II. Von Neumann algebras with a bicyclic vector

Here we consider the case of an n.s.f (normal faithful) state φ on a von Neumann algebra, that is, via the GNS construction, the following setting:

| $M \subset B(H)$ von Neumann algebra with a (unit) bicyclic vector $\xi_0 \in H$ and $\varphi = \omega_{\xi_0}$ the corresponding n.s.f state on M.

1° Consider the conjugate linear operators

$$S_0 : M\xi_0 \ni x\xi_0 \longmapsto x^*\xi_0 \in M\xi_0 \subset H \;\; ; \; D(S_0) = M\xi_0$$

$$F_0 : M'\xi_0 \ni x'\xi_0 \longmapsto x'^*\xi_0 \in M'\xi_0 \subset H \;\; ; \; D(F_0) = M'\xi_0$$

We repeat the argument that shows that $S_0 \subset F_0^*$, that is

$$(S_0\xi \mid \eta) = (F_0\eta \mid \xi), \; (\forall)\, \xi \in D(S_0), \eta \in D(F_0),$$

that is

$$(S_0 x\xi_0 \mid x'\xi_0) = (F_0 x'\xi_0 \mid x\xi_0), \; (\forall)\, x \in M, x' \in M'$$

which is obvious:

$$(S_0 x\xi_0 \mid x'\xi_0) = (x^*\xi_0 \mid x'\xi_0) = (\xi_0 \mid x x'\xi_0)$$

$$= (\xi_0 \mid x' x\xi_0) = (x'^*\xi_0 \mid x\xi_0) = (F_0 x'\xi_0 \mid x\xi_0).$$

Since every adjoint operator is closed, it follows that S_0 is preclosed. Let $S = \overline{S_0}$ denote its closure:

$$S : D(S) \to H, \; \overline{S|M\xi_0} = S, \; Sx\xi_0 = x^*\xi_0, \; (x \in M).$$

Similarly

$$F : D(F) \to H, \; \overline{F|M'\xi_0} = F, \; Fx'\xi_0 = x'^*\xi_0, \; (x' \in M').$$

and from $S_0 \subset F_0^*$, $F_0 \subset S_0^*$ it follows that

$$S \subset F^*, \; F \subset S^*$$

Also, it is obvious that $S_0^2 \subset I$ and this yields

$S^2 \subset I$, i.e. $\xi \in D(S) \Rightarrow S\xi \in D(S)$ and $SS\xi = \xi$.

Let us recall the standard proof. Since $\xi \in D(S)$ and $S = \overline{S|M\xi_0}$, there is a sequence $\{x_n\} \subset M$ such that $x_n\xi_0 \to \xi$ and $x_n^*\xi_0 = Sx_n\xi_0 \to S\xi$. We read this as $x_n^*\xi_0 \to S\xi$ and $Sx_n^*\xi_0 = x_n\xi_0 \to \xi$. Since S is closed, it follows that $S\xi \in D(S)$ and $S(S\xi) = \xi$. Moreover,

S is invertible and

$$S^{-1} = S.$$

Consider now the <u>modular operator</u>

$$\Delta = S^*S.$$

By a well known theorem of J. von Neumann, Δ is a positive selfadjoint linear operator and

$$\overline{S \mid D(\Delta)} = S.$$

This enables us to consider the polar decomposition

$$S = J\Delta^{1/2}.$$

Since $S = S^{-1}$, Δ is a nonsingular operator and

$$J\Delta^{1/2} = S = S^{-1} = \Delta^{-1/2} J^{-1}$$

hence

$$\Delta^{-1/2} = J\Delta^{1/2} J = J^2 (J^* \Delta^{1/2} J)$$

and, by the uniqueness of the polar decomposition,

$$J^2 = I \quad \text{i.e.} \quad J = J^* = J^{-1}$$

that is, J is a conjugation, <u>the canonical conjugation</u>.

