROOF. Let $M \subset B(H)$. Then $A|_{s(A)H}$ is a non-singular positive self-adjoint operator on the Hilbert space $s(A)H$ so that we can consider the unitary operators

$e_n = \chi_{(\frac{1}{n},n)}(A)$

$$A^{it} = (A|_{s(A)})^{it} \text{ on } s(A)H, \quad (t\in\mathbb{R}).$$

Of course, the operator A^{it} can also be regarded as a partial isometry in M.

Let $n \in \mathbb{N}$ be fixed. Then φ_{e_n} is an n.s.f. weight on e_nMe_n and Ae_n is a bounded invertibe positive operator in $e_nM^{\varphi}e_n$ which is the centralizer of the weight φ_{e_n}. Using 5°(1) we see that $(\varphi_A)_{e_n} = (\varphi_{e_n})_{Ae_n}$ as weights on e_nMe_n. Thus, taking into account 1°.(3), we get for $x \in e_nMe_n$

$$\sigma_t^{\varphi_A}(x) = \sigma_t^{(\varphi_A)_{e_n}}(x) = \sigma_t^{(\varphi_{e_n})_{Ae_n}} = (Ae_n)^{it}\sigma_t^{\varphi_{e_n}}(x)(Ae_n)^{-it}$$
$$= A^{it}e_n\sigma_t^{\varphi}(x)e_nA^{-it} = A^{it}\sigma_t^{\varphi}(x)A^{-it}$$

Since $e_n \uparrow s(A)$, the set $\bigcup_n e_nMe_n$ is w-dense in $s(A)Ms(A)$ and the desired conclusion follows.

7°. COROLLARY. $[D\varphi_A : D\varphi]_t = A^{it}$, $t\in\mathbb{R}$.

PROOF. Consider the balanced weights $\theta = \theta(\varphi,\varphi)$ and $\tau = \theta(\varphi,\varphi_A)$ on $Mat_2(M)$. Then

$$\tau = \theta_B \quad \text{with} \quad B = \begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}$$

hence

$$\sigma_t^{\tau}(X) = B^{it}\sigma_t^{\theta}(X)B^{-it} \quad (\forall)\ X \in s(B)\,Mat_2(M)\,s(B).$$

In particular, for $X = \begin{pmatrix}0 & 0\\ s(A) & 0\end{pmatrix}$ we obtain the announced result.

From this Corollary it follows again that

$$\varphi_A = \varphi_B \iff A = B.$$

8° COROLLARY. Let φ be an n.s.f. weight on the von Neumann algebra M and let A, B be two <u>commuting</u> positive self-adjoint operators affiliated to M^{φ}. Then the weights $\varphi_{\overline{AB}}$, $(\varphi_A)_B$, $(\varphi_B)_A$ are defined and equal

$$(\varphi_A)_B = \varphi_{\overline{AB}} = (\varphi_B)_A.$$

PROOF. Since A and B commute, the closure $\overline{AB}$ of AB

(MOD. TH. A6) is a positive self-adjoint operator and $(AB)^{it} = A^{it}B^{it}, t \in \mathbb{R}$. It is easy to check that $\overline{AB}$ (resp. A, B) is affiliated to the centralizer of φ (resp. φ_B, φ_A), hence the weights from the statement are indeed defined. Using 7° and the chain rule for the Connes derivative of these weights with respect to φ coincide and hence these weights are equal.

9° THEOREM (PEDERSEN-TAKESAKI). Let φ be an n.s.f weight and ψ a normal semifinite weight on M. The following conditions are equivalent

(i) $\psi \circ \sigma_t^{\varphi} = \psi \quad (\forall) t \in \mathbb{R}$.

(ii) $[D\psi : D\varphi]_t \in M^{\psi} \quad (\forall) t \in \mathbb{R}$

(iii) $[D\psi : D\varphi]_t \in M^{\varphi} \quad (\forall) t \in \mathbb{R}$

(iv) $\{[D\psi : D\varphi]_t\}_{t \in \mathbb{R}}$ is an s-continuous group of unitary elements of $s(\psi) M s(\psi)$

(v) there exists a positive self-adjoint operator A affiliated to M^{φ} such that

$$\psi = \varphi_A.$$

PROOF. Let $u_t = [D\psi : D\varphi]_t$, $(t \in \mathbb{R})$.

