PLANAR ALGEBRAS AND KUPERBERG'S 3-MANIFOLD INVARIANT

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Abstract. We recapture Kuperberg's numerical invariant of 3-manifolds associated to a semisimple and cosemisimple Hopf algebra through a 'planar algebra construction'. A result of possibly independent interest, used during the proof, which relates duality in planar graphs and Hopf algebras, is the subject of a final section.

1. Introduction

Throughout this paper, the symbol $\mathbf{k}$ will always denote an algebraically closed field and H will always denote a semisimple and cosemisimple Hopf algebra over $\mathbf{k}$. We use S to denote the antipodes of both H and its dual Hopf algebra H^*. The notations h and ϕ will be reserved for the unique two-sided integrals of H and H^* normalised to satisfy $\epsilon(h) = (dim\ H)1 = \phi(1)$ (in which case $\phi(h) = (dim\ H)1$). We will identify H with H^{**} and write the scalar obtained by pairing $x \in H$ with $\psi \in H^*$ as one of $\psi(x)$, $x(\psi)$, $\langle \psi, x\rangle$, or $\langle x, \psi\rangle$. Thus for instance, $\langle \psi, Sx\rangle = \langle x, S\psi\rangle$.

We will need the formalism of Jones' planar algebras. The basic reference is [Jns]. A somewhat more leisurely treatment of the basic notions may also be found in [KdySnd1]. (Mostly, we will follow the latter where, for instance, the $*$'s are attached to 'distinguished points' on boxes rather than to regions.)

While Vaughan Jones (who introduced planar algebras) mainly looked at 'C^*-planar algebras', which are *à fortiori* defined over $\mathbb{C}$, we will need to discuss planar algebras over fields possibly different from $\mathbb{C}$. We will, in particular, require some results from [KdySnd2] about the planar algebra $P = P(H)$ associated to a semisimple and cosemisimple Hopf algebra H over an arbitrary (algebraically closed) field. (To be entirely precise, we should call it $P(H, \delta)$, where δ is a solution in $\mathbf{k}$ of the equation $\delta^2 = (dim\ H)1$, as we have in [KdySnd2]; but we shall be sloppy and just write $P(H)$, with the understanding that one choice of a δ has been made as above.) In the sequel, we shall freely use 'planar algebra terminology' without any apology; explanations of such terminology can be found in [KdySnd1] or [KdySnd2].

This paper is devoted to showing that a 'planar algebra construction', when one works with the planar algebra $P = P(H)$, yields an alternative construction of Kuperberg's 'state-sum invariant' - see [Kpr] - of a closed 3-manifold associated with H.

We start with a recapitulation of Kuperberg's construction, which involves working with a Heegaard decomposition of the manifold. We describe Heegaard diagrams in some detail in the short §2. Another short section, §3, describes our planar algebra construction. A long §4 contains the details of the verification that the result of our construction agrees with that of Kuperberg's, and is consequently an invariant of the manifold.

Given a directed graph G embedded in an oriented 2-sphere, and a semisimple co-semisimple Hopf algebra H, we associate, in §5, two elements $V(G,H)$ and $F(G,H)$ of appropriate tensor powers of H. We show that $V(G,H)$ and $F(G,H^*)$ are related via the Fourier transform of the Hopf algebra H. Our initial verification that Kuperberg's invariant could be obtained by our planar algebraic prescription depended on this graph-theoretic result; what we have presented here is a shorter, cleaner version of the verification which only uses a special case (Corollary 3) of this result.

2. Kuperberg's invariant of 3-manifolds

In this section we describe Kuperberg's construction of his invariant. In addition to Kuperberg's original paper [Kpr], a very clear description of the invariant can be found in [BrrWst] which gives yet another construction.

The only 3-manifolds discussed here will be closed and oriented. Kuperberg's invariant (which is also defined for 3-manifolds that are not necessarily closed, though we restrict ourselves to these) is constructed from a Heegaard diagram of the 3-manifold. We recall - see [PrsSss] - that a Heegaard diagram consists of an oriented smooth surface Σ, say, of genus g, and two systems of smoothly embedded circles on Σ, which we will denote by $U^1,...,U^g$ and $L_1,...,L_g$ (to conform to Kuperberg's upper and lower circles), such that each is a non-intersecting system of curves that does not disconnect Σ. (Note that a system of g non-intersecting simple closed curves on a genus g surface will fail to disconnect it precisely when the complement of the union of small tubular neighbourhoods of the curves is a 2-sphere with $2g$-holes). However the U-circles and L-circles may well intersect but only transversally. There is a well-known procedure for constructing a 3-manifold from such data, and a theorem of Reidemeister and Singer specifies a set of moves under which two such Heegaard diagrams determine the same 3-manifold. It is a fact that either (i) reversing the orientation of Σ or (ii) interchanging the systems of U- and L-circles determines the oppositely oriented 3-manifold.

Consider now a genus g Heegaard diagram $(\Sigma, U^1,...,U^g, L_1,...,L_g)$. The computation of Kuperberg's invariant requires a choice of orientation and base-point on each of the circles $U^1,...,U^g, L_1,...,L_g$, so fix such a choice. We assume that none of the base-points is a point of intersection of a U- and an L-circle. Set $K^i_t = U^i \cap L_t, K^i = \coprod_t K^i_t, K_t = \coprod_i K^i_t, K = \coprod_{i,t} K^i_t$ and let k^i_t, k^i, k_t, k denote their cardinalities respectively[1]. Traverse the circles L_1 to L_g in order beginning from their base-points according to their orientation and index the points of intersection by the set $I_L = \{(t,p) : 1 \leq t \leq g, 1 \leq p \leq k_t\}$, with the lexicographic ordering of I_L agreeing with the order in which the points of K are encountered. Refer to this as the 'lower numbering' of the points of intersection. Next, traverse the circles U^1 to U^g the same way and index the points of intersection by the set $I^U = \{(i,j) : 1 \leq i \leq g, 1 \leq j \leq k^i\}$, with the lexicographic ordering of I^U agreeing with the order in which the points of K are encountered. Refer to this as the 'upper numbering' of the points of intersection. These give bijections $l : I_L \to K$ and $u : I^U \to K$.

Consider now the elements $\Delta_{k_1}(h) \otimes \cdots \otimes \Delta_{k_g}(h) \in H^{\otimes k}$ and $\Delta_{k^1}(\phi) \otimes \cdots \otimes \Delta_{k^g}(\phi) \in (H^*)^{\otimes k}$. Also consider, for each $q \in K$, the endomorphism T_q of H^* (or

[1] $\coprod$ denotes disjoint union.

of H) defined to be id or S according as the tangent vectors of the lower and upper circles at the point q, in that order, form a positively or negatively oriented basis for the tangent space at q to Σ. Kuperberg's invariant is obtained by pairing these off using the bijections l and u after twisting by the T_q.

Here, and elsewhere in this paper, we will find it convenient to use two bits of Hopf algebra notation: (i) superscripts indicate that multiple copies of Haar integrals are being used, while (ii) subscripts indicate use of our version of the so-called Sweedler notation for comultiplication - according to which we write, for example, $\Delta_n(x) = x_1 \otimes \cdots \otimes x_n$ rather than the more familiar $\Delta_n(x) = \sum_{(x)} x_{(1)} \otimes \cdots \otimes x_{(n)}$ in the interest of notational convenience.

Thus explicitly, suppose that c and d are the numbers of isolated U- and L-circles repectively in the Heegaard diagram. Then Kuperberg's invariant is given by the expression:

$$\delta^{-2g+2c+2d} \prod_{q \in K} \langle h^{t(q)}_{p(q)}, T_q \phi^{i(q)}_{j(q)} \rangle$$

where t, p and i, j are the obvious projection functions on I_L and I^U regarded as functions on K via the l and u identifications respectively. We may also rewrite this expression as

$$\delta^{-2g+2c} \prod_{t=1}^{g} h^t \left(\prod_{p=1}^{k_t} T_{l(t,p)} \phi^{i(l(t,p))}_{j(l(t,p))} \right). \tag{2.1}$$

Note that the δ^{2d} is absorbed into the product as those terms for which $k_t = 0$, each of which gives a $h^t(\epsilon) = \delta^2$.

That this expression is independent of the chosen base-points follows from the traciality of ϕ and h on H and H^* respectively while independence of the chosen orientations follows from the fact of S being an anti-algebra and anti-coalgebra map. The main result of [Kpr] is that this is a topological invariant of the 3-manifold determined by the Heegaard diagram and is, in a sense that is made precise there, complete. We note that Kuperberg's invariant is a 'picture invariant' in the sense of [DttKdySnd].