We also have

$$S = J\Delta^{1/2} = \Delta^{-1/2} J, \quad D(S) = D(\Delta^{1/2})$$

$$S^* = J\Delta^{-1/2} = \Delta^{1/2} J, \quad D(S^*) = D(\Delta^{-1/2})$$

Notice that $S\xi_0 = \xi_0 = S^*\xi_0$, $\Delta\xi_0 = \xi_0$, so $\Delta^{1/2}\xi_0 = \xi_0$ and hence $J\xi_0 = \xi_0$

Moreover, for any Borel function $f : [0, \infty) \to \mathbb{C}$ we have

$$J f(\Delta) J = \overline{f}(\Delta^{-1})$$

since $J\Delta J = \Delta^{-1}$ and $J(\lambda\Delta)J = \overline{\lambda} J\Delta J = \overline{\lambda}\Delta^{-1}$ and we conclude by functional calculus. In particular

$$J\Delta^{it} = \Delta^{it} J, \quad (t \in \mathbb{R}; \ i^2 = -1).$$

$\Delta^\alpha \xi_0 = \xi_0 = J\xi_0$

TOMITA'S THEOREM:

$$JMJ = M' \quad \text{and} \quad \Delta^{it} M \Delta^{-it} = M \quad (\forall) t \in \mathbb{R}$$

Both statements will be proved simultaneously by the metod due to ALFONS VAN DAELE.

The original proof of Tomita presented by Takesaki is more natural but more involved and considerably longer.

2° The problem at this primary stage is to construct some „bridge" between $M\xi_0$ and $M'\xi_0$ (Once we prove $JMJ = M'$ it follows that $M'\xi_0 = JMJ\xi_0 = JM\xi_0$, so J is the „good" bridge, but first we shall find a more primitive „wood" bridge). This is based on the following

LEMA (S. SAKAI). Let M be a von Neumann algebra and $\varphi, \psi \in M_*^+$ normal positive forms such that $\psi \leq \varphi$ and let $\lambda \in \mathbb{C}$ with $\lambda + \bar{\lambda} = 1$. There exists $a \in M$, $0 \leq a \leq 1$ such that

$$\psi(x) = \varphi(\lambda a x + \bar{\lambda} x a), \ (\forall) x \in M.$$

PROOF. The set $X = \{\varphi(\lambda a \cdot + \bar{\lambda} \cdot a);\ a = a^* \in M, \|a\| \leq 1\}$ is a convex subset of M_*^{sa} and $\sigma(M_*, M)$-compact. If $\psi \notin X$, then, by the Hahn-Banach theorem, there exists $b \in M$, $b = b^*$, and $t \in \mathbb{R}$ with

$$\psi(b) > t \text{ while } \varphi(\lambda a b + \bar{\lambda} b a) \leq t \ (\forall) a = a^* \in M, \|a\| \leq 1.$$

Let $b = v|b|$ be the polar decomposition. Then

$$v \in M,\ v = v^*,\ \|v\| \leq 1 \text{ and } |b| = vb = bv$$

hence

$$t < \psi(b) \leq \psi(|b|) \leq \varphi(|b|) = \varphi(\lambda v b + \bar{\lambda} b v) \leq t$$

which is a contradiction. Thus, there exist $a_1 \in M$, $a_1^* = a_1$, $\|a_1\| \leq 1$ such that

$$\psi(x) = \varphi(\lambda a_1 x + \bar{\lambda} x a_1), \ x \in M.$$

Since $\varphi \geq 0$, $\psi \geq 0$, it follows that

$$\psi(a_1^-) = \varphi(-(a_1^-)^2) = 0$$

hence $\varphi(a_1^- x) = 0 = \varphi(x a_1^-)$ by the Cauchy-Schwarz inequality and consequently $a = a_1^+ \in M$, $0 \leq a \leq 1$ and

$$\psi(x) = \varphi(\lambda a x + \bar{\lambda} x a), \ x \in M.$$

3° LEMMA. For $\omega \in \mathbb{C}$, $|\omega| = 1$, $\omega \neq -1$, we have

$$(\Delta + \omega)^{-1} M'\xi_0 \subset M\xi_0$$

PROOF. It suffices to consider $a' \in M'$, $0 \leq a' \leq 1$, and to show that $(\Delta + \omega)^{-1} a'\xi_0 \in M\xi_0$. To this end, consider $\lambda = (1+\omega)^{-1} \in \mathbb{C}$, notice that $\lambda + \bar{\lambda} = 1$, and consider $\varphi, \psi \in M_*^+$ defined by

$$\varphi(x) = (x\xi_0 | \xi_0) \,,\quad \psi(x) = (x\xi_0 | a'\xi_0) \,,\quad (x \in M).$$