(i) $\Rightarrow$ (ii) From (i) it follows that $s(\psi)$ and M^{ψ} are σ^{φ}-invariant. In particular, for any $t \in \mathbb{R}$ we have

$$\sigma_t^{\varphi}(u_t) \sigma_t^{\varphi}(u_t)^* = s(\psi) = \sigma_t^{\varphi}(u_t)^* \sigma_t(u_t)$$

Then, for every $x \in (s(\psi) M s(\psi))^+$ and every $t \in \mathbb{R}$ we obtain

$$\psi(s(\psi) x s(\psi)) = \psi(x) = \psi(\sigma_t^\psi(x)) = \psi(u_t \sigma_t^\varphi(x) u_t^*) =$$
$$= \psi(\sigma_t^\varphi(\sigma_{-t}^\varphi(u_t) x \sigma_{-t}^\varphi(u_t^*))) = \psi(\sigma_{-t}^\varphi(u_t) x \sigma_{-t}^\varphi(u_t)^*)$$

By LEMMA VII.4° it follows that $\sigma_{-t}^\varphi(u_t) \in M^\psi$ and hence

$$u_t = \sigma_t^\varphi(\sigma_{-t}^\varphi(u_t)) \in M^\psi$$

(ii) $\Rightarrow$ (iv) From (ii) it follows that

$$u_t = \sigma_s^\psi(u_t) = u_s \sigma_s^\varphi(u_t) u_s^* = u_{s+t} u_s^*$$

hence $u_{s+t} = u_t u_s$, $(\forall)\, s,t \in \mathbb{R}$.

(iii) $\Leftrightarrow$ (iv) If $u_t \in M^\varphi$, then $u_{s+t} = u_s \sigma_s^\varphi(u_t) = u_s u_t$.
Conversely, if $u_s u_t = u_{s+t} = u_s \sigma_s^\varphi(u_t)$ then $u_t = \sigma_s^\varphi(u_t)$ hence $u_t \in M^\varphi$

(iv) $\Rightarrow$ (v) If (iv) holds, then by the Stone's Theorem there is a positive self-adjoint operator A affiliated to M^φ, with $s(A) = s(\psi)$ such that

$$[D\psi : D\varphi]_t = A^{it} = [D\varphi_A : D\varphi]_t$$

hence $\psi = \varphi_A$.

(v) $\Rightarrow$ (i) This follows from the definition of φ_A. Roughly:

$$\varphi_A(\sigma_t^\varphi(x)) = \varphi(A^{1/2} \sigma_t^\varphi(x) A^{1/2}) = \varphi(\sigma_t^\varphi(A^{1/2} x A^{1/2})) =$$
$$= \varphi(A^{1/2} x A^{1/2}) = \varphi_A(x).$$

Notice that, if ψ is also faithful then also the condition

(vi) $\qquad \varphi(\sigma_t^\psi(x)) = \varphi(x) \quad (\forall)\, x \in M^+, (\forall)\, t \in \mathbb{R}$

is equivalent to the above conditions, due to the symmetry of conditions (ii) and (iii)

In this case we say that the n.s.f. weights φ and ψ <u>commute</u>.

Thus, the n.s.f. weights of the form φ_A are exactly the n.s.f. weights commuting with φ.

10° COROLLARY. Let φ be an n.s.f. weight on the von Neumann algebra M with center $Z(M)$ and ψ be a normal semifinite weight on M.

The following conditions are equivalent:

(i) ψ satisfies the KMS conditions w.r.t. $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$

(ii) $s(\psi)\in Z(M)$ and $\sigma_t^\psi = \sigma_t^\varphi|_{Ms(\psi)}$, $(\forall)\, t\in\mathbb{R}$

(iii) there exists a positive self-adjoint operator A affiliated to $Z(M)$ such that $\psi = \varphi_A$

FOR SIMPLICITY ASSUME $q=1, p=0$

PROOF. Let $q = s(\psi)$ and $p = 1-q$.

(i) $\Rightarrow$ (ii) Since $\psi\circ\sigma_t^\varphi = \psi$, we have $p, q\in M^\varphi$. Let $x, y\in \mathfrak{N}_\psi\cap\mathfrak{N}_\psi^*$. Then

$\psi((xp)^*(xp)) = \psi(px^*xp) = 0$ and

$\psi((py)^*(py)) = \psi(y^*py) \le \psi(y^*y) < \infty$

hence $xp, py\in\mathfrak{N}_\psi\cap\mathfrak{N}_\psi^*$. By assumption there exists a function f, defined, continuous and bounded on the strip $\{\alpha\in\mathbb{C};\ 0\le \operatorname{Re}\alpha\le 1\}$, analytic in the interior, such that

$$\left.\begin{aligned} f(it) &= \psi(\sigma_t^\varphi(xp)py) = \psi(\sigma_t^\varphi(x)py) \text{ and}\\ f(1+it) &= \psi(py\,\sigma_t^\varphi(xp)) = 0 \end{aligned}\right\} (\forall)\, t\in\mathbb{R}$$

