0.5 setg
.5 setgray1

When is a knot not the unknot?

V.S. Sunder

IMSc

Chennai

Outline of talk

- What are knots?

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- Equivalence of (oriented) knots/links

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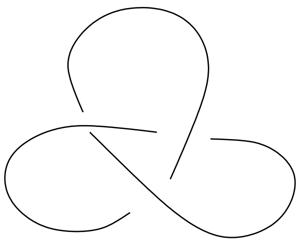
- Knot invariants
- Skein relations
- The Jones polynomial invariant

Some knots

right–handed trefoil

T_+



Some knots

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  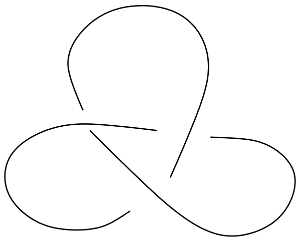

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  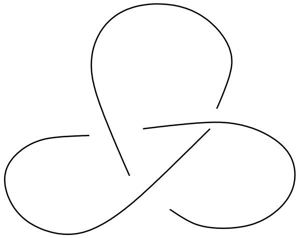

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  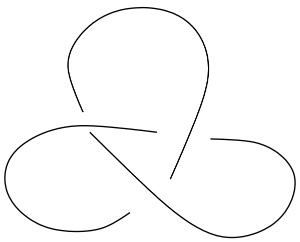

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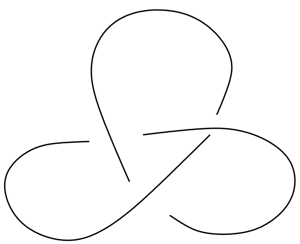
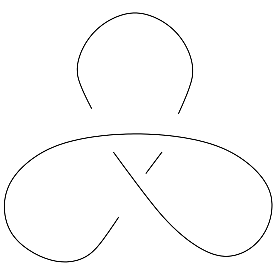
  

- *unknot*

  U_1

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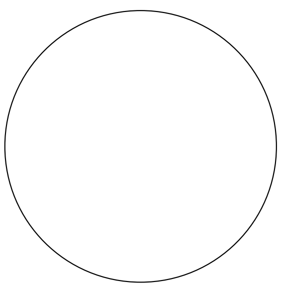
  

Questions

When is $K \sim U_1$?

More generally, when is $K_1 \sim K_2$?

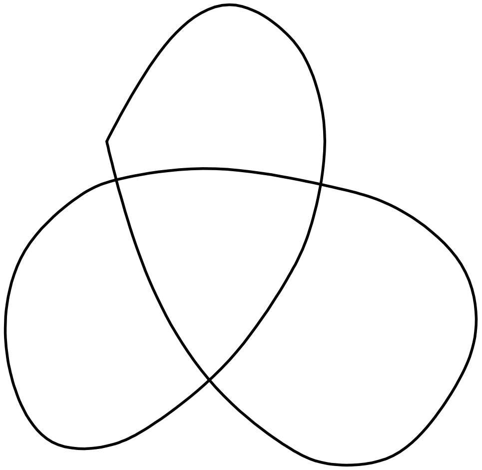
(here, $\sim$ denotes *ambient isotopy*.)

i.e., when can you jiggle K_1 into K_2?

Knot-projections *vs* knot-diagrams

As above, we employ *plane projections* to denote knots.

Many links may have the same ‘projection’:

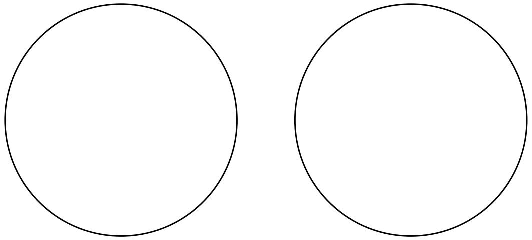


Use device of *over-* and *under-crossings*.