3. A planar algebra construction

In this section, we will describe our method of starting with a connected, spherical, non-degenerate planar algebra P with non-zero modulus δ, and associating a number to a Heegaard diagram with data $(\Sigma, U^1, ..., U^g, L_1, ..., L_g)$ as above.

Associated to such a Heegaard diagram is a certain planar diagram that conveys the same information. This is also often called a Heegaard diagram but in order to distinguish the two, we will refer to the latter picture as a planar Heegaard diagram. The planar Heegaard digram is obtained from the Heegaard diagram in the following way. Remove thin tubular neighbourhoods of the L-circles from Σ to get an oriented 2-sphere with $2g$ holes. Now a U-circle U^i becomes either (a) a simple closed curve on this sphere with holes - in case $k^i = 0$, or (b) a collection of k^i arcs with endpoints on the boundaries of the holes, if $k^i > 0$.

Fix a point on the sphere, and identify its complement with the plane - with anti-clockwise orientation - and finally arrive at the associated planar Heegaard diagram, which consists of the following data:

(1) a set of $2g$ of circles (the boundaries of the tubular neighbourhoods of the L-circles) that comes in pairs - two circles being paired off if they come from the same L-circle - and denoted $L_1^+, L_1^-, ..., L_g^+, L_g^-$ (with L_i^+ and L_i^- being paired for each i, and the choice of which to call $+$ and which $-$ being arbitrary);
(2) diffeomorphisms of L_t^+ onto L_t^- which reverse the orientations inherited by $L_t^\pm$ from the plane;
(3) collections of k_t distinguished points on each of L_t^+ and L_t^- - that are points of intersection with the U-curves - which are mapped to one another by the diffeomorphism of 2 above;
(4) a collection of curves - which we shall refer to as the strings of the diagram - which are either (a) entire U-circles which intersect no L-circles, or (b) arcs of U-curves terminating at distinguished points on the L-circles.

It is to be noted that the planar Heegaard diagram is specified by the associated Heegaard diagram together with a 'choice of point at infinity'.

From a planar Heegaard diagram we create a planar network in the sense of Jones. For this, we will first make a choice of base-points on all the circles $L_t^\pm$, taking care to ensure that (i) the base-points on L_t^+ and L_t^- correspond under the diffeomorphism (of 2 above) between $L_t^\pm$, and (ii) the base-points are not on the U-curves.

Next, thicken the U-curves of the planar Heegaard diagram to black bands. If the bands are sufficiently thin, no base-point on the L-circles will lie in a black region. We will refer to the L_t^+ as 'positive circles' and the L_t^- as 'negative circles'. Each of the positive and negative circles now has an even number of distinguished points on its boundary - these being the points of intersection of the boundaries of the black bands, i.e., the doubled U-curves, with the circles. For each circle $L_t^\pm$, start from its base-point and move clockwise until the first band is hit - at a distinguished point - and mark that point with a $*$. This yields a planar network in Jones' sense. Call it N.

The boxes of this network are the holes bounded by the circles $L_t^\pm$. There are $2g$ of them with colours $k_1, ..., k_g$, each occuring twice, and we denote these boxes by $B_t^\pm$. (Recall that k_t is the number of points of intersection of L_t with all the U-curves in the original Heegaard diagram.) Suppose that the boxes of N are ordered as $B_1^+, B_1^-, ..., B_g^+, B_g^-$. The number we wish to associate to the Heegaard diagram is given by the expression

$$\delta^{-(k_1+k_2+...+k_g)} Z_N^P(c_{k_1} \otimes ... \otimes c_{k_g}) \tag{3.2}$$

where Z_N^P is the partition function of the planar network N for the planar algebra P and $c_k \in P_k \otimes P_k$ is the unique element satisfying $(id \otimes \tau_k)((1 \otimes x)c_k) = x$ for all $x \in P_k$, and τ_k is the normalised 'picture trace' on the P_k. The element c_k is sometimes referred to as a quasi-basis for the functional τ_k on P_k - see [BhmNllSzl] - and its existence and uniqueness are guaranteed by the non-degeneracy of τ_k. It is true and easy to see that

$$c_k = \sum_{j \in J} f_j \otimes f^j. \tag{3.3}$$

whenever $\{f_j : j \in J\}$ and $\{f^j : j \in J\}$ are any pair of bases for P_k which are dual with respect to the trace τ_k meaning that

$$\tau_k(f_i f^j) = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases} .$$

We will show that when $P = P(H)$, the expression given by (3.2) agrees with Kuperberg's invariant.

We would like to remark that the expression given by (3.2) is independent of the chosen base-points (because c_k is invariant under $Z_R \otimes Z_{R^{-1}}$, where R is the k-rotation tangle) and also independent of the choice of which circles to call positive and which negative, due to the symmetry of c_k under the flip (which is an easy consequence of the traciality of τ_k).

4. Concordance with Kuperberg's construction

Our aim in this section is to show that when $P = P(H)$, the construction of §4 yields the same result as that of §3.

We begin by observing that the construction of the previous section makes perfectly good sense at the following level of generality. Let us say that a planar network is *box doubled* if there is given a fixed-point free involution on the set of its boxes which preserves colours, i.e., its boxes are paired off with each k-box being paired with another such. Suppose P is a connected, spherical, non-degenerate planar algebra with non-zero modulus δ and N is a box doubled planar network with $2g$ boxes; let $\sigma \in \Sigma_{2g}$ be any permutation with the property that the boxes $D_{\sigma(2l-1)}(N)$ and $D_{\sigma(2l)}(N)$ are paired off, and are of colour k_l, say, for $1 \leq l \leq g$. Then define

$$\tau^P(N) := \delta^{-(k_1+k_2+\ldots+k_g)} Z^P_{\sigma^{-1}(N)}(c_{k_1} \otimes \cdots \otimes c_{k_g}) \tag{4.4}$$

where, for $\pi \in \Sigma_n$, $\pi(N)$ refers to the network which is N, but with its boxes re-numbered according to π - see [KdySnd1].Thus, again by equation (2.3) of [KdySnd1], we have

$$\tau^P(N) := \delta^{-(k_1+k_2+\ldots+k_g)} Z^P_N(U_\sigma(c_{k_1} \otimes \cdots \otimes c_{k_g}) ,$$

where the notation U_σ refers, as in [KdySnd1], to the invertible operator $U_\sigma : \otimes_{i=1}^n V_i \to \otimes_{i=1}^n V_{\sigma^{-1}(i)}$ - between n-fold tensor products - defined by

$$U_\sigma(\otimes_{i=1}^n v_i) = \otimes_{i=1}^n v_{\sigma^{-1}(i)} .$$

The motivation for this definition, and in particular for the normalisation, comes from the $(1+1)$ TQFT of [KdyPtiSnd]. Symmetry of the c_{k_j} under the flip implies - as in §3 - that the definition $\tau^P(N)$ depends only on N, P, and on the pairing between the boxes of N, and not on the choice of the permutation σ above.

For the rest of this section, we assume that

(1) $P = P(H)$. (Recall that in this case $H = P_2$ with non-degenerate trace given by $\tau_2 = \delta^{-2}\phi$.)
(2) N is obtained from a planar Heegaard diagram D - and we assume that the choices of $L_t^{\pm}$ are made in such a way as to ensure that the orientation inherited by L_t^+ (resp., L_t^-) from the choice of orientation made for L_t in Kuperberg's construction is the clockwise (resp., anticlockwise) one.

(3) N has $2g$ boxes $B_1^+, B_1^-, ..., B_g^+, B_g^-$ in that order, where the $B_t^\pm$ have colour k_t and have been paired off as above, with the boundary of $B_t^\pm$ being identified with $L_t^\pm$. Thus, the boxes of N are naturally indexed by $X = \{(t,\epsilon) : 1 \leq t \leq g, \epsilon \in \{+,-\}\}$. (So, we may choose σ to be the identity permutation in the computation of $\tau^P(N)$.)

We will proceed to calculate $\tau^P(N)$ in several steps. Our first step will be to relate $\tau^P(N)$ and $\tau^P(\widetilde{N})$, where $\widetilde{N}$ is a box doubled planar network that contains only 2-boxes (and is built from N).