It is clear that $0 \le \psi \le \varphi$ so, by the Sakai's Lemma, there exists $a \in M$, $0 \le a \le 1$, such that

$$\begin{aligned}(x\xi_0 | a'\xi_0) = \psi(x) &= \varphi(\lambda a x + \bar{\lambda} x a) = \\ &= (\lambda a x \xi_0 | \xi_0) + (\bar{\lambda} x a \xi_0 | \xi_0) = \\ &= (x\xi_0 | \bar{\lambda} a \xi_0) + (a\xi_0 | \lambda x^* \xi_0)\end{aligned}$$

that is,

$$(x\xi_0 | a'\xi_0 - \bar{\lambda} a \xi_0) = (\bar{\lambda} a \xi_0 | S x \xi_0) \quad (\forall)\, x \in M$$

$$[\text{read: } (\xi | \zeta) \qquad = (\eta | S\xi) \,,\ (\forall)\, \xi \in D(S)]$$

which means that $[\eta \in D(S^*),\ S^*\eta = \zeta]$:

$$\begin{aligned}a'\xi_0 - \bar{\lambda} a \xi_0 &= S^*(\bar{\lambda} a \xi_0) = \lambda S^*(a\xi_0) = \\ &= \lambda S^* S a \xi_0 = \lambda \Delta a \xi_0\end{aligned}$$

that is:

$$a'\xi_0 = (\bar{\lambda} + \lambda\Delta) a \xi_0 = (\Delta + \omega)(1+\omega)^{-1} a \xi_0$$

$$(\Delta + \omega)^{-1} a'\xi_0 = (1+\omega)^{-1} a \xi_0 \in M\xi_0$$

4° We shall also need the following approximation result:

LEMMA. Let $\xi \in D(\Delta^{1/2}) \cap D(\Delta^{-1/2})$. There is a sequence

$$\xi_n \in (\Delta + 1)^{-1} M'\xi_0 \subset M\xi_0 \ [\subset D(\Delta^{1/2}) \cap D(\Delta^{-1/2})]$$

such that

$$\xi_n \to \xi \qquad \Delta^{1/2}\xi_n \to \Delta^{1/2}\xi \qquad \Delta^{-1/2}\xi_n \to \Delta^{-1/2}\xi$$

PROOF. Since $J\Delta^{-1/2} M'\xi_0 = S^* M'\xi_0 = F M'\xi_0 = M'\xi_0$ is dense in H, we see that

$$\Delta^{-1/2} M'\xi_0 \text{ is dense in } H$$

hence there exist:

$$\eta_n \in M'\xi_0 \text{ such that } \Delta^{-1/2}\eta_n \to \Delta^{1/2}\xi + \Delta^{-1/2}\xi$$

Consider then:

$$\xi_n = (1+\Delta)^{-1}\eta_n \in (1+\Delta)^{-1} M'\xi \subset M\xi_0 \ (\text{by Lemma } 3°)$$

Notice that $\xi_n \in M\xi_0 \subset D(S) = D(\Delta^{1/2})$ and $M'\xi_0 \subset D(S^*) = D(\Delta^{-1/2})$, so that

$$\xi_n = (1+\Delta)^{-1}\eta_n = \Delta^{-1/2}(1+\Delta^{-1})^{-1}\Delta^{-1/2}\eta_n \text{ and } \xi_n \in D(\Delta^{-1/2})$$

since $\Delta^{-1}(1+\Delta^{-1})^{-1}$ is bounded. We have

$$\xi_n = (1+\Delta)^{-1}\eta_n = \Delta^{1/2}(1+\Delta)^{-1}\Delta^{-1/2}\eta_n \longrightarrow$$
$$\longrightarrow \Delta^{1/2}(1+\Delta)^{-1}(\Delta^{1/2}\xi + \Delta^{-1/2}\xi) = \dots = \xi$$

$$\Delta^{-1/2}\xi_n = (1+\Delta)^{-1}\Delta^{-1/2}\eta_n \longrightarrow (1+\Delta)^{-1}(\Delta^{1/2}\xi + \Delta^{-1/2}\xi) = \dots = \Delta^{-1/2}\xi$$

$$\Delta^{1/2}\xi_n = \Delta(1+\Delta^{-1})\Delta^{-1/2}\eta_n \longrightarrow \Delta(1+\Delta)^{-1}(\Delta^{1/2}\xi + \Delta^{-1/2}\xi) = \dots = \Delta^{1/2}\xi$$