Links

unlink on two components

U_2



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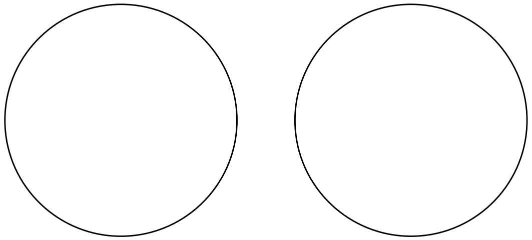
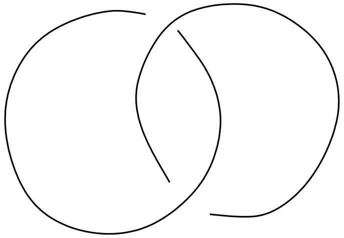


- *Hopf link* H_+



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  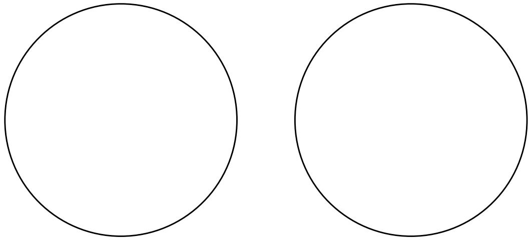

- *Hopf link*

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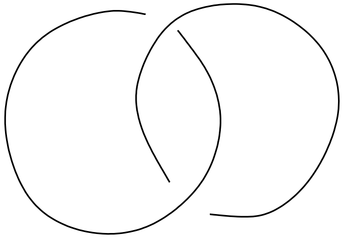
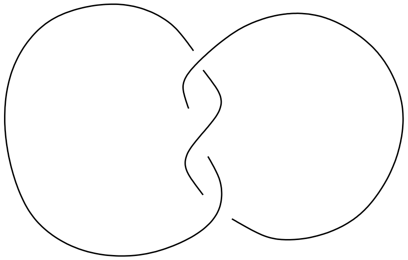
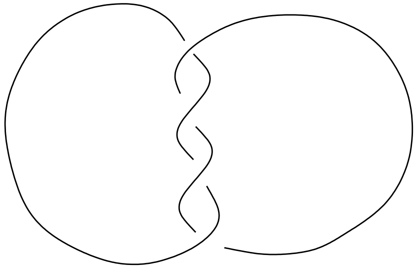
  

- *T_+*

  Q

Link invariants

An $\mathcal{S}$-valued **link invariant** is an assignment

$$\mathcal{L} \ni \mathcal{L} \mapsto \phi_L \in \mathcal{S}$$

such that

$$L_1 \sim L_2 \;\Rightarrow\; \phi_{L_1} = \phi_{L_2}.$$

$\mathcal{L}$ = set of ‘oriented link diagrams’
$\mathcal{S}$ = any set

So if $\phi_{L_1} \neq \phi_{L_2}$ then L_1 and L_2 are not equivalent.

link invariants may tell inequivalent links apart

Examples of Link invariants

- $c(L)$ = no. of components of L

$$c(U_n) = n$$

But, $c(K) = 1$ for *every* knot K!