In an obviously suggestive notation, we set $\widetilde{N}$ to be the planar network defined by

$$\widetilde{N} = N \circ_{\{B_t^\epsilon : (t,\epsilon) \in X\}} (\{S(t,\epsilon)\}) ,$$

where $S(t,\epsilon)$ is defined to be C_{k_t} or $C_{k_t}^*$ according as $\epsilon = +$ or $\epsilon = -$, and the tangles C_k are defined in Figure 1 and their adjoint tangles are illustrated in Figure 2. Note that $\widetilde{N}$ is box doubled, by pairing off the p^{th} box of C_{k_t} with the p^{th} box

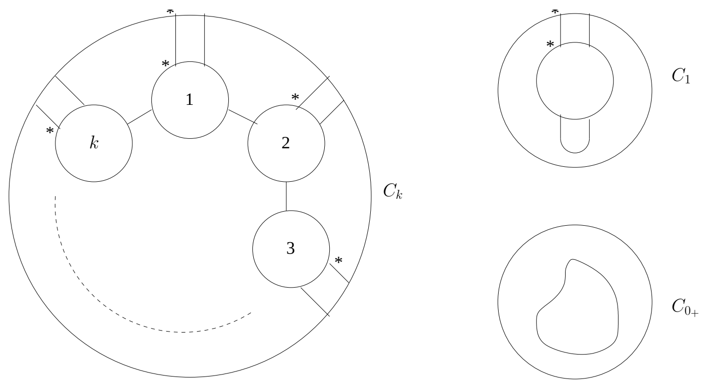


FIGURE 1. The tangles C_k for $k \geq 2$, $k = 1$ and $k = 0_+$

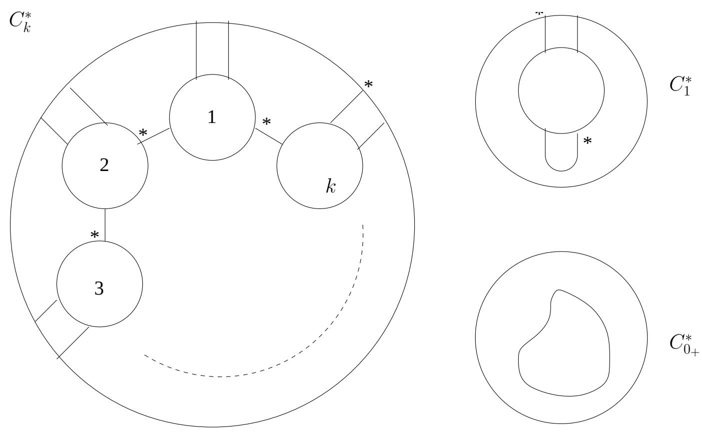


FIGURE 2. The tangles C_k^* for $k \geq 2$, $k = 1$ and $k = 0_+$

of $C^*_{k_t}$.

Our immediate aim is to prove, with the foregoing notation, that

$$\delta^{k_1+\cdots+k_g}\tau^P(\widetilde{N}) = \delta^{2g}\tau^P(N). \tag{4.5}$$

For this, we begin by noting that in P_2, we have

$$c_2 = h_1 \otimes Sh_2 = Sh_2 \otimes h_1. \tag{4.6}$$

In order to prove equation (4.6), note that, for all $x \in H$, we have

$$\begin{aligned}
(id_H \otimes \frac{1}{n}\phi)((1\otimes x)(h_1 \otimes Sh_2)) &= (id_H \otimes \frac{1}{n}\phi)(h_1 \otimes xSh_2)\\
&= (id_H \otimes \frac{1}{n}\phi)(h_1 x \otimes Sh_2)\\
&= \frac{1}{n}\phi(Sh_2)h_1 x\\
&= x\ ;
\end{aligned}$$

The second identity of equation (4.6) is established in similar fashion.

The next step towards proving equation (4.5) is to establish the following identity for $k = 0_+, 1, 2, \cdots$:

$$\delta^2 c_k = (Z_{C_k} \otimes Z_{C_k^*})(U_{\sigma_k}(c_2^{\otimes k})) \tag{4.7}$$

where $\sigma_k \in \Sigma_{2k}$ is the permutation defined by

$$\sigma_k = \left(\begin{array}{ccccccc} 1 & 2 & 3 & 4 & \cdots & 2k-1 & 2k \\ 1 & k+1 & 2 & k+2 & \cdots & k & 2k \end{array}\right).$$

Note that U_{σ_k} maps $H^{\otimes 2k}$ into itself, and we find from the definition that

(4.8)

$$U_{\sigma_k}(a(1)\otimes b(1)\otimes a(2)\otimes b(2)\otimes\cdots\otimes a(k)\otimes b(k)) = a(1)\otimes\cdots\otimes a(k)\otimes b(1)\otimes\cdots\otimes b(k)$$

for any $a(i), b(i) \in H$.

We shall now prove equation (4.7) for $k \geq 2$. The verification of the equation in the cases $k = 0_+$ and $k = 1$ is easy - and is a consequence of the facts $Z_{C_0}(1) = Z_{C_0^*}(1) = \delta 1_{0_+}$ and $Z_{C_1} = \epsilon(\cdot)1_1 = (\epsilon \circ S)(\cdot)1_1 = Z_{C_1^*}$.

We now wish to observe that what was called X_k in Lemma 5 of [KdySnd2] is nothing but the tangle $C_k \circ_k (1^2)$, so that $Z^P_{X_k}(\otimes_{i=1}^{k-1}(a(i)) = Z^P_{C_k}\big((\otimes_{i=1}^{k-1}(a(i))\otimes 1_H\big)$. It follows from Lemma 5 of [KdySnd2], that for $k \geq 1$, the LHS of equation (4.7) is given by

$$\begin{aligned}
\delta^2 c_k &= \delta^2 \sum_{\mathbf{i}\in I^{k-1}} Z_{C_k}(e_{i_1}\otimes\cdots\otimes e_{i_{k-1}}\otimes 1)\otimes Z_{C_k^*}(e^{i_1}\otimes\cdots\otimes e^{i_{k-1}}\otimes 1)\\
&= (Z_{C_k}\otimes Z_{C_k^*})\left[U_{\sigma_k}\left((\otimes_{j=1}^{k-1}(e_{i_j}\otimes e^{i_j}))\otimes(1\otimes 1)\right)\right] \text{ (by eq. (4.8))}\\
&= (Z_{C_k}\otimes Z_{C_k^*})\left[U_{\sigma_k}\left(c_2^{\otimes(k-1)}\otimes(1\otimes 1)\right)\right] \text{ (by eq. (3.3))}\\
&= (Z_{C_k}\otimes Z_{C_k^*})\left[U_{\sigma_k}\left(\otimes_{j=1}^{k-1}(h_1^j\otimes Sh_2^j)\otimes(1\otimes 1)\right)\right] \text{ (by eq. (4.6))}\\
&= \delta^2 Z_{C_k}(h_1^1\otimes h_1^2\otimes\cdots\otimes h_1^{k-1}\otimes 1)\otimes Z_{C_k^*}(Sh_2^1\otimes Sh_2^2\otimes\cdots\otimes Sh_2^{k-1}\otimes 1)\ .
\end{aligned}$$

(4.9)

On the other hand, equations (4.8) and (4.6) imply that the RHS of equation (4.7) is given by

$$Z_{C_k}(h_1^1 \otimes h_1^2 \otimes \cdots h_1^k) \otimes Z_{C_k^*}(Sh_2^1 \otimes Sh_2^2 \otimes \cdots Sh_2^k).$$

To proceed further, we need the following consequences of the so-called 'exchange relation' (see [Lnd] and [KdySnd2]) in $P(H)$:

$$\begin{aligned}
&Z_{C_k}(a(1) \otimes a(2) \otimes \cdots \otimes a(k)) \\
&\quad = \; Z_{C_k}(a(1)Sa(k)_{k-1} \otimes a(2)Sa(k)_{k-2} \otimes \cdots \otimes a(k-1)Sa(k)_1 \otimes 1) \\
&Z_{C_k^*}(a(1) \otimes a(2) \otimes \cdots \otimes a(k-1) \otimes Sa(k)) \\
&\quad = \; Z_{C_k^*}(a(k)_1 a(1) \otimes a(k)_2 a(2) \otimes \cdots \otimes a(k)_{k-1} a(k-1) \otimes 1)
\end{aligned}$$

for arbitrary $a(1), \cdots, a(k) \in H$. (It is still assumed that k is larger than 1.)

