Kaç algebras, quantum doubles and planar algebras

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June 23, 2008

Abstract

This note contains the announcement, and a description, but wihout any proofs, of results that will go into the doctoral thesis written by the first author under the guidance of the second author. The goal is to describe the planar algebra of the asymptotic inclusion of the subfactor of fixed points under an outer action of a finite-dimensional Kaç algebra on the hyperfinite II_1 factor.

When it was suggested recently that the second author might contribute an article to the special issue being brought out to commemorate (Varada)Raja(n)'s turning seventy, he did not immediately have anything handy, but did not want to miss out on the opportunity of raising a toast to one who has been a source of inspiration for many an aspiring Indian mathematician, and has been a dear friend for decades now. As a compromise solution, we submit something in the nature of a note one may find in *Comptes Rendus*, where we announce what will be the backbone of the doctoral thesis being written by the first author under the guidance of the second, with the interested reader being referred to the thesis (which now exists only in preprint form) for proofs of most facts stated here.

1 Introduction

The *raison d'être* for this investigation lies in the following three facts:

1. *(Ocneanu-Szymanski)*(cf. [Oc],[Sz])

   Finite-dimensional Kaç algebras (=Hopf C^*-algebras) are in bijective correspondence with a certain class of subfactors (specifically, those of *depth two*).

2. *(Ocneanu)*(cf. [EK])

   The subfactor analogue of the quantum double construction is the *asymptotic inclusion*.

3. *(Jones)*(cf. [Jon])

   'Good' subfactors are determined by their *planar algebras*.

Our goal is to describe the planar algebra of the asymptotic inclusion of a Kaç algebra subfactor. This paper is organised as follows: after a preliminary section (§2) devoted to recalling some basic definitions and facts about subfactors and planar algebras (and slightly expanding on what exactly the three facts above say), we devote the next section (§3) to describing a model for *the outer action of a Kaç algebra on the hyperfinite II_1 factor* which makes transparent the nature of the higher relative commutants; the next section (§4) uses the model developed in §3 of the Kaç algebra subfactor $N \subset M$ to obtain an explicit description of the members $\mathcal{M}_n$ of the basic construction tower of its asymptotic inclusion subfactor $\mathcal{N} \subset \mathcal{M}$. The final section (§5) reinterprets the slightly clumsy description (Lemma 4.1) obtained in §4 via the planar algebra formalism to obtain an aesthetically much more satisfactory description (Theorem 5.1) of the planar algebra $\mathcal{P}(H)$ of the asymptotic inclusion.

2 On subfactors

To every subfactor $N \subset M$ - by which we always mean a unital inclusion of II_1 factors - which has finite index, Jones showed how a *basic construction* led to a to a canonical tower

$$M_{-1} = N \subset M_0 = M \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots \tag{2.1}$$

of II_1 factors, which yielded a grid

$$\begin{array}{ccccccccc} \mathbb{C} = N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \cdots \\ & & \cup & & \cup & & \cup & \\ & & M' \cap M & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \cdots \end{array} \tag{2.2}$$

of finite-dimensional C^*-algebras, that comes equipped with a canonical consistent trace; and this 'traced grid' is referred to as *the standard invariant* of the subfactor $N \subset M$. A less cryptic formulation of the first fact of the introduction is: if $N' \cap M_2$ is isomorphic to a full matrix algebra $M_n(\mathbb{C})$ then (and only then) there exists an n-dimensional Kaç algebra H admitting an 'outer action' on M such that $N = M^H$ is the 'fixed point subalgebra'. (Definitions and more explanation can be found in the next section.) In case the factor M is *hyperfinite*, the (isomorphism class of the) subfactor $M^H \subset M$ depends only on the Kaç algebra H, and we shall call it the Kaç algebra subfactor associated to H.

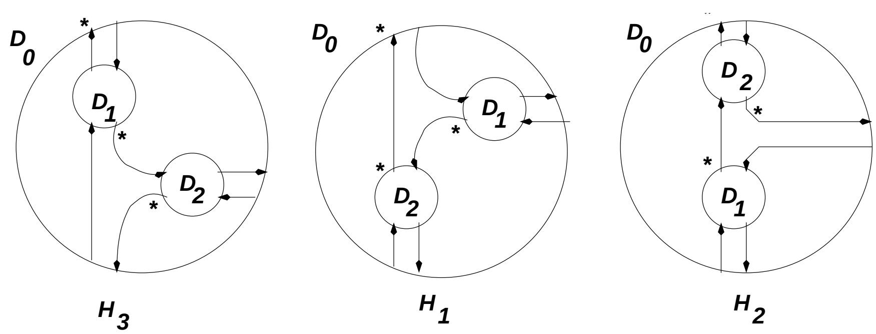
The two rows of the above grid can be completely described, in view of a certain 'reflection symmetry' possessed by the associated Bratteli diagrams, by a pair of bipartite graphs (usually called the *principal graphs*). The subfactor is said to have *finite depth* if either (equivalently both) of these graphs is finite. (A Kaç algebra subfactor has finite depth - and diameter two, in fact.)