It follows that f is identically zero, in particular

$$\psi(\sigma_t^\varphi(x)py) = 0,\ (t\in\mathbb{R})$$

Consequently, $\psi(xpx^*) = 0$ for every $x\in\mathfrak{N}_\psi\cap\mathfrak{N}_\psi^*$, so that $xpx^* = pxpx^*p$ $(\forall)\, x\in M$. In particular, for $x = v\in M$ unitary, we have $vpv^*\le p$, $vpv^* = p$ and therefore $p\in Z(M)$ and $s(\psi) = q = 1-p\in Z(M)$.

Now the restriction of ψ to Mq is an n.s.f weight on Mq which satisfies the KMS condition w.r.t. $\{\sigma_t^\varphi|Mq\}_t$ so that $\sigma_t^\varphi|Mq = \sigma_t^\psi$.

(ii) $\Rightarrow$ (iii) From (ii) it follows that $\psi\circ\sigma_t^\varphi = \psi$, $(\forall)\, t\in\mathbb{R}$, so by the Theorem 9°,

$$\psi = \varphi_A$$

for some positive self-adjoint operator A affiliated to M^{φ}.
Then, for every $x \in M_{\varphi}$ and every $t \in \mathbb{R}$ we have

$$\sigma_t^{\varphi}(x) = \sigma_t^{\psi}(x) = A^{it}\sigma_t^{\varphi}(x)A^{-it} \quad (\forall) x \in M, (\forall) t \in \mathbb{R}$$

hence $A^{it} \in Z(M)$, $(\forall) t \in \mathbb{R}$, which means that A is affiliated to $Z(M)$

Finally, the implication (iii) => (ii) follows from 6° since $s(\psi) = s(A) \in Z(M)$, while the implication (ii) => (i) is obvious.

11° If φ is an n.s.f. trace on M, then $S_{\varphi} \equiv J_{\varphi}$ is a conjugation and $\Delta_{\varphi} = I$ so, identifying $x \equiv \pi_{\varphi}(x)$,

$$\sigma_t^{\varphi}(x) = \Delta_{\varphi}^{it} x \Delta_{\varphi}^{-it} = x \quad (\forall) x \in M, (\forall) t \in \mathbb{R}$$

hence $M^{\varphi} = M$ and $\sigma_t^{\varphi} = id$ $(\forall) t \in \mathbb{R}$. It follows that any other n.s.f. weight ψ commutes with φ.

Thus, from the Pedersen-Takesaki theorem 9° we get the following result originally due to I.E. Segal (for normal positive forms - not weights).

If φ is an n.s.f. trace on M, then any other normal semifinite weight ψ is of the form

$$\psi = \varphi_A \quad \text{with } A \text{ affiliated to } M \text{ a positive self-adjoint operator.}$$

12° If there is an n.s.f. weight φ on M and an s-continuous one parameter group $\mathbb{R} \ni t \mapsto u_t \in M$ of unitary elements such that

$$\sigma_t^{\varphi}(x) = u_t x u_t^* \quad (t \in \mathbb{R}, x \in M)$$

then the algebra M is semifinite.

Indeed let A^{-1} be the analytic generator of $(u_t)_{t \in \mathbb{R}}$, i.e. the unique positive self-adjoint operator A such that

$$u_t = A^{-it}; \quad t \in \mathbb{R}.$$

Then $\mu = \varphi_A$ satisfies

$$\sigma_t^{\mu}(x) = A^{it}\,\sigma_t^{\varphi}(x)\,A^{-it} = A^{it}\,u_t\,x\,u_t^{-1}\,A^{-it} =$$
$$= A^{it}A^{-it}\,x\,A^{it}A^{-it} = x \qquad (\forall)\, x\in M, (\forall)\, t\in \mathbb{R}$$

hence the centralizer $M^{\mu} = M$ which means that

$$a\in M,\ x\in \mathfrak{M}_{\mu} \Rightarrow ax, xa \in \mathfrak{M}_{\mu} \text{ and } \mu(ax) = \mu(xa)$$

i.e. μ is an n.s.f. trace on M, and hence M is semifinite.

There is also a different proof (see LECTURES 10.29).