We may now deduce that the RHS of equation (4.7) is given by

$$\begin{aligned}
&Z_{C_k}(h_1^1 Sh_{k-1}^k \otimes h_1^2 Sh_{k-2}^k \otimes \cdots h_1^k Sh_1^k \otimes 1) \\
&\qquad \otimes Z_{C_k^*}(h_k^k Sh_2^1 \otimes h_{k+1}^k Sh_2^2 \otimes \cdots h_{2k-2}^k Sh_2^{k-1} \otimes 1) \\
&\quad = \; Z_{C_k}(h_1^1 h_k^k Sh_{k-1}^k \otimes h_1^2 h_{k+1}^k Sh_{k-2}^k \otimes \cdots h_1^{k-1} h_{2k-2}^k Sh_1^k \otimes 1) \\
&\qquad \otimes Z_{C_k^*}(Sh_2^1 \otimes Sh_2^2 \otimes \cdots Sh_2^{k-1} \otimes 1)
\end{aligned} \tag{4.10}$$

where we have used the Hopf algebra fact $xSh_2 \otimes h_1 y = Sh_2 \otimes h_1 xy$ in the last line above. Yet another Hopf algebra fact guarantees the equality of the right sides of equations (4.9) and (4.10); this other (easily established) fact is that

$$h_k Sh_{k-1} \otimes h_{k+1} Sh_{k-2} \otimes \cdots h_{2k-2} Sh_1 = \delta^2 \; 1^{\otimes(k-1)} \; .$$

Now for proving equation (4.5), note that

$$\begin{aligned}
\delta^{2g}\tau^P(N) &= \delta^{2g-(k_1+\cdots+k_g)} Z_N^P(c_{k_1} \otimes \cdots \otimes c_{k_g}) \\
&= \delta^{-(k_1+\cdots+k_g)} Z_N^P \left[\otimes_{t=1}^g \left(Z_{C_{k_t}}^P \otimes Z_{C_{k_t}^*}^P\right)(U_{\sigma_{k_t}}(c_2^{\otimes k_t}))\right] \text{(by eq. (4.7))} \\
&= \delta^{-(k_1+\cdots+k_g)} Z_{\widetilde{N}}^P(U_\sigma(c_2^{\otimes k})) \qquad (4.11) \\
&= \delta^{(k_1+\cdots+k_g)} \tau^P(\widetilde{N}) \; ,
\end{aligned}$$

where the last step uses the fact that one choice for the permutation $\sigma \in \Sigma_{2k}$ that is needed in the computation of $\tau^P(\widetilde{N})$ is given by $\sigma = \coprod_{i=1}^g \sigma_{k_i}$; and equation (4.5) has finally been established.

Next, note that K^i splits the U-circle U^i into k^i strings if $k^i > 0$ or into a single closed string if $k^i = 0$. For $(i,j) \in I^U$, define $e(i,j)$ to be the string bounded by $u(i,j-1)$ and $u(i,j)$. (The symbols l and u refer, of course, to the lower and upper numbering defined in §3. Further, we adopt the cyclic convention that $u(i,0) = u(i,k^i)$.) Orient each string of the diagram to agree with the choice of orientation of the U-circles in computing Kuperberg's invariant.

We shall use the symbol E to denote the set of non-closed strings of the diagram D and C to denote the set of closed strings. Thus $|C|$ is the number of isolated U-circles, which was earlier denoted by c. Note that each $e \in E$ comes equipped with the data of various features of its source and range; specifically, we shall write:

- $a(e)$ (resp., $z(e)$) for the point in K at which the string of the Heegard diagram which corresponds to e originates (resp., terminates); (these depend only on the original Heegaard diagram.)
- $\alpha(e)$ (resp., $\zeta(e)$) for 1 or 2 according as the string in D which corresponds to e originates (resp., terminates) in a positive or negative box; (these depend on the planar Heegaard diagram derived from the original Heegaard diagram.)

Note that, by definition,

$$z(e(i,j)) = a(e(i,j+1)) = u(i,j)\ \forall 1 \leq i \leq g, 1 \leq j \leq k^i \ , \tag{4.12}$$

with the convention that $e(i, k^i + 1) = e(i, 1)$. Note also that the maps

$$z, a : E \to K$$

are bijections and in particular, that $|E| = k$.

We will need to recall the definition and some basic properties of the Fourier transform map for a semisimple and cosemisimple Hopf algebra. This is the map $F : H \to H^*$ defined by $F(x) = \delta^{-1}\phi_1(x)\phi_2$. The properties that will be relevant for us are (i) $F \circ F = S$, (ii) $F \circ S = S \circ F$, (iii) $F(1) = \delta^{-1}\phi$ and $F(h) = \delta\epsilon$. An easily proved Hopf algebra result is:

$$(F \otimes F)(h_1 \otimes Sh_2) = (\phi_1 \otimes \phi_2). \tag{4.13}$$

We refer the reader to [KdySnd2] for an explanation of the notations involved and a proof of the following result which appears as Corollary 10 there.

Proposition 1. *Let $P = P(H)$ and $Q = P(H^*)$ for a semisimple and cosemisimple Hopf algebra H. Suppose that N is a planar network with g boxes all of which are 2-boxes. Then:*

$$Z_N^P = Z_{N^-}^Q \circ F^{\otimes g},$$

where both sides are regarded as $\mathbf{k}$-valued functions on $H^{\otimes g}$.

It follows from Proposition 1, equation (4.11) and equation (4.13) that

$$\begin{aligned} \tau^P(N) &= \delta^{-(2g+k_1+\cdots+k_g)} Z_{\widetilde{N}}^P(U_\sigma(c_2^{\otimes k})) \\ &= \delta^{-(2g+k_1+\cdots+k_g)} Z_{\widetilde{N}^-}^Q(U_\sigma((\phi_1 \otimes \phi_2)^{\otimes k})). \end{aligned} \tag{4.14}$$

We next apply Corollary 3 of [KdySnd2] in order to evaluate $Z_{\widetilde{N}^-}^Q(U_\sigma((\phi_1 \otimes \phi_2)^{\otimes k}))$. According to this, given a planar network with only 2-boxes that are labelled by elements of H, its partition function is computed by first replacing each 2-box labelled by a with a pair of strands, where the one going through $*$ is labelled a_1 and the other Sa_2. (This prescription was first outlined in the case of the group planar algebra in [Lnd].) The labels on each loop so formed are read in the order opposite to the orientation of the loop and $\delta^{-1}\phi$ evaluated on the product. The product of these terms over all loops is the required scalar. We assert that applied to $\widetilde{N}^-$, the number of loops formed is given by $2g + k + 2c$.

For instance consider the planar Heegaard diagram of $L(3,1)\#(S^2 \times S^1)$ - the connected sum of the lens space $L(3,1)$ and $S^2 \times S^1$ - shown in Figure 3. It consists of 2 U- and 2 L-curves. The L curves have their $\pm$ versions and are shown as dark circles along with basepoints chosen on $L_1^{\pm}$ (the others are irrelevant), while the U-curves are shown by lighter lines. One of the U curves is isolated (the one

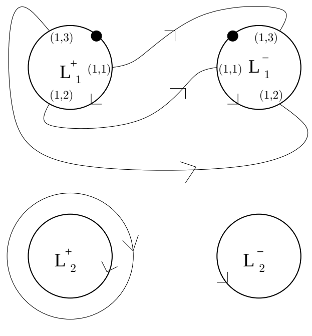


FIGURE 3. The planar Heegaard diagram for $L(3,1)\#(S^2 \times S^1)$

around L_2^+) while the other breaks up into 3 strings. The labellings of the points of intersection between the L- and U-curves is the 'lower numbering'.

The planar network $\widetilde{N}$ corresponding to this Heegaard diagram is shown in Figure 4. The planar network $\widetilde{N}^-$ is, by definition, obtained from $\widetilde{N}$ by moving all

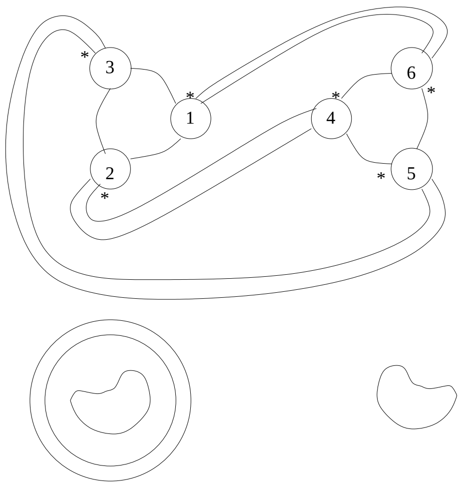


FIGURE 4. The planar network $\widetilde{N}$ for $L(3,1)\#(S^2 \times S^1)$

the $*$'s anticlockwise by one and therefore $Z^Q_{\widetilde{N}^-}(U_\sigma((\phi_1 \otimes \phi_2)^{\otimes 3}))$ in this example is given by the labelled planar network in Figure 5. Applying the procedure of Corollary 3 of [KdySnd2] to this labelled planar network yields the labelled loops as in Figure 6. It should now be clear why even in the general case, the number of loops obtained is $2g + k + 2c$.