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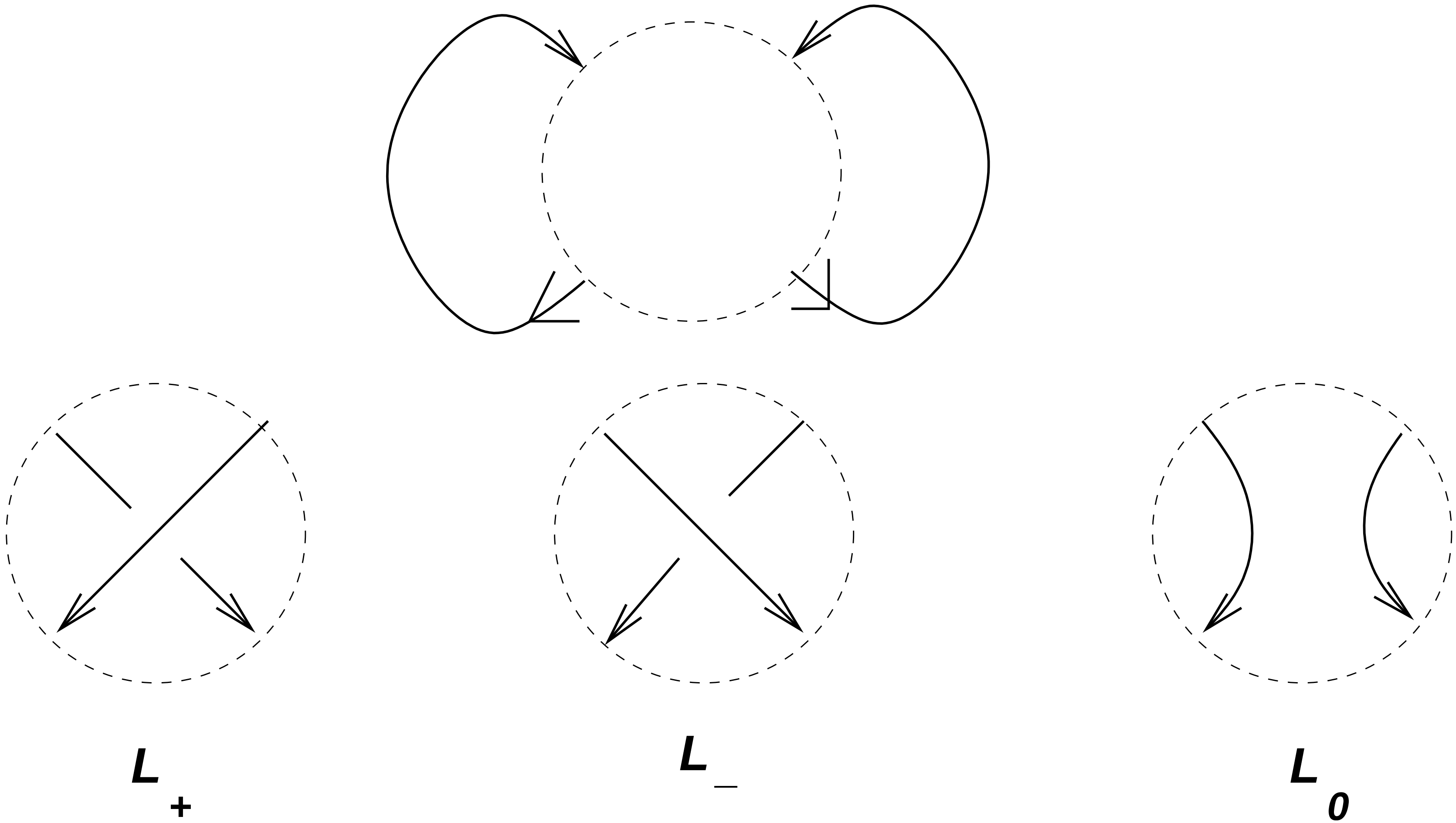
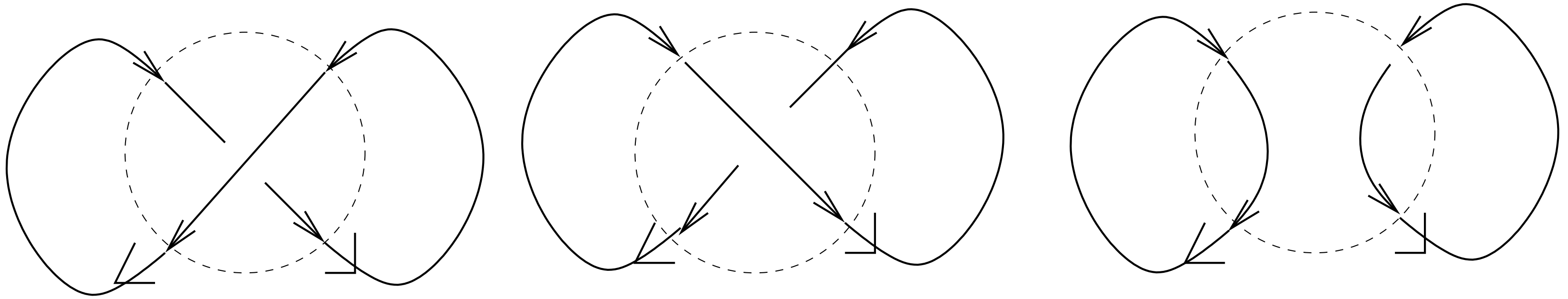
- Useful link invariants must be **discriminating** and **computable.**