13° If φ is an n.s.f. weight on M and $\sigma \in \mathrm{Aut}(M)$ is a $*$-automorphism of M, then

$$\sigma_t^{\varphi\circ\sigma} = \sigma^{-1}\circ\sigma_t^{\varphi}\circ\sigma\ , \quad (t\in\mathbb{R}).$$

as can be checked using the KMS-conditions. In particular,

$$M^{\varphi\circ\sigma} = \sigma^{-1}(M^{\varphi}).$$

Thus:

(1) $$\varphi\circ\sigma = \varphi \implies \sigma_t^{\varphi}\circ\sigma = \sigma\circ\sigma_t^{\varphi}\ ,\ (\forall)\, t\in\mathbb{R}$$

(2) $$\psi \text{ commutes with } \varphi \implies \sigma_t^{\varphi}\circ\sigma_s^{\psi} = \sigma_s^{\psi}\circ\sigma_t^{\varphi}$$

In general the converses are not true. We will provide an example and, on the other hand, supplimentary conditions that allow the converses.

14° Let $H = L^2(\mathbb{R})$ and $u_t, V_s \in B(H)$ unitary operators defined by

$$(u_t\xi)(r) = \xi(r+t)\ , \qquad (V_s\xi)(r) = e^{isr}\xi(r).$$

which satisfy the <u>anticommutation relations</u>:

(1) $$V_s u_t = e^{-its}u_t V_s \qquad (s,t\in\mathbb{R}).$$

By the Stone's theorem there are uniquely determined non-singular positive selfadjoint operators A, B in H such that

$$u_t = A^{it},\ (t\in\mathbb{R}),\quad V_s = B^{is},\ (s\in\mathbb{R}).$$

Thus $B^{is}A^{it} = e^{-its}A^{it}B^{is}$ and, using the definition of the analytic generators A, B (LECTURES, 9.20), we infer that

(2) $$B^{is}AB^{-is} = e^{s}A\ , \quad (s\in\mathbb{R}),$$

Consider the von Neumann algebra $M = B(H)$ with the canonical trace tr and the n.s.f. weights

$$\varphi = tr_A \ , \ \psi = tr_B \ \text{ on } M.$$

Then $\sigma_t^\varphi = Ad(u_t)$, $\sigma_s^\psi = Ad(v_s)$ and from (1) we get

(3) $$\sigma_t^\varphi \circ \sigma_s^\psi = \sigma_s^\psi \circ \sigma_t^\varphi$$

Moreover, φ and ψ do <u>not</u> commute, more precisely we have

(4) $$\varphi \circ \sigma_t^\psi = e^s \varphi \ , \ (s \in \mathbb{R}).$$

Indeed, let $x \in M^+$ and $A_\varepsilon = A(1+\varepsilon A)^{-1}$, $(\varepsilon > 0)$. From (2) it follows that

$$B^{-is} A_\varepsilon B^{is} = (e^s A)_\varepsilon$$

and therefore for $0 \leq x \in B(H)$:

$$\begin{aligned}\varphi(\sigma_s^\psi(x)) &= tr_A(B^{is} x B^{-is}) = \lim_{\varepsilon \to 0} tr\left(A_\varepsilon^{1/2} B^{is} x B^{-is} A_\varepsilon^{1/2}\right) \\ &= \lim_{\varepsilon \to 0} tr(x^{1/2} B^{-is} A_\varepsilon B^{is} x^{1/2}) \\ &= \lim_{\varepsilon \to 0} tr(x^{1/2} (e^s A)_\varepsilon x^{1/2}) \\ &= \lim_{\varepsilon \to 0} tr((e^s A)_\varepsilon^{1/2} x (e^s A)_\varepsilon^{1/2}) = tr_{e^s A}(x) = e^s \varphi(x)\end{aligned}$$

By extension two nsf weights φ, ψ whose modular automorphisms commute but φ, ψ do not commute, are said to anticommute.

14°. The following results are relevant mainly for factors.

PROPOSITION. Let φ be an nsf weight on the von Neumann algebra M and $\sigma : G \to Aut(M)$ an action of the group G on M by $*$-automorphisms σ_g, $(g \in G)$, acting identically on the center $Z(M)$. Then the following statements are equivalent:

(i) $\sigma_g \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \sigma_g$, $(\forall) g \in G$, $(\forall) t \in \mathbb{R}$

(ii) There is a group morphism $g \mapsto A_g$ of G into the group of all non-singular positive selfadjoint operators affiliated to $Z(M)$ such that

$(A,B) \mapsto \overline{AB}$

(1) $$\varphi \circ \sigma_g = \varphi_{A_g}$$

If, moreover, $G = \mathbb{R}$ and the action σ is continuous, then also the following statement is equivalent with (i) and (ii)

