Transcendental L^2-Betti numbers Atiyah's question

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Analytic definition

Definition (Atiyah)

M = closed Riemannian manifold, $\pi_1(M) = \Gamma$, universal covering $\tilde{M}$ ($M = \tilde{M}/\Gamma$) with fundamental domain F.

L^2-Betti numbers:= normalized dimension(space of L^2-harmonic forms):
$pr\colon L^2\Omega^k(\tilde{M}) \to L^2\Omega^k(\tilde{M})$ be orthogonal projection onto the space of harmonic L^2-forms = $\ker(\Delta)$. It has a smooth integral kernel, and

$$b^k_{(2)}(\tilde{M};\Gamma) := \int_F tr_x pr(x,x)\, dx.$$

(Here: use a lifted Riemannian metric).

Examples

- $\Gamma = 1$ or more generally $|\Gamma| < \infty$:

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- harmonic 0-forms are constant functions which are not L^2 if $\tilde{M}$ is non-compact, therefore

$$b^0_{(2)}(\tilde{M};\Gamma) = 0; \qquad \text{if } |\Gamma| = \infty.$$

Properties

- **Multiplicative under coverings**. If M' is a finite covering of M with fundamental group $\Gamma' \subset \Gamma$, then $\tilde{M}$ is the universal covering also of M' and
$$b^k_{(2)}(\tilde{M};\Gamma') = [\Gamma : \Gamma'] b^k_{(2)}(\tilde{M};\Gamma).$$
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- in particular, if M covers itself non-trivially (like the torus T^n) then $b^k_{(2)}(\tilde{M}; \Gamma) = 0$ for all k.
- $b^k_{(2)}(\overline{M}, \Gamma)$ only depends on the homotopy type of M.

Euler characteristic

A special case of Atiyah's L^2-index theorem states

$$\chi(M) = \sum_{k=0}^{\dim M} (-1)^k b_{(2)}^k(\tilde{M};\Gamma) \in \mathbb{Z}.$$

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- If Γ is torsion-free then $b^k_{(2)}(\tilde{M};\Gamma) \in \mathbb{Z}$.
- Let $A_\Gamma \subset \mathbb{Q}$ be the additive subgroup generated by $1/|F|$ where F runs through the finite subgroups of Γ. Then $b^K_{(2)}(\tilde{M};\Gamma) \in A_\Gamma$.
- the above assertion, but only if there is a bound on the orders of finite subgroups of Γ.

Generalization

Instead of working with the universal covering with action of Γ be deck transformations, one can use any normal covering $\bar{M} \to M$ with deck transformation action by π (and $M = \bar{M}/\pi$).
One the gets

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Remark:

- universal covering — $\Gamma = \pi_1(M)$ — Γ finitely presented
- general normal covering — arbitrary quotient of $\pi_1(M)$ — group finitely generated

Combinatorial description

K = finite simplicial (or CW-) complex (e.g. triangulation of M). Let $\tilde{K}$ = induced cell decomposition of universal covering.
Consider the cellular cochain complex, and the subcomplex of square summable cochains.
The combinatorial L^2-cohomology is its reduced cohomology

$$H^k_{(2)}(\tilde{K};\Gamma) := \ker(d)/\overline{im(d)} \cong \ker(d^*d + dd^*).$$

Γ acts simplicially and freely on $\tilde{K}$:

$$C^k_{(2)}(\tilde{K}) \cong \oplus_{d_k} l^2(\Gamma),$$

d_k = numer of k-cells in K.
Under this identification, differential $d_k\colon C^k_{(2)} \to C^{k+1}_{(2)}$ is left (convolution) multiplication with a matrix A over $\mathbb{Z}[\Gamma]$.

Example: S^1

$K = S^1$, with one 0-cell and one 1-cell. Then $\tilde{K} = \mathbb{R}$, $\Gamma = \mathbb{Z}$ with one orbit $\{0\} \times \mathbb{Z}$ of 0-cells and one orbit $[0,1] \times \mathbb{Z}$ of 1-cells. We obtain cellular L^2-cochain complex

$$l^2(\mathbb{Z}) \xrightarrow{z-1} l^2(\mathbb{Z})$$

(z the generator of $\mathbb{Z}$).

Combinatorial L^2-Betti numbers

$pr \in B(C^k_{(2)}(\tilde{K})) \cong B(l^2(\Gamma)^{n_k}) :=$ orthogonal projection onto $\ker(d^*d + dd^*)$.
This is a measurable function of $d^*d + dd^* \in M_{n_k}(\mathbb{C}\Gamma) \subset B(l^2\Gamma^{n_k})$, therefore lies in the von Neumann closure $M_n(\mathbb{C}) \otimes L\Gamma$.
This is a finite von Neumann algebra with trace $\tau = Tr \otimes \tau_e$ ($\tau_e(f) = \langle f(\delta_e), \delta_e \rangle_{l^2\Gamma}$ the standard trace on $L\Gamma$).
Define $b^k_{(2)}(\tilde{K}, \Gamma) := \tau(pr) := \dim_\Gamma(\ker(d^*d + dd^*))$.