5° Let $x' \in M'$ and $\omega \in \mathbb{C}$, $|\omega| = 1$, $\omega \neq -1$. We show that

(1) $$(x'\eta|\zeta) = (Jx^*J\Delta^{-1/2}\eta|\Delta^{1/2}\zeta) + \omega(Jx^*J\Delta^{1/2}\eta|\Delta^{-1/2}\zeta)$$

for any $\eta, \zeta \in D(\Delta^{1/2}) \cap D(\Delta^{-1/2})$ where $x \in M$ is given by

$$x'\xi_0 = (\Delta + \omega)x\xi_0 \quad (\text{cf. LEMMA } 3^\circ)$$

By the approximation LEMMA 4°, it suffices to consider

$$\eta = (\Delta+1)^{-1}y'\xi_0 = y\xi_0, \quad (y' \in M', y \in M)$$
$$\zeta = (\Delta+1)^{-1}z'\xi_0 = z\xi_0, \quad (z' \in M', z \in M)$$

Since $x'\xi_0 = \Delta x\xi_0 + \omega x\xi_0 = S^*S x\xi_0 + \omega x\xi_0$, we have

$$(x'\eta|\zeta) = (x'y\xi_0|z\xi_0) = (yx'\xi_0|z\xi_0) = (yS^*Sx\xi_0|z\xi_0) + \omega(yx\xi_0|z\xi_0)$$
$$= (Sy^*z\xi_0|Sx\xi_0) + \omega(yx\xi_0|z\xi_0)$$
$$= (z^*y\xi_0|x^*\xi_0) + \omega(yx\xi_0|z\xi_0)$$

But we want to get back $\eta = y\xi_0$ and $\zeta = z\xi_0$. Thus:

$$(z^*y\xi_0|x^*\xi_0) = (y\xi_0|zx^*\xi_0) = (y\xi_0|Sxz^*\xi_0) = (y\xi_0|SxSz\xi_0) =$$
$$= (y\xi_0|\Delta^{-1/2}JxJ\Delta^{1/2}z\xi_0) = (Jx^*J\Delta^{-1/2}y\xi_0|\Delta^{1/2}z\xi_0) =$$
$$= (Jx^*J\Delta^{-1/2}\eta|\Delta^{1/2}\zeta)$$

$$(yx\xi_0|z\xi_0) = (Sx^*y^*\xi_0|z\xi_0) = (Sx^*Sy\xi_0|z\xi_0) =$$
$$= (\Delta^{-1/2}Jx^*J\Delta^{1/2}y\xi_0|z\xi_0) = (Jx^*J\Delta^{1/2}y\xi_0|\Delta^{-1/2}z\xi_0) =$$
$$= (Jx^*J\Delta^{1/2}\eta|\Delta^{-1/2}\zeta)$$

These prove (1).

6° The main technical tool is the following operator theoretical rezult:

THEOREM (A. VAN DAELE) Let A, B be nonsingular positive self-adjoint operators densely defined on the Hilbert space H, and $\omega \in \mathbb{C}$, $|\omega| = 1$, $\omega \neq -1$. Let $x, y \in B(H)$ and assume that

$$(x\eta | \zeta) = (yB^{-1/2}\eta | A^{1/2}\zeta) + \omega (yB^{1/2}\eta | A^{-1/2}\zeta)$$

for all $\zeta \in D(A^{1/2}) \cap D(A^{-1/2})$, $\eta \in D(B^{1/2}) \cap D(B^{-1/2})$

Then

(1) $$y = \Phi_\omega(x) = \int_{-\infty}^{\infty} \frac{\omega^{it-\frac{1}{2}}}{e^{\pi t}+e^{-\pi t}} A^{-it} x B^{it}\, dt$$

We postpone the proof of this Theorem

COROLLARY. In the same situation

(2) $$A^{1/2}(A+\omega)^{-1} = \int_{-\infty}^{\infty} \frac{\omega^{it-\frac{1}{2}}}{e^{\pi t}+e^{-\pi t}} A^{-it}\, dt$$

PROOF of the Corollary. Take $B = 1$, $x = 1$ and $y = A^{1/2}(A+\omega)^{-1}$. It is easy to check that for every $\eta \in H$ and every $\zeta \in D(A^{1/2}) \cap D(A^{-1/2})$ we have

$$(\eta | \zeta) = (y\eta | A^{1/2}\zeta) + \omega (y\eta | A^{-1/2}\zeta)$$

So, the Corollary follows indeed from the Theorem.