The asymptotic inclusion associated to a finite depth subfactor $N \subset M$ is the inclusion given by $\mathcal{N} \subset \mathcal{M}$ where (i) $\mathcal{M}$ is the von Neumann closure of $\cup_{n=1}^{\infty} M_n$ of the image of the 'GNS representation' resulting from the unique consistent trace on this union, and (ii) $\mathcal{N} = (M \cup (M' \cap \mathcal{M}))''$. Ocneanu proved (see [Oc], and also, [EK)]) that the inclusion $\mathcal{N} \subset \mathcal{M}$ is also a subfactor of finite depth, which has reason (see, for instance, [Kaw]) to be likened to the quantum double construction of Drindel'd [Dri].

It turns out (see [Po1]) that the standard invariant is a complete isomorphism invariant for finite-depth subfactors. This has been studied and viewed in manifold equivalent perspectives - as a *paragroup* by Ocneanu ([Oc]), as a λ-*lattice* by Popa ([Po2]), and as a *subfactor planar algebra* by Jones ([Jon]). We shall say a few words about the last, since we will be using this version to decribe the asymptotic inclusion of a Kaç algebra subfactor.

A subfactor planar algebra should be viewed as an algebra over the coloured operad of planar tangles; thus it is a collecition $\{P_k : k \geq 0\}$ of finite-dimensional C^*-algebras with the property that each planar k-tangle T with 'internal boxes' of colours $k_1, \cdots, k_b$ gives rise to a linear map $Z_T : \otimes_{i=1}^{b} P_{k_i} \to P_k$; and the association $T \mapsto Z_T$ is required to satisfy some natural conditions. (Among them are the requirments that $P_0 = \mathbb{C}$ and that a certain natural 0-tangle with one internal k-box induces a faithful normal trace on P_k.) For a detailed account of planar algebras, please see [Jon] or [KS].

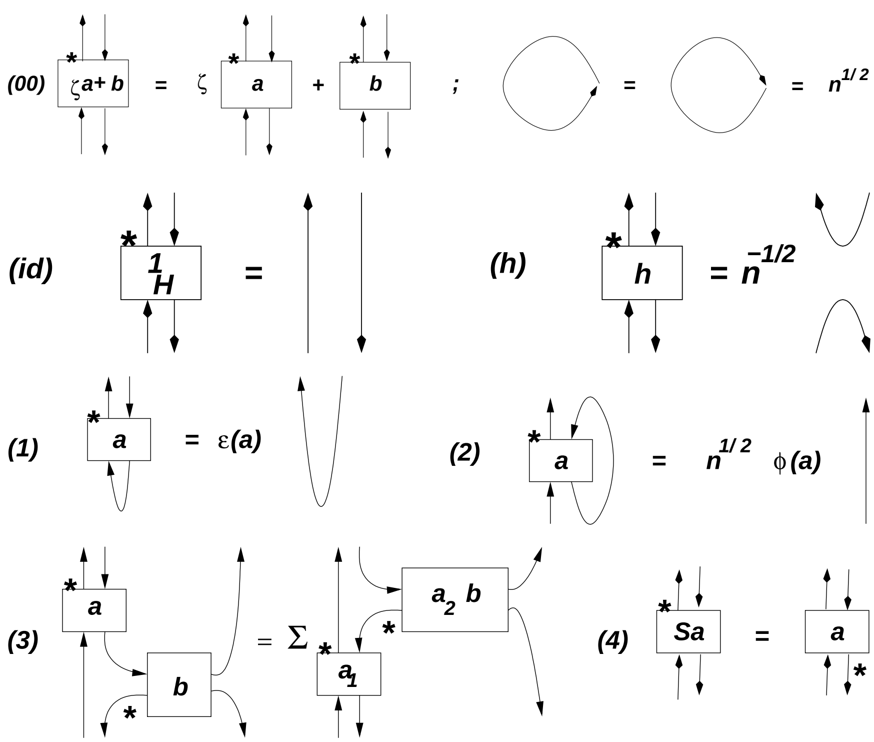
For our purposes, it will suffice to look at an example of some tangles:



All three tangles have the property that each of their inner boxes is a '2-box' (ie, has colour 2). Our main theorem depends on a presentation, given in [KLS], of the subfactor planar algebra associated to a Kaç algebra H, which we briefly describe now. Consider the space $\mathcal{U}_n(H)$ of spanned by all n-tangles each of whose internal box is a 2-box and is labelled by some element of H. Then $\mathcal{U} = \{\mathcal{U}_n : n \geq 0\}$ is the 'universal planar algebra' with labelling set given by

$$L_k = \begin{cases} H & \text{if } k = 2 \\ \emptyset & \text{if } k \neq 2 \end{cases}$$

It was shown in [KLS] that the subfactor planar algebra associated to the Kaç algebra H - which we shall consistently denote by $P(H)$ - is the quotient of $\mathcal{U}$ by the *planar ideal* generated by the following relations, which relate to (00) vector space structurein H, and the vector space dimension, say n, of H, (id) unit 1_H, (h) Haar integral h on H^*, (1) co-unit ϵ of H, (2) Haar integral[1] ϕ on H (3) multiplication and co-multiplication on H, and (4) antipode on H:



3 Kaç algebras

We begin by recalling the notion of an action of a Kac algebra $H(= (H, \mu, \eta, \Delta, \epsilon, S))$ on a C^*-algebra A.

DEFINITION 3.1. *By a left action of H (or simply, an action) on A, we mean a linear map $\alpha : H \to End_C(A)$ satisfying the following conditions.*

1. $\alpha_1 = Id_A$

[1] ϕ and h are assumed to be so normalised that each is a self-adjoint projection; they satisfy $\phi(h) = \frac{1}{n}$.

2. $\alpha_a(1_A) = \epsilon(a)1_A, \forall a \in H$

3. $\alpha_{ab} = \alpha_a \circ \alpha_b$

4. $\alpha_a(xy) = \Sigma\alpha_{a_1}(x)\alpha_{a_2}(y)$

5. $\alpha_a(x)^* = \alpha_{sa^*}(x^*)$

For such an action of H on A, the 'fixed subalgerbra' is defined by

$$A^H = \{x \in A : \alpha_a(x) = \epsilon(a)x \ \forall a \in H\}$$

EXAMPLE 3.2. *H^* acts on H by the rule $\alpha_f(a) = f(a_2)a_1$ Interchanging the roles of f and a we have a similiar action of H on H^**

DEFINITION 3.3 (the * algebra $A \rtimes H$). *The vector space $A \rtimes H = A \otimes H$ has algebra structure given by the following multiplication*

$$(x \rtimes a)(y \rtimes b) = x\alpha_{a_1}(y) \rtimes a_2 b.$$

It is easy to see that the multiplication is associative and that if A has an identity 1_A, then $A \rtimes H$ has identity given by $1_A \rtimes 1_H$. The $*$ structure on this associative unital algebra is given by $(x \rtimes a)^* = \alpha_{a_1^*}(x^*) \rtimes a_2^*$. For instance, the verification that this * is product-reversing is as follows:

$$\begin{aligned}
\{(x \rtimes a)(y \rtimes b)\}^* &= [x\alpha_{a_1}(y) \rtimes a_2 b]^* \\
&= \alpha_{(a_2 b)_1^*}[x\alpha_{a_1}(y)]^* \rtimes (a_2 b)_2^* \\
&= \alpha_{b_1^* a_2^*}\left[\alpha_{sa_1^*}(y^*)x^*\right] \rtimes b_2^* a_3^* \\
&= \alpha_{b_1}^*\left[\alpha_{a_2^* sa_1^*}(y^*)\alpha_{a_3^*}(x^*)\right] \rtimes b_2^* a_4^* \\
&= \alpha_{b_1}^*\left[\alpha_{a_2^* sa_1^*}(y^*)\alpha_{a_3^*}(x^*)\right] \rtimes b_2^* a_4^* \\
&= \alpha_{b_1^*}\left[y^*\alpha_{a_1^*}(x^*)\right] \rtimes b_2^* a_2^* \\
&= \alpha_{b_1^*}(y^*)\alpha_{b_2^* a_1^*}(x^*) \rtimes b_3^* a_2^* \\
&= \left[\alpha_{b_1^*}(y^*) \rtimes b_2^*\right]\left[\alpha_{a_1^*}(x^*) \rtimes a_2^*\right] \\
&= (y \rtimes b)^*(x \rtimes a)^*
\end{aligned}$$