(iii) there is a nonsingular positive self-adjoint operator A affiliated $Z(M)$ such that

$$\varphi \circ \sigma_s = \varphi_{A^s}, \quad (s \in \mathbb{R}).$$

PROOF. By 10° and 13°.(1) it follows that (ii) $\Rightarrow$ (i) and from (i) it follows that there is a map $G \ni g \mapsto A_g \in{}'' Z(M)$ such that (1) holds and then, since each σ_g acts identically on the center

$$\varphi_{A_{gh}} = (\varphi \circ \sigma_g) \circ \sigma_h = \varphi_{A_g} \circ \sigma_h \overset{!}{=} (\varphi \circ \sigma_h)_{A_g} =$$

$$= (\varphi_{A_h})_{A_g} = \varphi_{\overline{A_g A_h}}$$

Concerning (iii) it is obvious that (iii) $\Rightarrow$ (ii). We show that (ii) $\Rightarrow$ (iii). By assumption there is a morphism $\mathbb{R} \ni s \longmapsto A_s$ such that $\varphi \circ \sigma_s = \varphi_{A_s}$, $(s \in \mathbb{R})$. Let $A = A_1$. Since $A_{s+t} = \overline{A_s A_t}$, it follows that $A_s = A^s$ for every rational number s. Let $e_n = \chi_{[\frac{1}{n}, n]}(A)$ and $x \in \mathfrak{N}_\varphi$. The function

$$(2) \qquad \mathbb{R} \ni s \longmapsto \varphi(A^s e_n x^* x) = \| \pi_\varphi(A^s e_n)^{1/2} x_\varphi \|_\varphi^2$$

is continuous and bounded. On the other hand we have:

$$\varphi(\sigma_s(e_n x^* x)) = \lim_{\varepsilon \to 0} \varphi((A_s)_\varepsilon e_n x^* x) =$$

$$= \lim_{\varepsilon \to 0} \| \pi_\varphi(A_s)_\varepsilon^{1/2} \pi_\varphi(e_n) x_\varphi \|_\varphi^2 = \| \pi_\varphi(A_s e_n)^{1/2} x_\varphi \|_\varphi^2 .$$

Since φ is normal, it follows that the function

$$(3) \qquad \mathbb{R} \ni s \longmapsto \varphi(\sigma_s(e_n x^* x)) = \| \pi_\varphi(A_s$$

is lower w-semicontinuous. As the functions (3) and (4) coincide for all rational s, it follows that

$$A_s e_n \leq A^s e_n, \quad (\forall)\, s \in \mathbb{R}, \ (\forall) n.$$

Consequently $s \longmapsto (A_s e_n)(A^s e_n)^{-1}$ is a one parameter group of positive operators with norm ≤ 1 on $e_n H$. Since the only such group is the trivial one it follows that

$$A_s e_n = A^s e_n \text{ and, since } e_n \uparrow 1, \ A_s = A^s.$$

a single $*$-automorphism corresponds to an action of $\mathbb{Z}$.

15° COROLLARY. Let φ be an normal faithful <u>state</u> on M and $\sigma : G \to Aut(M)$ an action of the group G on M which is trivial on $Z(M)$. Then are equivalent:

(i) $\varphi \circ \sigma_g = \varphi$, $(\forall)\, g \in G$

(ii) $\sigma_t^{\varphi} \circ \sigma_g = \sigma_g \circ \sigma_t^{\varphi}$, $(\forall)\, g \in G$, $(\forall)\, t \in \mathbb{R}$

<u>PROOF</u>. We must prove (ii) $\Rightarrow$ (i). By PROPOSITION 13° we can write

$$\varphi \circ \sigma_g = \varphi_{A_g} \quad (g \in G)$$

Fix $g \in G$, $\varepsilon > 0$ and $e_\varepsilon = \chi_{[1+\varepsilon,\infty)}(A_g)$. Then, for every $k \in \mathbb{N}$,

$$\|\varphi\| \geq \varphi(\sigma_g^k(e_\varepsilon)) = \varphi(A_g^k e_\varepsilon) \geq (1+\varepsilon)^k \varphi(e_\varepsilon)$$

hence $e_\varepsilon = 0$. It follows that $A_g \leq 1$ so $1 \leq A_g^{-1} = A_{g^{-1}} \leq 1$ and hence $A_g = 1$, $(\forall)\, g \in G$, which proves (i)

16° COROLLARY. Let φ, ψ be n.s.f weights on M. If φ is finite, or if $\varphi \leq \psi$, then are equivalent:

(i) φ commutes with ψ

(ii) $\sigma_t^{\varphi} \circ \sigma_s^{\psi} = \sigma_s^{\psi} \circ \sigma_t^{\varphi}$ $(\forall)\, s, t \in \mathbb{R}$.