Skein relation



A triple (L_+, L_-, L_0) as above is said to be **skein-related**

Example U of a skein-related triple



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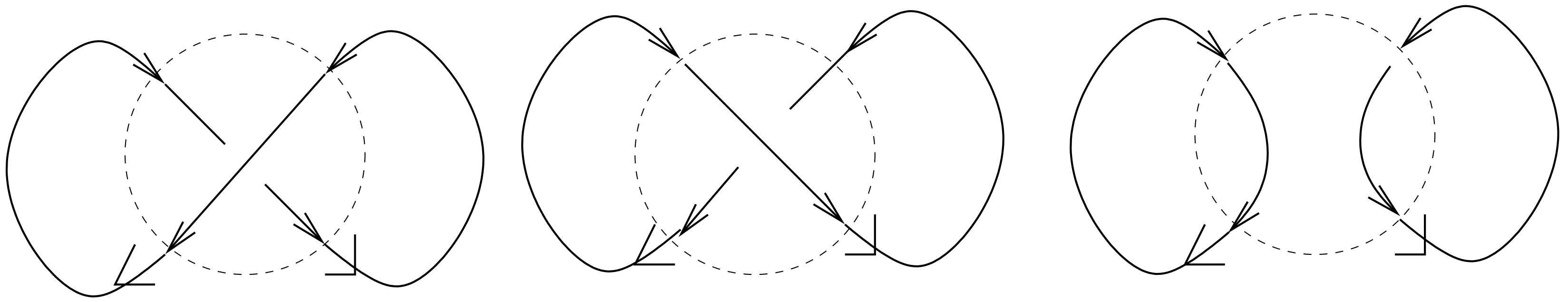
- 