L^2-de Rham theorem

Theorem (Dodziuk's L^2-Hodge-de Rham theorem)

Analytic and combinatorial L^2-Betti numbers of a closed manifold coincide.

Algebraic reformulation of the Atiyah conjecture

If Γ is finitely presented, for every matrix A over $\mathbb{Z}[\Gamma]$ one can construct a closed M with $\pi_1(M) = \Gamma$ and such that A^*A is a combinatoral Laplacian. Therefore, equivalent to the above Atiyah conjecture is:

- $\dim_\Gamma(\ker(A^*A)) \in \mathbb{Q}$ for all $A \in M_n(\mathbb{Z}\Gamma)$
- $\dim_\Gamma(\ker(A^*A)) \in \mathbb{Z}$ if Γ is torsion-free
- $\dim_\Gamma(\ker(A^*A) \in A_\Gamma$ for general Γ.

If Γ' is finitely generated (but not finitely presented), one can still construct M with a Γ'-covering such that A^*A is a combinatoral of this covering.

relevant algebras

$$\begin{array}{ccccccc}
\mathbb{Z}[\Gamma] & \longrightarrow & \mathbb{C}[\Gamma] & \longrightarrow & C^*_{red}\Gamma & \longrightarrow & L\Gamma \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D_{\mathbb{Q}}\Gamma & \longrightarrow & D_{\mathbb{C}\Gamma} & \longrightarrow & & \longrightarrow & U\Gamma
\end{array}$$

$U\Gamma$ is the algebra of *affiliated operators*, i.e. densely defined operators on $l^2(\Gamma)$ all whose spectral projections belong to $L\Gamma$ (needs that $L\Gamma$ is finite to define addition and multiplication). $D_{\mathbb{Q}}\Gamma$ is the division closure: the smallest subalgebra of $U\Gamma$ containing $\mathbb{Q}\Gamma$ and closed under taking inverses in $U\Gamma$.

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We are interested in special projections in $L\Gamma$, namely kernel projectons for $A \in \mathbb{Z}\Gamma$. Without this condition, always projections in $L\Gamma$ with arbitrary trace exist. On the other hand, in $C^*\Gamma$, and certainly in $\mathbb{Q}\Gamma$ almost no traces exist.

In some sense this is non-commutative algebrac geometry: understand solution sets of non-commutative polynomial equations.

Example

Specialize to $\Gamma = \mathbb{Z}$. Then we obtain (via the Fourier transform isomorphism) the diagram

$$\begin{array}{ccccccc}
\mathbb{Z}[z,z^{-1}] & \longrightarrow & \mathbb{C}[z,z^{-1}] & \longrightarrow & C(S^1) & \longrightarrow & L^\infty(S^1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{Q}(z) & \longrightarrow & \mathbb{C}(z) & \longrightarrow & & \longrightarrow & \{f\colon S^1 \xrightarrow{\text{measurable}} \mathbb{C}\}.
\end{array}$$

Atiyah conjecture and algebra

Theorem (Linnell)

If Γ is torsion free, then the Atiyah conjecture for Γ holds if and only if $D_{\mathbb{Q}}\Gamma$ is a skew field.
In this case, the L^2-Betti numbers are dimensions over this skew-field (and therefore integers).

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Theorem (Knebusch-Linnell-Schick)

If Γ has a bound on the orders of finite subgroups, then the Atiyah conjecture (slightly refined) holds if and only if $D_{\mathbb{Q}}\Gamma$ is a (certain) finite direct sum of matrix algebras over skew fields (number and size given by the lattice of finite subgroups).

Positive results

The Atiyah conjecture is known for the following groups with bound on the order of finite subgroups

- for $\mathbb{Z}^n$ (folklore)
- for free groups (Linnell)
- for (extensions of free by) elementare amenable groups (Linnell)
- for residually torsion-free elementary amenable groups (S.)
- for braid groups (Linnell-S.)
- for congruence subgroups of $Sl(n, \mathbb{Z})$ (Farkas-Linnell)

With bound on the orders of finite subgroups, there is no known counterexample.

Negative results

Let $\Gamma = (\oplus_{\mathbb{Z}}\mathbb{Z}/2) \rtimes \mathbb{Z}$ be the lamplighter group.
Then "the" Markov operator

$$A = \sum_{g\in\mathbb{Z}/2} gt + t^{-1} \sum_{g\in\mathbb{Z}/2} g \in \mathbb{Q}\Gamma$$

satisfies

$$\dim_\Gamma(\ker(A)) = \frac{1}{3}.$$

In particular, the strong Atiyah conjecture does not hold for Γ (because all its finite subgroups of Γ are 2-groups).