7° Thus, we started with arbitraries $x' \in M'$ and $\omega \in \mathbb{C}$, $|\omega| = 1$, $\omega \neq -1$ and we obtained 5°(1) with $x \in M$ such that

$$x'\xi_0 = (\Delta + \omega) x \xi_0 .$$

Consider now another arbitrary $y' \in M'$. By the Theorem 6°(1),

$$Jx^*J = \int_{-\infty}^{\infty} \frac{\omega^{it-\frac{1}{2}}}{e^{\pi t}+e^{-\pi t}} \Delta^{-it} x' \Delta^{it}\, dt$$

hence

$$x^* = \int_{-\infty}^{\infty} \frac{\omega^{it-\frac{1}{2}}}{e^{\pi t}+e^{-\pi t}} J\Delta^{-it} x' \Delta^{it} J\, dt$$

$$x^* y'\xi_0 = \int_{-\infty}^{\infty} \frac{\omega^{it-\frac{1}{2}}}{e^{\pi t}+e^{-\pi t}} J\Delta^{-it} x' \Delta^{it} J y' \xi_0\, dt.$$

On the other hand, by using the Corollary 6°(2),

$$x^* y' \xi_0 = y' x^* \xi_0 = y' S x \xi_0 = y' J \Delta^{1/2} (\Delta + \omega)^{-1} x' \xi_0$$
$$= \int_{-\infty}^{\infty} \frac{\omega^{it - \frac{1}{2}}}{e^{\pi t} + e^{-\pi t}} \, y' J \Delta^{-it} x' \xi_0 \, dt$$

It follows that:

$$F(\omega) := \int_{-\infty}^{\infty} \frac{\omega^{it-\frac{1}{2}}}{e^{\pi t}+e^{-\pi t}} \left(J\Delta^{-it} x' \Delta^{it} J y' \xi_0 - y' \Delta^{-it} x' \xi_0 \right) dt = 0$$

for every $\omega \in \mathbb{C}$, $|\omega| = 1$, $\omega \neq -1$. This equality extends by analyticity to all $\omega \in \mathbb{C} \setminus [-\infty, 0)$, so it is valid for all $\omega \in [0, \infty)$. But this means that the Fourier transform of the continuous L^1-function

$$t \mapsto \frac{1}{e^{\pi t} + e^{-\pi t}} \left(J\Delta^{-it} x' \Delta^{it} J y' \xi_0 - y' \Delta^{-it} x' \xi_0 \right)$$

vanishes and then, by the uniqueness theorem in Fourier Analysis we get

(1) $$J\Delta^{-it} x' \Delta^{it} J y' \xi_0 = y' \Delta^{-it} x' \xi_0 \quad (\forall)\, x', y' \in M'$$

8° We now exploit this equality. For $x', y', z' \in M'$ we get

$$J\Delta^{-it} x' \Delta^{it} J y' z' \xi_0 = y' z' \Delta^{-it} x' \xi_0 = y' J \Delta^{-it} x' \Delta^{it} J z' \xi_0$$

and, since $M'\xi_0$ is dense in H, it follows that

$$\left(J\Delta^{-it} x' \Delta^{it} J \right) y' = y' \left(J \Delta^{-it} x' \Delta^{it} J \right)$$

Since $y' \in M'$ was abitrary and $(M')' = M$, we conclude

(1) $$J\Delta^{-it} x' \Delta^{it} J \in M, \quad (\forall\, x' \in M', (\forall)\, t \in \mathbb{R})$$

in particular $(t = 0)$:

(2) $$JM'J \subset M$$

We want to show that also $JMJ \subset M'$. Let $y, z \in M$ and $x' \in M'$. Then, using the above conclusion (!) we get

$$(zJy\xi_0 \mid x'\xi_0) = (Jy\xi_0 \mid z^* x' \xi_0) = (Jy\xi_0 \mid x' z^* \xi_0) =$$
$$= (x'^* J y \xi_0 \mid J\Delta^{1/2} z \xi_0) = (\Delta^{1/2} z \xi_0 \mid J x'^* J y \xi_0) =$$
$$! = (\Delta^{1/2} z \xi_0 \mid S\, y^* J x' J \xi_0)$$
$$= (\Delta^{1/2} z \xi_0 \mid \Delta^{-1/2} J y^* J x' \xi_0) = (z \xi_0 \mid J y^* J x' \xi_0)$$
$$= (JyJz\xi_0 \mid x'\xi_0)$$

and, since $M\xi_0$ is dense in H,

$$z\,Jy\xi_0 = JyJz\xi_0\,, \quad (y,z\in M).$$

So, for $x,y,z\in M$:

$$JyJxz\xi_0 = xz\,Jy\xi_0 = x\,JyJz\xi_0$$

and, as $M\xi_0$ is dense in H we conclude

$$(JyJ)x = x(JyJ), \quad (\forall)\ x,y\in M$$

that is $JyJ\in M'$ and therefore $JMJ\subset M'$

We thus get the first main conclusion

(3) $$JMJ = M'\,,\quad JM'J = M$$

Since $J\Delta^{it} = \Delta^{it}J$, $(t\in\mathbb{R})$, and $JM'J = M$, from (1) we get now also the second main conclusion

(4) $$\Delta^{it}M\Delta^{-it} = M \quad (\forall)\ t\in\mathbb{R}.$$

9°. The Tomita Theorem is now proved. The map

$$j: M\ni x\longmapsto Jx^*J\in M'$$

is a $*$-antiisomorphism of M onto M', and the mappings

$$\sigma_t: M\ni x\longmapsto \Delta^{it}x\Delta^{-it}\in M$$

is a one-parameter group of automorphisms of M, called the <u>modular automorphism group</u> of M associated to $\varphi = \omega_{\xi_0}$, while J is called the <u>canonical conjugation</u>

10° PROOF OF THEOREM 6°. By assumption $\omega\in\mathbb{C}$, $|\omega|=1$, $\omega\neq -1$, and

(1) $$(x\eta|\xi) = (yB^{-1/2}\eta|A^{1/2}\xi) + \omega\,(yB^{1/2}\eta|A^{-1/2}\xi)$$

for all $\xi\in D(A^{1/2})\cap D(A^{-1/2})$, $\eta\in D(B^{1/2})\cap D(B^{-1/2})$

and we want to show that

$$y = \Phi_\omega(x) = \int_{-\infty}^{\infty}\frac{\omega^{it-\frac12}}{e^{\pi t}+e^{-\pi t}}\,A^{-it}xB^{it}\,dt$$

a) Let $e_n = \chi_{(\frac1n,n)}(A) = \chi_{(\frac1n,n)}(A^{-1}) \longrightarrow$ the support $s(A)=1$

$$f_n = \chi_{(\frac1n,n)}(B) = \chi_{(\frac1n,n)}(B^{-1}) \longrightarrow s(B)=1$$

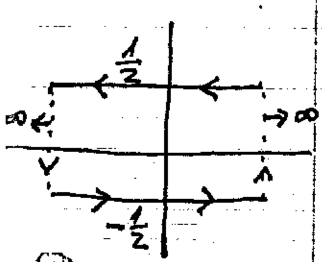
and consider the mappings:

$$\mathbb{C}\setminus i\mathbb{Z} \ni \alpha \longmapsto F_{n,m}(\alpha) = \frac{-i\,\omega^{i\alpha}}{e^{\pi\alpha}-e^{-\pi\alpha}}\, A^{-i\alpha} e_n x B^{i\alpha} f_n \in B(H)$$

which are analytic for the norm topology on $B(H)$.