Furthermore, a little thought shows that, as in this example, $Z^Q_{\widetilde{N}^-}(U_\sigma((\phi_1 \otimes \phi_2)^{\otimes k}))$ is the product of the following 4 types of terms:

(a) For each circle of the form L_t^+, a term $\delta^{-1}h^{(t,+)}(\prod_{p=1}^{k_t} \phi_1^{(t,p)})$,

(b) For each circle of the form L_t^-, a term $\delta^{-1}h^{(t,-)}(\prod_{p=1}^{k_t} S\phi_4^{(t,k_t+1-p)})$,

(c) For each closed string in C, a multiplicative factor of $(\delta^{-1}h(\epsilon))^2 = \delta^2$, and

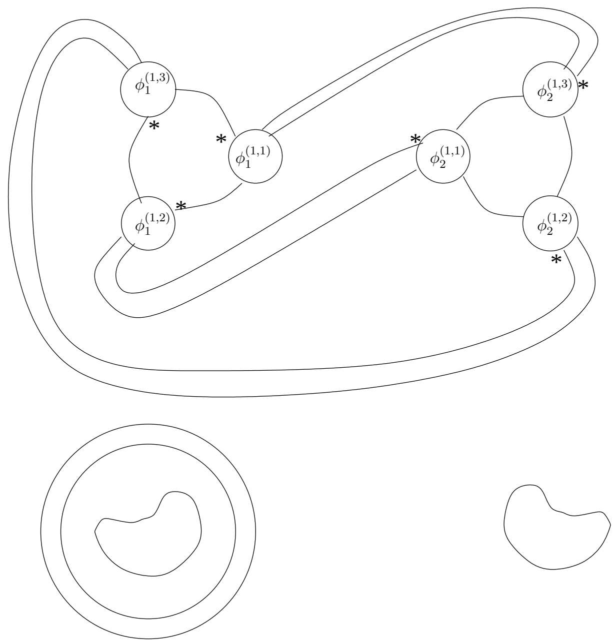


FIGURE 5. $Z^Q_{\tilde{N}^-}(U_\sigma((\phi_1 \otimes \phi_2)^{\otimes 3}))$

(d) For each non-closed string $e \in E$, a term of the form $\delta^{-1} h^e(T_a \phi^{(t_a,p_a)}_{\alpha(e)+1} T_z \phi^{(t_z,p_z)}_{\zeta(e)+1})$, where $l^{-1}(a(e)) = (t_a, p_a)$ and $l^{-1}(z(e)) = (t_z, p_z)$ and T_a (resp. T_z) is S or id according as e originates (resp. terminates) at a positive or negative box.
Note that (i) since the computation is being done in $Q = P(H^*)$, h and ϕ have interchanged roles, as have 1 and ϵ and (ii) the prescriptions of (a) and (b) also work for L_t's where $k_t = 0$ with the obvious interpretation of the empty product.

To summarise, we have seen that

$$
\begin{aligned}
&Z^Q_{\tilde{N}^-}(U_\sigma((\phi_1 \otimes \phi_2)^{\otimes k}))\\
&= \delta^{-2g+2c-k} \prod_{t=1}^{g} h^{(t,+)}(\prod_{p=1}^{k_t} \phi_1^{(t,p)}) \prod_{t=1}^{g} h^{(t,-)}(\prod_{p=1}^{k_t} S\phi_4^{(t,k_t+1-p)}) \prod_{e\in E} h^e(T_a \phi^{(t_a,p_a)}_{\alpha(e)+1} T_z \phi^{(t_z,p_z)}_{\zeta(e)+1})\\
&= \delta^{-2g+2c-k} \prod_{t=1}^{g} h^{(t,+)}(\prod_{p=1}^{k_t} \phi_4^{(t,p)}) \prod_{t=1}^{g} h^{(t,-)}(\prod_{p=1}^{k_t} S\phi_3^{(t,k_t+1-p)}) \prod_{e\in E} h^e(T_a \phi^{(t_a,p_a)}_{\alpha(e)} T_z \phi^{(t_z,p_z)}_{\zeta(e)})\\
&= \delta^{-2g+2c-k} \prod_{t=1}^{g} h^{(t,+)}(\prod_{p=1}^{k_t} \phi_4^{(t,p)}) \prod_{t=1}^{g} h^{(t,-)}(\prod_{p=1}^{k_t} S\phi_3^{(t,k_t+1-p)}) \prod_{e\in E} \phi^{(t_a,p_a)}_{\alpha(e)}(T_a h^e_1) \phi^{(t_z,p_z)}_{\zeta(e)}(T_z h^e_2)
\end{aligned}
$$

where the second equality is a consequence of an application of $\phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4 = \phi_4 \otimes \phi_1 \otimes \phi_2 \otimes \phi_3$ to each $\phi^{(t,p)}$. We are guilty of a little sloppiness in the equations above, since actually, $t_a, p_a, t_z, p_z, T_a, T_z$ are all functions of e; for instance, $t_a(e) = t(a(e))$ while

$$T_a(e) = ST_{a(e)}. \tag{4.15}$$

(The $T_{a(e))}$ on the right side of the last equation refers to the T_q used in §2.)

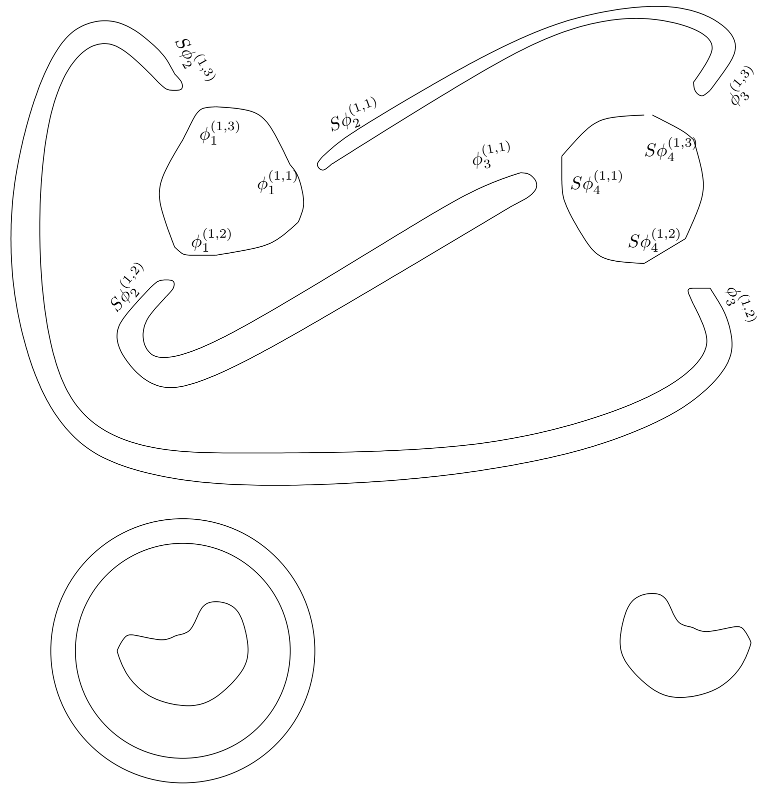


Figure 6. The labelled loops for Figure 5

Using the relations $Sh = h$ and $h^2 = \delta^2 h$, it is easy to see that

$$\prod_{t=1}^{g} h^{(t,+)} \left(\prod_{p=1}^{k_t} \phi_4^{(t,p)} \right) \prod_{t=1}^{g} h^{(t,-)} \left(\prod_{p=1}^{k_t} S\phi_3^{(t,k_t+1-p)} \right) = \delta^{2g} \prod_{t=1}^{g} h^t \left(\prod_{p=1}^{k_t} \phi_3^{(t,p)} \right),$$

and therefore we have:

$$\begin{aligned} (4.16) \qquad & Z^Q_{\widetilde{N}^-}(U_\sigma((\phi_1 \otimes \phi_2)^{\otimes k})) \\ & \quad = \delta^{2c-k} \prod_{t=1}^{g} h^t \left(\prod_{p=1}^{k_t} \phi_3^{(t,p)} \right) \prod_{e \in E} \phi_{\alpha(e)}^{(t_a,p_a)}(T_a h_1^e) \phi_{\zeta(e)}^{(t_z,p_z)}(T_z h_2^e). \end{aligned}$$

We will next analyse the terms in the product coming from $e \in E$ by grouping together those terms where the e's come from a single U-curve. In other words we write:

$$\prod_{e \in E} \phi_{\alpha(e)}^{(t_a,p_a)}(T_a h_1^e) \phi_{\zeta(e)}^{(t_z,p_z)}(T_z h_2^e) = \prod_{\{i: 1 \leq i \leq g, U^i \notin C\}} \prod_{e \subset U^i} \phi_{\alpha(e)}^{(t_a,p_a)}(T_a h_1^e) \phi_{\zeta(e)}^{(t_z,p_z)}(T_z h_2^e)$$

and for a fixed i such that $U^i \notin C$ (so that $k^i \neq 0$), consider the product $\prod_{e \subset U^i} \phi_{\alpha(e)}^{(t_a,p_a)}(T_a h_1^e) \phi_{\zeta(e)}^{(t_z,p_z)}(T_z h_2^e)$.