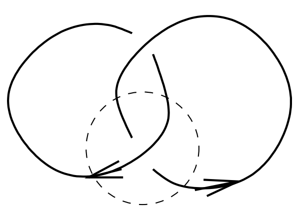
- $$L_+ = L_- = U_1,\ L_0 = U_2$$

  and more generally, for any $n \geq 1$,

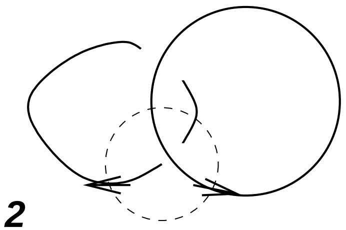
  $$L_+ = L_- = U_n,\ L_0 = U_{n+1}$$

Example H of a skein-related triple

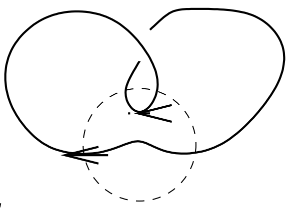
- H_+



U_2



U_1



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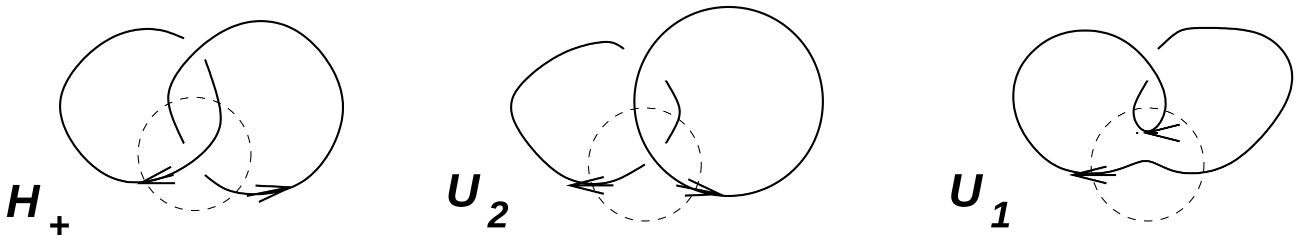
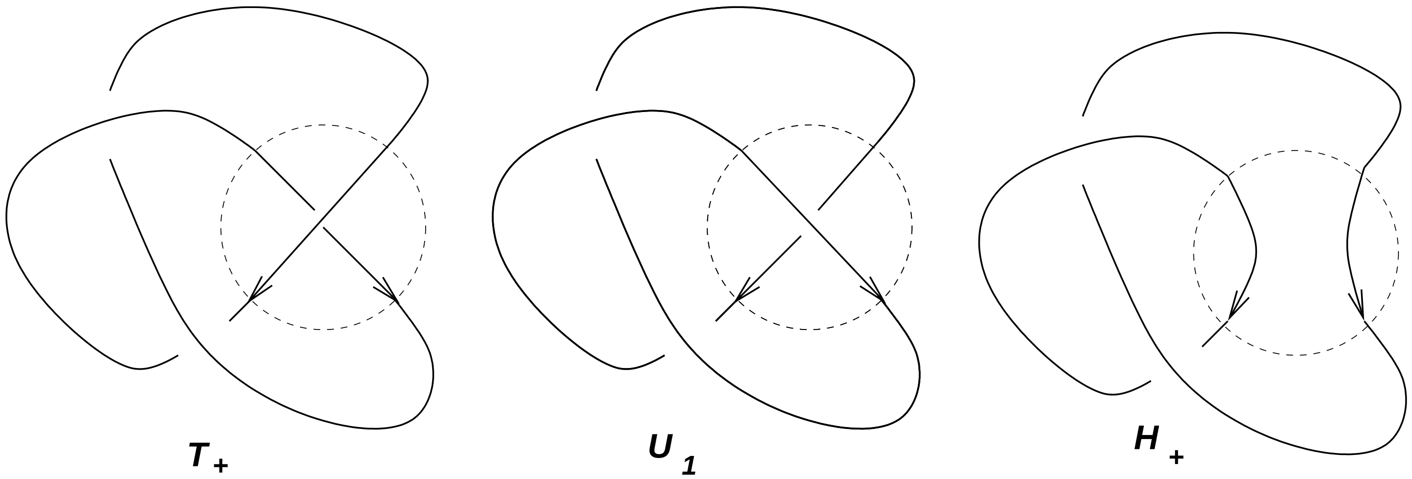


$$L_+ = H_+, L_- = U_2,\ L_0 = U_1$$

Example T of a skein-related triple

- 

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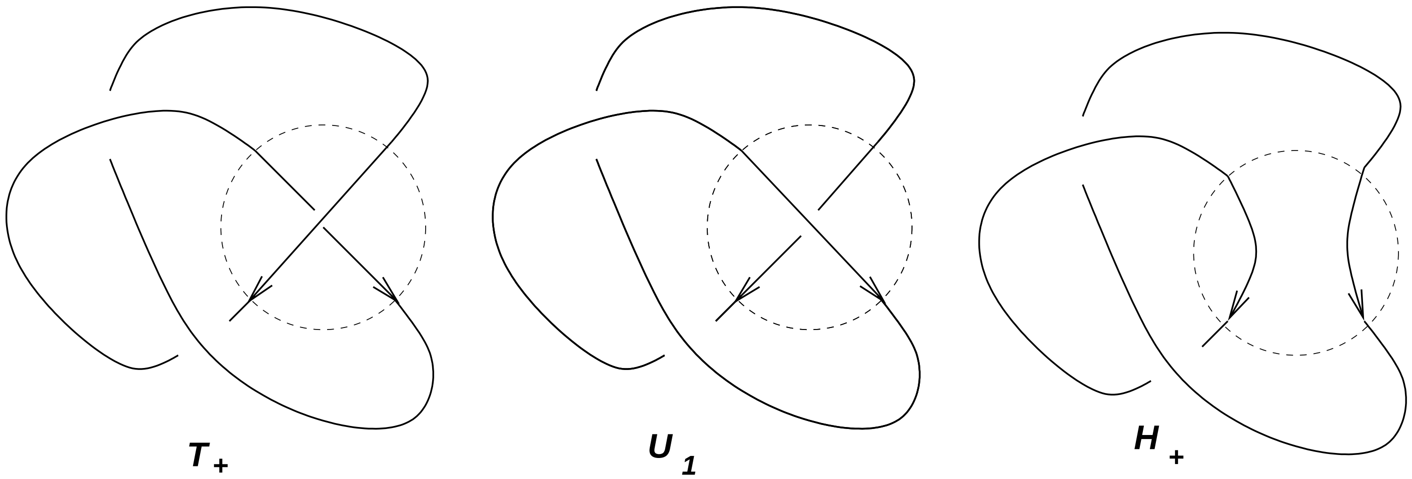

$$L_+ = T_+, L_- = U_1,\ L_0 = H_+$$

Aside on Laurent polynomials

Here is an example of a (usual) polynomial:

$$3 - 4t + 17t^3 - 50.7t^{419}$$

Here is an example of a Laurent polynomial:

$$\begin{aligned} & \frac{2}{t^6} - \frac{3}{t} + 7 + 9t^5 \\ = \ & t^{-6} \times (2 - 3t^5 + 7t^6 + 9t^{11}) \end{aligned}$$