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In particular, the strong Atiyah conjecture does not hold for Γ (because all its finite subgroups of Γ are 2-groups).
First proof by Grigorchuk-Zuk. Later, Dicks-S. generalize to other groups.
Moreover, a complete diagonalization with explicit computation of eigenspaces is given.
Indeed, the structure of the operator (and the group) lets the kernel break up as a sum of countably many contributions whose L^2-dimensions adds up to $1/3$.

More on decomposition

One gets a contribution for each finite connected subgraph (containing the origin) of the line, the Cayley graph of the quotient group $\mathbb{Z}$.More precisely, we have to understand a chosen rational eigenvalue of the of the graph Laplacian.

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Lehner-Neuhauser-Woess show that the same applies if $\mathbb{Z}$ is replaced by any other group.

Curious example

Dicks-S. construct another example with

$$\dim_\Gamma(\ker(A^*A)) = b^k_{(2)}(\tilde{M};\Gamma) = \sum_{k=1}^{\infty} \frac{\phi(k)}{(2^k-1)^2}.$$

Question: is this real number rational?
(if so, the denominator is larger than 10^{100}).

finitely presented groups

The lamplighter group does not admit a finite presentation. However, there is an easy induction principle: if $\Gamma \subset H$, also $\mathbb{Z}[\Gamma] \subset \mathbb{Z}[H]$. The von Neumann dimension of the kernel of $a \in M_n(\mathbb{Z}\Gamma)$ remains the same if we consider it via this embedding as element of $M_n(\mathbb{Z}[H])$. The lamplighter group can easily (and explicitly) be embedded into a finitely presented group which is 2-step solvable and we work with that one.

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More generally, every group with a recursive presentation can (by Higman's embedding result) be explicitly embedded into a finitely presented group.

Tim Austin's idea I

Continue to work with wreath products $\oplus_H \mathbb{Z}/2 \rtimes H$. But change from the usual Markow operator to a better operator:
Via Fourier transform in the abelian base group $\oplus_H \mathbb{Z}/2$ we have to look at pointwise multiplication operators acting on $L^2(\prod_H \mathbb{Z}/2)$ (rational linear combinations of characteristic functions of cylinder sets). Then useful operators are sums of such multiplication ops composed with translation by the generators of H.
Result: for H free get contributions only from (certain) locally determined paths in the Cayley graph whose graph Laplacian can be understood.

Tim Austin's idea II

This is still not good enough, as the contributions stack up to regularly and one gets probably still rational L^2-Betti numbers.
Change the base group $\oplus_H \mathbb{Z}/2$ to

$$(\oplus_H \mathbb{Z}/2) / V; \qquad \Gamma_V = (\oplus_H \mathbb{Z}/2) / V \rtimes H$$

for an H-invariant sub-vector space V: then we can still form the semidirect product. In the dual picture, we pass to a subgroup of the dual group. Explicit eigenspace calculatoins are still possible: we get the same with slightly shifted weights.

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Howeverk, in Austin's work the combinatorics was too complicated to really carry out the calcualations. Still, he has estimates to see: using different V, there is a Canter set (i.e. uncountably many) different L^2-betti numbers, among them therefore transcendental ones.

Finite presentation

Problem: The groups Austin produces can not be embedded into finitely presented groups (counting argument: in total there are only countably many matrices over the integral group rings of finitely presented groups!

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Theorem (Pichot-S.-Zuk)

Taking up the basic constructoin of Austin, but with a couple of crucial improvements to make explicit calculations possible:

1. *with finite generated groups, every non-negative real number is an L^2-Betti number*
2. *with finitely presented groups, there are explicit transcendental L^2-Betti numbers (like $e, \pi, \ldots$), also every algebraic number is an L^2-Betti number*
3. *these examples can be realized with solveable groups.*

Other work

Grabowski developped the Dicks-S. ideas in a different direction, combined with algebraic implementations of Turing machines, to obtain the same results as above with different groups (which are slightly easier).

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Lehner and Wagner managed to push through the conbinatorics/linear algebra for the standard Markov operator on the wreath product of $\mathbb{Z}/k$ with F_l and for suitable k, l obtain irrational algebraic L^2-Betti numbers.

Open questions

- All the examples have arbitrarily large finite subgroups. What about the torsin-free case?
- What precisely is the set of real numbers obtained as L^2-Betti numbers for finitely presented groups (i.e. for universal coverings)? It is known that this set is countable, and there are some weak computability restrictions.