Now U^i comprises of the edges $e(i,j)$ where $1 \leq j \leq k^i$; suppose $a(e(i,j)) = l(t^i_{j-1}, p^i_{j-1})$ so that $u(i,j) = z(e(i,j)) = l(t^i_j, p^i_j)$ (with the convention that $(t^i_0, p^i_0) = (t^i_{k^i}, p^i_{k^i})$).

It follows - from equation (4.15) - that

$$\begin{aligned}
&\prod_{e \in U^i} \phi^{(t_a,p_a)}_{\alpha(e)}(T_a h^e_1)\phi^{(t_z,p_z)}_{\zeta(e)}(T_z h^e_2) \\
&= \prod_{j=1}^{k^i} \phi^{(t^i_{j-1},p^i_{j-1})}_{\alpha(e(i,j))}(ST_{a(e(i,j))}h^{e(i,j)}_1)\phi^{(t^i_j,p^i_j)}_{\zeta(e(i,j))}(ST_{z(e(i,j))}h^{e(i,j)}_2)\ .
\end{aligned}$$

After some minor rearrangement, this product may be rewritten as

$$\prod_{j=1}^{k^i} \phi^{(t^i_j,p^i_j)}_{\zeta(e(i,j))}(ST_{z(e(i,j))}h^{e(i,j)}_2)\phi^{(t^i_j,p^i_j)}_{\alpha(e(i,j+1))}(ST_{a(e(i,j+1))}h^{e(i,j+1)}_1)$$

The definitions show that the j^{th} term of the above product is $\phi_1^{(t^i_j,p^i_j)}(h^{e(i,j)}_2 S h^{e(i,j+1)}_1)$ or $\phi_1^{(t^i_j,p^i_j)}(h^{e(i,j+1)}_1 S h^{e(i,j)}_2)$ according as $e(i,j)$ terminates at a positive or negative circle. Finally, the product above may be written as:

$$\prod_{j=1}^{k^i} \phi_1^{(t^i_j,p^i_j)}(ST_{l(t^i_j,p^i_j)}(h^{e(i,j)}_2 S h^{e(i,j+1)}_1)).$$

Next, we appeal to Corollary 3 $(g-c)$ times - once for each non-isolated U^i. From that corollary, we get: $\otimes_{j=1}^{k^i} h^{e(i,j)}_2 S h^{e(i,j+1)}_1$ as $\delta^{k^i} F^{\otimes k^i}(\Delta_{k^i}\phi^i) = \otimes_{j=1}^{k^i}\delta F(\phi^i_j)$, which implies that

$$\prod_{j=1}^{k^i} \phi_1^{(t^i_j,p^i_j)}(ST_{l(t^i_j,p^i_j)}(h^{e(i,j)}_2 S h^{e(i,j+1)}_1)) = \prod_{j=1}^{k^i} \phi_1^{(t^i_j,p^i_j)}(ST_{l(t^i_j,p^i_j)}(\delta F(\phi^i_j))\ .$$

Observe that $(t,p) = (t^i_j, p^i_j)$ iff $l(t,p) = u(i,j)$ iff $i = i(l(t,p))$ and $j = j(l(t,p))$. It now follows from equation (4.16) that

$$\begin{aligned}
&Z^Q_{\tilde{N}^-}(U_\sigma((\phi_1 \otimes \phi_2)^{\otimes k})) \\
&= \delta^{-k+2c}\prod_{t=1}^{g} h^t\left(\prod_{p=1}^{k_t}\phi_1^{(t,p)}(ST_{l(t,p)}(\delta F(\phi^{i(l(t,p))}_{j(l(t,p))})))\phi_2^{(t,p)}\right) \\
&= \delta^{2c}\prod_{t=1}^{g} h^t\left(\prod_{p=1}^{k_t}\phi_1^{(t,p)}(ST_{l(t,p)}(F(\phi^{i(l(t,p))}_{j(l(t,p))})))\phi_2^{(t,p)}\right) \\
&= \delta^{2c}\prod_{t=1}^{g} h^t\left(\prod_{p=1}^{k_t}\delta FST_{l(t,p)}F(\phi^{i(l(t,p))}_{j(l(t,p))})\right) \\
&= \delta^{k+2c}\prod_{t=1}^{g} h^t\left(\prod_{p=1}^{k_t}T_{l(t,p)}(\phi^{i(l(t,p))}_{j(l(t,p))})\right)
\end{aligned}$$

Finally, a perusal of equations (2.1) and (4.14) completes the verification that Kuperberg's invariant is indeed given by $\tau^P(N)$.

5. On spherical graphs and Hopf algebras

Throughout this section, the symbol G will denote an oriented graph embedded on an oriented smooth sphere S^2. Thus G comprises of a finite subset $V \subset S^2$ of vertices and a finite set E of edges. We regard an edge $e \in E$ as a smooth map from the unit interval I to S^2 such that $e(0), e(1) \in V$ and such that e is injective except possibly that $e(0) = e(1)$. Two (images of) distinct edges do not intersect except possibly at vertices. Thus multiple edges and self-loops are allowed. An edge e is regarded as being oriented from $e(0)$ to $e(1)$. We regard G as the subset of S^2 given by the union of its edges and isolated vertices, if any. By a face of G, we mean a connected component of the complement of G in S^2.

We will use the terms anticlockwise and clockwise to stand for "agreeing with the orientation of" and "opposite to the orientation of" S^2 respectively. If u is the direction of the oriented edge e at a point p, and if v is a perpendicular direction such that $\{u, v\}$ is positively (resp., negatively) oriented (according to the orientation of the underlying S^2), we shall call the points near p on the side indicated by v as the 'left' (resp., 'right') of the edge e.

We digress now with a discussion of tensor products of indexed families of vector spaces. We consider only finite indexing sets. For a family $\{V_p : p \in K\}$ of vector spaces (over some field $\mathbf{k}$), which is indexed by the finite set K, we define $\otimes_{p \in K} V_p$ to be the quotient of the vector space, with basis consisting of functions $f : K \to \coprod_{p \in K} V_p$ such that $f(p) \in V_p$ for all $p \in K$, by the subspace spanned by

$$\{f - \alpha_1 f_1 - \alpha_2 f_2 : \exists p_0 \in K \text{ such that} f(p) = \begin{cases} f_1(p) = f_2(p) & if\ p \neq p_0 \\ \alpha_1 f_1(p) + \alpha_2 f_2(p) & if\ p = p_0 \end{cases} \} .$$

We denote the image in $\otimes_{p \in K} V_p$ of the function f by $\otimes_{p \in K} f(p)$. If $\{T_p : V_p \to W_p\}_p$ is an indexed family of vector space maps, there is a natural induced map $\otimes_{p \in K} T_p : \otimes_{p \in K} V_p \to \otimes_{p \in K} W_p$.

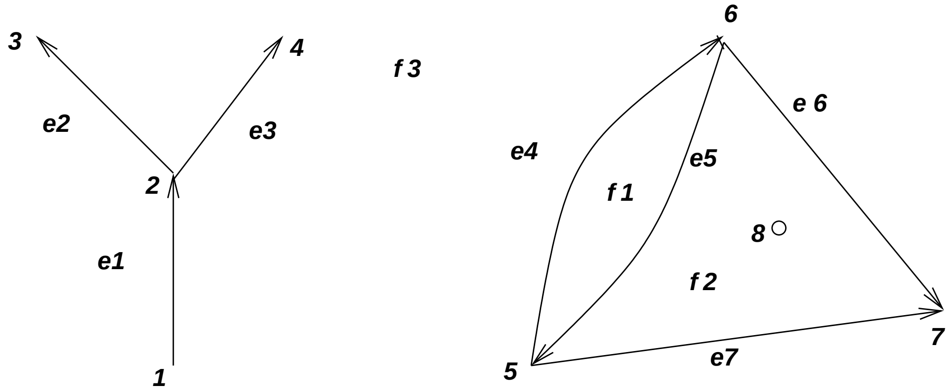
In the important special case of this indexed tensor product when $V_p = V$ for all $p \in K$, we will also denote $\otimes_{p \in K} V_p$ by $V^{\otimes K}$. We adopt a similar convention for tensor product of vector space maps.