PROOF. We already know that (i) $\Rightarrow$ (ii) and that (ii) $\Rightarrow$ (i) if φ is finite. Suppose $\varphi \leq \psi$. By the PROPOSITION 13° there is a nonsingular positive selfadjoint operator $A \,\eta\, Z(M)$ such that

$$\varphi \circ \sigma_s^{\psi} = \varphi_{A^s}, \quad (s \in \mathbb{R}).$$

Let $e_\varepsilon = \chi_{[1+\varepsilon,\infty)}(A)$. For $x \in \mathfrak{M}_\psi \cap M^+$ and $s \geq 0$ we have

$$+\infty > \psi(x) \geq \psi(e_\varepsilon x) = \psi(e_\varepsilon \sigma_s^{\psi}(x)) \geq \varphi(e_\varepsilon \sigma_s^{\psi}(x)) =$$

$$= \varphi(A^s e_\varepsilon x) \geq (1+\varepsilon)^s \varphi(e_\varepsilon x).$$

Consequently, $\varphi(e_\varepsilon x) = 0$. Since φ is normal and semifinite it follows that $\varphi(e_\varepsilon) = 0$, hence $e_\varepsilon = 0$, i.e. $A \leq I$. Similarly, using $f_\varepsilon = \chi_{[0,1-\varepsilon]}(A)$, we get $A \geq I$, hence $A = I$ which means that $\varphi \circ \sigma_s^{\psi} = \varphi$ $(\forall)\, s \in \mathbb{R}$, i.e. φ commutes with ψ.

17° Finally we record without proof (see MOD. TH. 4.19 - 4.22, pp. 71-74) the following results:

(1) If φ, ψ are normal positive forms on M and $s(\varphi)=1$, then are equivalent

(i) ψ commutes with φ

(ii) $|\varphi + i\psi| = |\varphi - i\psi|$

(iii) [assume $s(\psi)=1$] $\sigma_t^{\varphi} \circ \sigma_s^{\psi} = \sigma_s^{\psi} \circ \sigma_t^{\varphi}$

Thus, the commutation of normal faithful positive forms is the same as the commutation of the corresponding modular automorphisms. Condition (ii) recals a similar commutation condition in operator theory.

(2) If φ is an n.s.f. weight on M and $\sigma : G \to \mathrm{Aut}(M)$ is a continuous action of the <u>compact</u> group G on M, then are equivalent

(i) $\varphi \circ \sigma_g = \varphi$

(ii) $\sigma_t^{\varphi} \circ \sigma_g = \sigma_g \circ \sigma_t^{\varphi}$ $(\forall) g \in G$, $(\forall) t \in \mathbb{R}$.

The proof is similar to the proof in 16°.

(3) If φ, ψ are n.s.f weights on M and $\sigma_t^{\varphi} \circ \sigma_s^{\psi} = \sigma_s^{\psi} \circ \sigma_t^{\varphi}$ $(\forall) t, s \in \mathbb{R}$ then the sum $\varphi + \psi$ is a <u>semifinite</u> weight.

IX The Converse of the Connes Theorem

1°. Let G be a locally compact group and $\sigma: G \to Aut(M)$ a continuous action of G on M. Let e be the neutral element in G.

A σ-cocycle is an s^*-continuous function $w: G \to M$ such that:

NOTATION $Z_\sigma(G,M)$ all cocycles

$$w(gh) = w(g)\,\sigma_g(w(h)), \quad w(g^{-1}) = \sigma_g^{-1}(w(g)^*) \ ; \ g,h \in G.$$

In this case $w(g)$ are partial isometries and

$$w(g)w(g)^* = w(e), \quad w(g)^* w(g) = \sigma_g(w(e)) \ ; \ g \in G.$$

2° THEOREM. (A. CONNES) Let φ be a fixed n.s.f weight on M. For every cocycle $w \in Z_{\sigma^\varphi}(\mathbb{R},M)$ there is a unique normal semifinite weight ψ on M such that

$$[D\psi : D\varphi] = w.$$

This Theorem is a strong motivation to consider n.s.f weights, not only normal positive forms.

The proof is contained in the sections 2°–7°. We will consider

$$M \subset B(H), \quad F_\infty = B(L^2(\mathbb{R})), \quad M \bar{\otimes} F_\infty = Mat_\infty(M).$$

Thus every $x \in M \bar{\otimes} F_\infty$ is determined by a certain matrix $[x_{ij}]$ with entries in M:

$$x = \sum_{i,j} x_{ij} \otimes e_{ij}$$

where e_{ij} are the matrix units in F_∞. Let $e_n \in M\bar{\otimes} F_\infty$ be the element corresponding to the matrix $[x_{ij}]$ where

$x_{ij} = 0$ whenever $i \neq j$ or $i > n$ and $x_{11} = x_{22} = \dots = x_{nn} = 1$, that is $e_n = \sum_{k=1}^{n} 1 \otimes e_{kk}$.