So a Laurent polynomial (in q) is an expression of the form $q^{-m} \times P(q)$

The Jones polynomial

Theorem (V. Jones) There exists an invariant of oriented links

$$L \mapsto V_L(q)$$

taking, as values, Laurent polynomials in $q^{\frac{1}{2}}$, which is uniquely determined by the properties

- $$V_{U_1}(q) = 1$$
- and

$$q^{-1}V_{L_+}(q) - qV_{L_-}(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})V_{L_0}(q)$$

V_{U_n}

The first equation in Example U of a skein related triple gives:

$$(q^{\frac{1}{2}} - q^{-\frac{1}{2}})V_{U_2}(q) = (q^{-1} - q)V_{U_1}(q)$$

and so

$$V_{U_2}(q) = \left(\frac{q^{-1} - q}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\right) = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \ ;$$

and similarly the second equation of that example yields

$$V_{U_{n+1}}(q) \ = \ -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})V_{U_n}(q)$$

V_{H_+}

Example H of a skein related triple yields

$$\begin{aligned} V_{H_+}(q) &= q\left(qV_{U_2}(q) + (\sqrt{q} - \frac{1}{\sqrt{q}})V_{U_1}(q)\right) \\ &= q\left(-\frac{q(q+1)}{\sqrt{q}}) + \frac{q-1}{\sqrt{q}}\right) \\ &= -\sqrt{q}(q^2+1) \end{aligned}$$

V_{T_+}

Example T of a skein related triple yields

$$
\begin{aligned}
V_{T_+}(q) &= q\left(qV_{U_1}(q) + (\sqrt{q} - \frac{1}{\sqrt{q}})V_{H_+}(q)\right) \\
&= q\left(q + (\sqrt{q} - \frac{1}{\sqrt{q}})(-\sqrt{q})(q^2+1)\right) \\
&= q\left(q - (q-1)(q^2+1)\right) \\
&= q\left(q + 1 - q + q^2 - q^3\right) \\
&= q + q^3 - q^4
\end{aligned}
$$

Properties of $V_L(q)$

The relation between the Jones polynomials associated to skein-related links, together with a sort of induction argument, can be used to prove the following properties of the Jones polynomial:

- If $c(L)$ is odd, then $V_L(q)$ is a Laurent polynomial in q

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- If $c(L)$ is even, then then $V_L(q)$ is $\sqrt{q}$ times a Laurent polynomial in q
- If $\widetilde{L}$ denotes the 'mirror-reflection' of L, then $V_{\widetilde{L}}(q) = V_L(q^{-1})$

Conclusion

1. $V_{T_+}(q) = q + q^3 - q^4$.
2. $V_{T_-}(q) = q^{-1} + q^{-3} - q^{-4}$.
3. $V_{H_+}(q) = -\sqrt{q}(q^2 + 1)$
4. $V_{U_n}(q) = \left(-(q^{\frac{1}{2}} + q^{-\frac{1}{2}})\right)^{n-1}$

Hence, T_+, T_-, H_+, U_n all have different Jones polynomials; and we may deduce that they are all pairwise inequivalent links!

Open problem

Though the Jones polynomial can detect all this, the following problem is still open.

Can the Jones polynomial decide if a knot is not the unknot?

If you can crack this problem, Vaughan Jones would be only too happy to split his Leff with you.

Vaughan Jones and his Leff

References

[1] *Knots*, V.S. Sunder, Resonance, Vol. **1**, no. 7, (1996), 31-43.

(This contains details of many things discussed in this talk.)

[2] *On the Jones polynomial*, Pierre de la Harpe, Michael Kervaire and Claude Weber, l'Enseignement Mathématique, **32**, (1986), 271-335.

(This is much more *meaty*; it includes a proof of the fact that the Jones polynomial is indeed an invariant of oriented links.)