Note that if $K = \{1, 2, ..., k\}$, then $\otimes_{p \in K} V_p$ can be naturally identified with $\otimes_{p=1}^k V_p = V_1 \otimes \cdots \otimes V_k$, and in particular, we will write $V^{\otimes K} = V^{\otimes k}$. More generally, if K is a totally ordered finite set with $|K| = k$, then $V^{\otimes K}$ can be naturally identified with $V^{\otimes k}$. Even more generally, a bijection, say θ, from a set L to a set K, induces a functorial isomorphism, which we will denote by $\tilde{\theta}$, from $\otimes_{l \in L} V_l$ to $\otimes_{k \in L} V_k$, and in particular from $V^{\otimes L}$ to $V^{\otimes K}$. In the sequel, we will use without explicit mention, the canonical identifications

$$\begin{aligned} V^{\otimes (\coprod_{i \in I} K_i)} &\sim \otimes_{i \in I} V^{\otimes K_i} \\ (V^{\otimes K})^{\otimes L} &\sim V^{\otimes (L \times K)}. \end{aligned}$$

To the pair (G, H) (of a graph and a Hopf algebra), we shall associate two elements of $H^{\otimes E}$. One of these is computed using the faces of G and is denoted by $F(G, H)$ and the other is computed using the vertices of G and is denoted by $V(G, H)$. The main result of the section relates $F(G, H^*)$ and $V(G, H)$.

We will make use of the following example - of a directed graph G with eight vertices and three faces - with multiple edges ($e4$ and $e5$ between vertices 5 and 6) and an isolated vertex (vertex 8) - to clarify our definitions:



Let $D(V)$ denote the set $E \times \{0,1\}$. For a vertex $v \in V$, let D_v denote the set $\{(e,i) \in D(V) : e(i) = v\}$ and let d_v denote its cardinality which is the degree of v. Consider an enumeration of D_v in clockwise order around the vertex v. This is, of course, determined once one of the edges at v is chosen as the first.[2] Denote this bijection by $\theta_v : \{1, \cdots, d_v\} \to D_v$. Note that $D(V)$ is the disjoint union of D_v as v varies over V and consider $\otimes_{v \in V} \widetilde{\theta_v}(\delta^{-1}\Delta_{d_v}(h)) \in H^{\otimes D(V)}$. The traciality of h implies that this element is independent of the choice of clockwise ordering of the edges around each vertex.

Now consider the map $\mu \circ (id \otimes S) : H^{\otimes \{0,1\}} = H^{\otimes 2} \to H$ and the tensor product map $\otimes_{e \in E}(\mu \circ (id \otimes S)) : H^{\otimes D(V)} = H^{E \times \{0,1\}} \to H^E$. Define $V(G,H)$ to be the image under this map of $\otimes_{v \in V} \widetilde{\theta_v}(\delta^{-1}\Delta_{d_v}(h))$. Explicitly, we have

$$V(G,H) = \delta^{\rho(G)} \otimes_{e \in E} h^{s(e)}_{m(e)} S h^{r(e)}_{n(e)} \tag{5.17}$$

where (i) $\rho(G) = -|V| + 2|\{v \in V : d_v = 0\}|$;[3] and (ii) s, r, m, n are functions defined on E and with appropriate ranges, so that $(e,0)$ is the $m(e)$-th element of $D_{s(e)}$ while $(e,1)$ the $n(e)$-th element of of $D_{r(e)}$, for any edge $e \in E$. (Thus, for example, $s, r : E \to V$ are the 'source' and 'range' maps.)

For our example, $V(G,H) \in H^{\otimes 7}$ - since there are 7 edges; the prescription unravels to yield

(5.18)
$$V(G,H) = \delta^{-6} \left(h^1 S h^2_1 \otimes h^2_2 S h^3 \otimes h^2_3 S h^4 \otimes h^5_1 S h^6_1 \otimes h^6_3 S h^5_2 \otimes h^6_2 S h^7_1 \otimes h^5_3 S h^7_2 \right) .$$

A similar construction using the faces yields $F(G,H)$. For this, begin with the set $D(F) = E \times \{l, r\}$. Consider a pair (f,c) where f is a face of G and c is a component of the boundary of f. By $\widetilde{F}$, we will refer to the set of all such pairs. (This set is the set 'dual' to the vertex set V in case the graph G is disconnected.) Let $D_{(f,c)} = \{(e,d) \in D(F) : e(t) \in c$ for all $t \in [0,1]$ and there exist points in f sufficiently close to c where the orientation agrees or disagrees with the orientation of e according as d is l or r $\}$. We pause to explain this mouthful of a definition. A pair consisting of an edge e and a direction d is put into $D_{(f,c)}$ exactly when

[2]For our example, the sets D_v, with their elements listed in a possible order, are: $D_1 = \{(e1,0)\}, D_2 = \{(e1,1),(e2,0),(e3,0)\}, D_3 = \{(e2,1)\}, D_4 = \{(e3,1)\}, D_5 = \{(e4,0),(e5,1),(e7,0)\}, D_6 = \{(e4,1),(e6,0),(e5,0)\}, D_7 = \{(e6,1),(e7,1)\}, D_8 = \emptyset$.

[3]The reason for the correction term '$+2|\{v : d_v = 0\}|$' is that $\Delta_0(h) = \epsilon(h) = n = \delta^2$.

the image of the edge is part of c and some parts of f lie to the left or right of e according as d is l or r. Note that it is quite possible for points of f to lie on both sides of the image of e. Set $d_{(f,c)}$ to be the cardinality of $D_{(f,c)}$.

In our example, there are three faces $f1, f2, f3$, and these boundaries have $1, 2$ and 2 components respectively, and we have

$$\widetilde{F} = \{\tilde{f}1 = (f1, 4\bar{5}), \tilde{f}2 = (f2, \bar{5}6\bar{7}), \tilde{f}3 = (f2, \cdot), \tilde{f}4 = (f3, 12\bar{2}3\bar{3}\bar{1}), \tilde{f}5 = (f3, \bar{4}7\bar{6})\} \,,$$

with the notation $(f1, 4\bar{5})$ signifying the pair consisting of the face $f1$ and the component given by the traversing the edge $e4$ followed by the reverse of the edge $e5$.

We will need the notion of a thickening of G - by which we will understand a sufficiently small neighbourhood of G with respect to some Riemannian metric on S^2. A moment's thought shows that there is a natural bijection between the set of boundary components of such a thickening of G and what we earlier called $\widetilde{F}$. A clockwise traversal of the boundary component corresponding to $(f, c) \in \widetilde{F}$ (under the above bijection) leads naturally to what we would like to term a clockwise enumeration of $D_{(f,c)}$. Denote this enumeration by $\rho_{(f,c)} : \{1, \cdots, d_{(f,c)}\} \to D_{(f,c)}$.

In our example, the sets $D_{(f,c)}$, with their members listed in a choice of such a clockwise order, are as follows:

$$\begin{aligned} D_{\tilde{f}1} &= \{(e4, r), (e5, r)\} \\ D_{\tilde{f}2} &= \{(e5, l), (e6, r), (e7, l)\} \\ D_{\tilde{f}3} &= \emptyset \\ D_{\tilde{f}4} &= \{(e1, r), (e3, r), (e3, l), (e2, r), (e2, l), (e1, l)\} \\ D_{\tilde{f}5} &= \{(e4, l), (e7, r), (e6, l)\} \,. \end{aligned}$$

Now, $D(F)$ is the disjoint union of the $D_{(f,c)}$ as (f, c) range over $\widetilde{F}$ and so the element $\otimes_{(f,c)\in\widetilde{F}} \widetilde{\rho(f,c)}(\delta^{-1}\Delta_{d_{(f,c)}}(h))$ is a well-defined element of $H^{\otimes D(F)}$ which is independent of the choice of clockwise enumerations of the $D_{(f,c)}$'s.

Finally, consider the map $\mu \circ (id \otimes S) : H^{\otimes\{l,r\}} = H^{\otimes 2} \to H$. In this, $\{l, r\}$ is mapped to $\{1, 2\}$ by $l \mapsto 1$ and $r \mapsto 2$. The tensor product map $\otimes_{e\in E}(\mu \circ (id \otimes S))$: $H^{\otimes D(F)} = H^{E\times\{l,r\}} \to H^E$. Define $F(G, H)$ to be the image under this map of $\otimes_{(f,c)\in\widetilde{F}} \widetilde{\rho(f,c)}(\delta^{-1}\Delta_{d_{(f,c)}}(h))$. The element of interest is $F(G, H^*)$ which is obtained by replacing h by ϕ in the above expression. Explicitly, we have

$$F(G, H^*) = \delta^{\sigma(G)} \otimes_{e\in E} \phi^{L(e)}_{i(e)} S \phi^{R(e)}_{j(e)} \tag{5.19}$$

where (i) $\sigma(G) = -|\widetilde{F}| + 2|\{v \in V : d_v = 0\}|$; and (ii) L, R, i, j are functions defined on E and with appropriate ranges, so that (e, l) is the $i(e)$-th element of $D_{L(e)}$ while (e, r) the $j(e)$-th element of of $D_{R(e)}$, for any edge $e \in E$. (Thus, for example, $L, R : E \to \widetilde{F}$.)