We shall successively construct weights Φ, Φ', Ψ, Ψ' on $M\bar{\otimes} F_\infty$ and weights ψ', ψ on M.

NOTE. The proof that follows is done in full generality but it is easier to follow it assuming $w(t)$ unitary operators – then ψ will be an n.s.f. weight.

2°. Define an n.s.f. weight Φ on $M\bar{\otimes}F_\infty$ by

$$\Phi(x) = \sum_{k=1}^{\infty} \varphi(x_{kk}) \quad , \quad \left(x = [x_{ij}] \in M\bar{\otimes}F_\infty\right).$$

Then

$$\sigma_t^{\Phi} = \sigma_t^{\varphi} \otimes id \tag{1}$$

Indeed, it is easy to check that e_n belongs to the centralizer of Φ (using VIII. 1°. (*)). Then, using the characterization of the centralizer:

$$a \in M^{\varphi} \iff a\mathfrak{M}_\varphi \subset \mathfrak{M}_\varphi \,,\, \mathfrak{M}_\varphi a \subset \mathfrak{M}_\varphi \text{ and } \varphi(ax) = \varphi(xa), \quad \forall x \in \mathfrak{M}_\varphi$$

and VII. 5°. (3) we obtain

$$\sigma_t^{\Phi}(x) = \left(\sigma_t^{\varphi} \otimes id\right)(x), \quad (\forall)\, x \in e_n(M\bar{\otimes}F_\infty)e_n.$$

Since $e_n \uparrow 1$, this proves (1).

Actually $\Phi = \varphi \bar{\otimes} tr$, but we did not define the tensor product of weights.

3°. Let $\{u_t\}_{t\in\mathbb{R}} \subset B(L^2(\mathbb{R}))$ be the so-continuous one parameter unitary group of translations

$$(u_t\xi)(r) = \xi(r+t) \quad , \quad \xi \in L^2(\mathbb{R})$$

with the analytic generator A, $u_t = A^{it}$, $(t\in\mathbb{R})$. Then the weight

$$\Phi' = \Phi_{(1\otimes A)}$$

satisfies

$$\sigma_t^{\Phi'} = Ad(1\otimes u_t)\circ\sigma_t^{\Phi} = \sigma_t^{\varphi} \otimes Ad(u_t)$$

4° The von Neumann algebra $M\bar{\otimes}F_\infty$ acts on the Hilbert space $H\bar{\otimes}L^2(\mathbb{R})$ which can be identified with $L^2(\mathbb{R}, H)$ via the mapping

$$H\otimes L^2(\mathbb{R}) \ni \xi\otimes f \longmapsto (t \mapsto f(t)\xi) \in L^2(\mathbb{R}, H).$$

Using the given cocycle $w \in Z_{\sigma^\varphi}(\mathbb{R}, M)$ we define a partial isometry W on this Hilbert space by

$$(W\zeta)(t) = w(t)\zeta(t) \quad , \quad (\zeta \in L^2(\mathbb{R}, H),\, t\in\mathbb{R}).$$

It is immediate to check that W commutes with the commutant $M'\otimes\mathbb{C}$ of $M\bar{\otimes}F_\infty$ hence, by the

7° Let $w'(t) = [D\psi' : D\varphi]_t$ and $a(t) = w'(t)^* w(t)$, $(t \in \mathbb{R})$. Since

$$Ad(w(t)) \circ \sigma_t^\varphi = \sigma_t^{\psi'} = Ad(w'(t)) \circ \sigma_t^\varphi, \quad t \in \mathbb{R}$$

it follows that $(a(t))_{t\in\mathbb{R}}$ is an so-continuous one-parameter group of unitary operators in the <u>centre</u> of the algebra $s(\psi')Ms(\psi')$.

By the Stone's theorem there is a positive selfadjoint operator A in H, affiliated to the <u>centre</u> of M such that

$$a(t) = A^{it}, \quad (t \in \mathbb{R}).$$

Let $\psi = \psi'_A$. Then ψ is a normal semifinite weight on M and we have by the chain rule

$$[D\psi : D\varphi]_t = [D\psi'_A : D\psi']_t \, [D\psi' : D\varphi]_t =$$
$$= a(t)\, w'(t) = w'(t)\, a(t) = w(t).$$

This ends the proof of the Theorem.