$F_{n,m}$ has a simple pole in 0 with residuum

$$\lim_{\alpha\to 0} \alpha F_{n,m}(\alpha) = -\frac{i}{2\pi}\, e_n x f_m .$$



Using the Cauchy residua theorem we obtain:

$$\int_{-\infty}^{\infty} F_{n,m}\left(t-\tfrac{i}{2}\right)dt - \int_{-\infty}^{\infty} F_{n,m}\left(t+\tfrac{i}{2}\right)dt = e_n x f_m$$

that is

$$\int_{-\infty}^{\infty} \frac{\omega^{it+\frac12}}{e^{\pi t}+e^{-\pi t}}\, A^{-\frac12} e_n (A^{-it} x B^{it}) B^{\frac12} f_m \, dt \, +$$

$$+ \int_{-\infty}^{\infty} \frac{\omega^{it-\frac12}}{e^{\pi t}+e^{-\pi t}}\, A^{\frac12} e_n (A^{-it} x B^{it}) B^{-\frac12} f_m \, dt = e_n x f_m .$$

Applying this so: $(\cdot\,\eta|\xi)$ we get

$$(x f_m \eta | e_n \xi) = (\Phi_\omega(x) B^{-\frac12} f_m \eta | A^{\frac12} e_n \xi) + \omega(\Phi_\omega(x) B^{\frac12} f_m \eta | A^{-\frac12} e_n \xi)$$

and with $n\to\infty$, $m\to\infty$ we see that $y=\Phi_\omega(x)$ satisfies (1).

b) It remains to show that from (1) with $x=0$ it follows $y=0$.
From (1) with $x=0$ we get

$$A^{\frac12} e_n y B^{-\frac12} f_m + \omega A^{-\frac12} e_n y B^{\frac12} f_m = 0 .$$

Multiply so: $f_m y^* A^{-\frac12} e_n (\cdots) B^{-\frac12} f_m$ to get

$$f_m y^* e_n y B^{-1} f_m = -\omega f_m y^* A^{-1} e_n y f_m$$

i.e.

$$(f_m y^* e_n y f_m)(B^{-1} f_m) = -\omega f_m y^* A^{-1} e_n y f_m$$

It follows that*) $f_m y^* e_n y B^{-1} f_m = 0$, $f_m y^* e_n y f_m = 0$, $e_n y f_m = 0$ and, with $n\to\infty$, $m\to\infty$, $y=0$, q.e.d.

*) $a,b,c \geq 0$, $|\lambda|=1$, $\lambda\neq 1$, $ab=\lambda c \Rightarrow c=0$. Indeed:

$ab=\lambda c \Rightarrow Sp(ab)\subset \lambda\mathbb{R}^+$
$ba=\bar\lambda c \Rightarrow Sp(ba)\subset \bar\lambda\mathbb{R}^+$] and, as $Sp(ab)=Sp(ba)$, $\Rightarrow$

$\Rightarrow (\exists)\ s,t\in\mathbb{R}^+$, $\lambda s=\bar\lambda t=\lambda^{-1}t \Rightarrow \lambda^2 s=t \Rightarrow$

$\Rightarrow s=|\lambda^2 s|=|t|=t \Rightarrow \lambda^2=1 \Rightarrow \lambda=-1$. And:

$ab=-c \Rightarrow ba=-c=ab \Rightarrow ab\geq 0$ while $-c\leq 0$. So $c=ab=0$.

11°. Using the GNS constuction we obtain the following result for an n.s.f. <u>positive form</u>:

THEOREM. There is a (canonical) conjugation $J_\varphi : H_\varphi \longrightarrow H_\varphi$ a nonsingular positive self-adjoint operator Δ_φ in H_φ (the modular operator) and a one parameter continuous automorphisms group

$$\sigma^\varphi : \mathbb{R} \longrightarrow Aut(M)$$

(the modular automorphisms group) such that

$$J\, \pi_\varphi(M)\, J = \pi_\varphi(M)' \quad \text{and} \quad \pi_\varphi(\sigma_t^\varphi(x)) = \Delta_\varphi^{it}\, \pi_\varphi(x)\, \Delta_\varphi^{-it}, \ (x \in M).$$

In particular M is $*$-isomorphic to a standard von Neumann algebra $\pi_\varphi(M) \subset B(H_\varphi)$.

<u>Exactly</u> <u>the</u> <u>same</u> <u>result</u> <u>is</u> <u>obtained</u> <u>if</u> φ <u>is</u> <u>an</u> <u>n.s.f. weight</u>.

The proof is similar to the proof in case of n.s.f states (the Theorem 6° plays the same role), but the preliminaries are a bit more involved. The proof can be read in

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where one starts with a left Hilbert algebra (instead of a bicyclic vector which may not exist).