In our example, $F(G, H^*) \in (H^*)^{\otimes 7}$; and the prescription unravels to yield (5.20)

$$F(G, H^*) = \delta^{-3} \left(\phi^4_6 S\phi^4_1 \otimes \phi^4_5 S\phi^4_4 \otimes \phi^4_3 S\phi^4_2 \otimes \phi^5_1 S\phi^1_1 \otimes \phi^2_1 S\phi^1_2 \otimes \phi^5_3 S\phi^2_2 \otimes \phi^2_3 S\phi^5_2\right) \,.$$

The remainder of this section is devoted to proving the following:

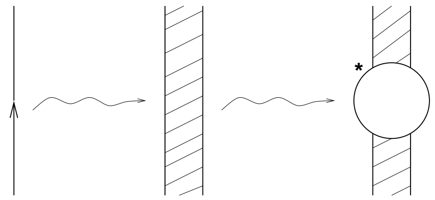
Proposition 2.

$$F(G, H^*) = F^{\otimes E}\ (V(G, H)) \tag{5.21}$$

for any spherical graph G.

Our proof goes through the machinery of planar algebras but it would be desirable to find a direct proof.

We use G to construct a network in the Jones sense on S^2. This network will be denoted $N = N(G)$. To construct N, choose a thickening of G, as described above. Colour this subset of S^2 black. Each edge of G now appears as a thin black band in this subset. Replace this portion of the band by introducing a 2-box as indicated below:



with the orientation of the edge determining the position of the $*$. This yields our network N on the sphere; note that N has only 2-boxes. From the construction it should be clear that there are natural bijections between the sets of black regions, white regions and 2-boxes of N and the sets of vertices, faces and edges of G respectively.

If P is any planar algebra, the partition function of $N(G)$ specifies a function from $(P_2)^{\otimes E}$ to P_{0_+}. In particular, if $P = P(H)$, this partition function may be identified with a linear map from $H^{\otimes E}$ to k or equivalently, with an element of $(H^*)^{\otimes E}$. We assert that this element is exactly $F(G, H^*)$. Explicitly, we need to verify that

$$Z_{N(G)}(\otimes_{e\in E} a^e) = (F(G,H^*))(\otimes_{e\in E} a^e) \ \forall a^e \in H \ . \tag{5.22}$$

By definition of $F(G, H^*)$, we have

$$\begin{aligned}
(F(G,H^*))(\otimes_{e\in E} a^e) &= \delta^{\sigma(G)} \prod_{e\in E} \left(\phi^{L(e)}_{i(e)} S \phi^{R(e)}_{j(e)}\right)(a^e) \\
&= \delta^{\sigma(G)} \prod_{e\in E} \phi^{L(e)}_{i(e)}(a^e_1)\phi^{R(e)}_{j(e)}(Sa^e_2) \\
&= \delta^{\sigma(G)} \prod_{Q\in\widetilde{F}} \left[\left(\prod_{e\in E:L(e)=Q} \phi^Q_{i(e)}(a^e_1)\right)\left(\prod_{e\in E:R(e)=Q} \phi^Q_{j(e)}(Sa^e_2)\right)\right] \\
&= \delta^{\sigma(G)} \prod_{Q\in\widetilde{F}} \phi^Q\left(\prod_{i=1}^{d_Q} T^Q_i a^{\rho_Q(i)}_{\epsilon^Q_i}\right)
\end{aligned}$$

where $(T^Q_i, \epsilon_Q(i)) = \begin{cases} (id, 1) & \text{if } (\rho_Q(i), l) \in D_Q \\ (S, 2) & \text{if } (\rho_Q(i), r) \in D_Q \end{cases}$.

The proof of the asserted equation (5.22) follows immediately from Corollary 3 of [KdySnd2]. (One only needs to note that the 'loops' of that prescription are in bijection with members of $\widetilde{F}$, and exercise a little caution - in case G has isolated vertices, so that $N(G)$ has isolated loops - to see that the powers of δ also match.)

We assert now that with identifications as above, that $Z_{N^-} = V(G, H^*)$. This assertion is proved exactly like the equation $Z_N = F(G, H^*)$ was proved - after having observed that the black and white regions for the network N^-, correspond to the white and black regions for N.

Applying Proposition 1 to $N = N(G)$,

$$F(G,H^*) = Z_N^{P(H)} = Z_{N^-}^{P(H^*)} \circ F^{\otimes E} = V(G,H) \circ F^{\otimes E} = F^{\otimes E}(V(G,H)) \ .$$

The first $V(G,H)$ is regarded as an element of $(H^{**})^{\otimes E}$ while the second is regarded as an element of $H^{\otimes E}$, and the last equality follows from $x(F(y)) = (F(x))(y)$. □

So, Proposition 2 has been finally proved.

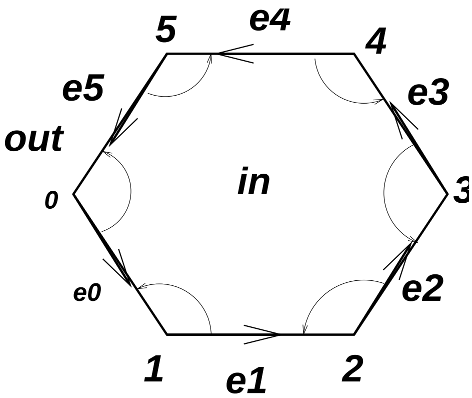
We finally wish to observe a consequence of this proposition.

Corollary 3. *In any semisimple co-semisimple Hopf algebra, we have*

$$\begin{array}{lcl} (a)\ h_1^0 S h_2^1 \otimes h_1^1 S h_2^2 \otimes \cdots \otimes h_1^{(n-1)} S h_2^0 & = & \delta^n F^{\otimes n}(\Delta_n \phi) \\ (b)\ h_1^1 S h_2^0 \otimes h_1^2 S h_2^1 \otimes \cdots \otimes h_1^0 S h_2^{(n-1)} & = & \delta^n F^{\otimes n}(\Delta_n^{op} \phi) \end{array}$$

for any $n \geq 1$.

Proof: (a) In fact, consider the special case of Proposition 2 corresponding to G being a cyclically oriented n-gon. Write $V = \{0, 1, \cdots, n-1\}, E = \{e0, e1, \cdots, e(n-1)\}, F = \{in, out\}$, and make 'cyclically symmetric' choices as below (where we illustrate the case $n = 6$:



We set

$$D_i = \{ei, e(i-1)\}, \ \forall\ 0 \leq i < n \ ,$$

with addition modulo n. Further, $\widetilde{F} = F$, and we choose

$$D_{in} = \{e(n-1), \cdots, e1, e0\} \text{ and } D_{out} = \{e0, \cdots, e(n-1)\} \ .$$

Our prescriptions yield

$$V(G,H) = \delta^{-n}(h_1^0 S h_2^1 \otimes h_1^1 S h_2^2 \otimes \cdots \otimes h_1^{(n-1)} S h_2^0) \ ;$$

and

$$F(G,H^*) = \delta^{-2}(\phi_n^{in} S \phi_1^{out} \otimes \phi_{n-1}^{in} S \phi_2^{out} \otimes \cdots \otimes \phi_1^{in} S \phi_n^{out}) \ .$$

Since $S^{\otimes n}(\Delta_n(a)) = \Delta_n^{op}(Sa)$ in any Hopf algebra, this simpliies to

$$\begin{array}{rcl} F(G,H^*) & = & \delta^{-2} \Delta_n^{op}(\phi^{in} S \phi^{out}) \\ & = & \Delta_n^{op}(\phi) \ , \end{array}$$

the final equality being a consequence of the fact that $\phi^2 = \delta^2 \phi$ and $S\phi = \phi$.

So we deduce from Proposition 2 that

$$F^{\otimes n}\left(\delta^{-n}(h_1^0 Sh_2^1 \otimes h_1^1 Sh_2^2 \otimes \cdots \otimes h_1^{(n-1)} Sh_2^0)\right) = \Delta_n^{op}(\phi) \ ;$$

and since $F^{-1} = F \circ S$, we conclude that

$$\begin{aligned} h_1^0 Sh_2^1 \otimes h_1^1 Sh_2^2 \otimes \cdots \otimes h_1^{(n-1)} Sh_2^0 &= \delta^n (F \circ S)^{\otimes n}(\Delta_n^{op}(\phi)) \\ &= \delta^n F^{\otimes n}(\Delta_n(\phi)) \ , \end{aligned}$$

as desired.

(b) This follows by applying $S^{\otimes n}$ to both sides of (a). □

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