8° Uffe Haagerup showed that the normality of a weight φ means that φ is the supremum of the normal positive forms majorized by φ. This is what we used in constructing the canonical left Hilbert algebra associated to an n.s.f. weight. The fact that for a normal semifinite weight we have the „property S":

$(\exists)$ f_i normal positive forms $(f_i)_{i\in I}$ such that
$$\varphi(x) = \sum_{i\in I} f_i(x) \; ; \; x \in M^+$$
was established later by Pedersen and Takesaki who noticed that

1) there is an n.s.f. weight with property S: if $\{\omega_i\}_{i\in I}$ is a maximal family of normal states with orthogonal supports, then

$$\varphi(x) = \sum_{i\in I} \omega_i$$

defines an n.s.f. weight with property S.

bicommutant theorem of von Neumann,

$$W \in M \overline{\otimes} F_\infty$$

Also;

$$W \sigma_t^{\Phi'}(W^*) = w(t) \otimes 1 \ , \ (t \in \mathbb{R}).$$

Indeed, W was defined by the function $s \mapsto w(s)$ and, similarly $\sigma_t^{\Phi'}(W^*)$ is defined by the function $s \mapsto \sigma_t^{\varphi}(w(s-t)^*)$ so that $W\sigma_t^{\Phi'}(W^*)$ is defined by the constant function

$$s \mapsto w(s)\,\sigma_t^{\varphi}(w(s-t)^*) = w(t)$$

so it is equal to $w(t) \otimes 1$.

We then define the normal semifinite weight

$$\Psi = \Phi'(W^* \cdot W) \text{ on } M \otimes F_\infty$$

Using the equality $\overline{VII}.4°.(2)$ we obtain

$$\sigma_t^{\Psi} = (Ad(w(t)) \circ \sigma_t^{\varphi}) \,\overline{\otimes}\, Ad(u_t)$$

5° Using the Pedersen - Takesaki construction again we obtain from Ψ a new normal semifinite weight Ψ' on $M \overline{\otimes} F_\infty$ such that

$$\sigma_t^{\Psi'} = (Ad(w(t)) \circ \sigma_t^{\varphi}) \otimes id, \ (t \in \mathbb{R}).$$

6° Let p be a minimal projection of F_∞. Using the mapping $x \mapsto x \otimes p$ we identify M with

$$(1 \otimes p)(M \overline{\otimes} F_\infty)(1 \otimes p)$$

It is clear that $1 \otimes p$ belongs to the centralizer of Ψ' hence

$$\psi' = \Psi'_{1 \otimes p} \qquad \text{(NOTATION AS IN } \overline{VII}.4°)$$

is a normal semifinite weight on

$$M = (1 \otimes p)(M \otimes F_\infty)(1 \otimes p)$$

and for $x \in s(\psi')Ms(\psi')$ we have ($\overline{VII}.4°$)

$$\sigma_t^{\psi'}(x) = \sigma_t^{\Psi'}(x \otimes p) = w(t)\,\sigma_t^{\varphi}(x)\,w(t)^* \otimes p =$$
$$= w(t)\,\sigma_t^{\varphi}(x)\,w(t)^*$$

hence

$$\sigma_t^{\psi'} = Ad(w(t)) \circ \sigma_t^{\varphi} \ , \ (t \in \mathbb{R}.$$

2) The Connes proof used the following operations with weights:

$$\varphi \mapsto \Phi = \varphi \bar{\otimes} tr$$
$$\varphi \mapsto \varphi_e \quad \text{for } e \in M^\varphi \text{ a projection}$$
$$\varphi \mapsto \varphi \circ \sigma \quad \text{for } \sigma \in Aut(M)$$
$$\varphi \mapsto \varphi_A \quad \text{for } A \text{ ,,}\in\text{'' } M^\varphi$$

and the point is that all these operations preserve the property S. This is obvious for the first three and also for the last if A is bounded. In the general case there is a sequence $\{e_n\}$ of mutually orthogonal spectral projections of A such that Ae_n are bounded and $\varphi_A = \sum_n \varphi_{Ae_n}$, thus every normal semifinite weight has property S. Indeed:

- Fix an n.s.f. weight φ on M with property S.
- Take another normal semifinite weight ψ on M and consider
$$w(t) = [D\psi : D\varphi]_t \in Z(\sigma^\varphi, M)$$
- Follow the Connes constuction to get a normal semifinite weight ψ' <u>with property</u> S such that
$$[D\psi' : D\varphi]_t = w(t)$$
- Then $[D\psi : D\varphi] = [D\psi' : D\varphi]$ hence $\psi = \psi'$ has property S.