The action of H^* on H can be promoted to an action - call it $f \mapsto \beta_f$ of H^* on $A \rtimes H$ by 'ignoring the A-component thus:

$$\beta_f(x \rtimes a) \quad = \quad x \rtimes \alpha_f(a) \ ,$$

and we can define

$$A \rtimes H \rtimes H^* = (A \rtimes H) \rtimes H^* \ .$$

If $k \leq l$ are integers, we shall write

$$A_{[k,l]} \quad = \quad H_k \rtimes H_{k+1} \rtimes .... \rtimes H_l$$

where $H_i = H$ or H^* according as i is odd or even; likewise, we shall write

$$\phi_i = \begin{cases} \phi & \text{if } i \text{ is odd} \\ h & \text{if } i \text{ is even} \end{cases}$$

and τ_i for the faithful tracial state on H_i defined by ϕ_i.

To start with, we have the following result:

THEOREM 3.4. *1. For $k \leq l$, there exists a unique faithful tracial state $\tau_{[k,l]}$ on $A_{[k,l]}$ satisfying*

$$\tau_{[k,l]}(x_k \rtimes x_{k+1} \rtimes .... \rtimes x_l) = \prod_{i=k}^{l} \tau_i(x_i).$$

2. *$A_{[k,l]} \subset A_{[k,l+1]} \subset A_{[k,l+2]}$ is an instance of Jones' basic construction, with a choice of 'Jones projection' being given by $(\phi_l)^{(l+2)}$, where we write*

$$H_i \ni x \mapsto x^{(\tilde{i})} = 1_{H_k} \rtimes \cdots \rtimes 1_{H_{(\tilde{i}-1)}} \rtimes x \rtimes 1_{H_{(\tilde{i}+1)}} \rtimes \cdots 1_{H_l} \quad (3.3)$$

for the natural inclusion maps of H_i into $A_{[k,l]}$ whenever $k \leq \tilde{i} \leq l$ and $\tilde{i} \equiv i(mod\ 2)$.

3. *The traces $\{\tau_{[k,l]} : k = \cdots, l-1, l\}\}$ patch up to yield a consistently defined trace $\tau_{(-\infty,l]}$ on $\cup_{k=-\infty}^{l} A_{[k,l]}$, and yield a model $A_{(-\infty,l]}(= (\pi_{\tau_{(-\infty,l]}}(\cup_{k=-\infty}^{l} A_{[k,l]})'')$ of the hyperfinite factor via the GNS construction.*

4. *If we write $M_l = A_{(-\infty,l]}$, for $l \geq -1$, then there exists a unique action α of H on M_0 such that $\alpha_a(\cdots x \rtimes f) = \cdots x \rtimes f_2(a)f_1$, where we use (our version of) the Sweedler notation whereby $\Delta(f) = f_1 \otimes f_2$; further*

   (a) *$(M_0)^H = M_{-1}$, and*

   (b) *the action α is outer in the sense that $M_0' \cap M_1 = \mathbb{C}$.*

5. $$M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots$$

   is the tower of Jones' basic construction.

6. *$M_k' \cap M_l = A_{[k+2,l]}\ \forall k \leq l$ (with the understanding that the right side is $\mathbb{C}$ if $k+2 > l$.*

4 The asymptotic inclusion

For a general finite index subfactor $N \subset M$, let the tower of Jones' basic construction be denoted, as usual, by

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots$$

Since all M_n's are II_1 factors and since the tracial state on a II_1 factor is unique, it follows that there is a consistently defined tracial state tr on $\cup_{n=1}^{\infty} M_n$, and that the von Neumann closure $(\pi_{tr}(\cup_{n=1}^{\infty} M_n)''$ in the GNS representation yields a II_1 factor $\mathcal{M} = M_\infty$. Ocneanu showed - see [EK], for instance - that if $N \subset M$ has finite depth, then $\mathcal{N} = (M \cup (M' \cap M_\infty))''$

is a subfactor of finite index in $\mathcal{M}$, and in fact, is of finte depth. He termed this subfactor $\mathcal{N} \subset \mathcal{M}$ the *asymptotic inclusion of* $N \subset M$.

We shall be interested in the case when $N = M^H$ as in the last section. We shall use the model described in the last section, and adopt the notation

$$\mathcal{N} = \mathcal{M}_{-1} \subset \mathcal{M} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_n \subset \cdots$$

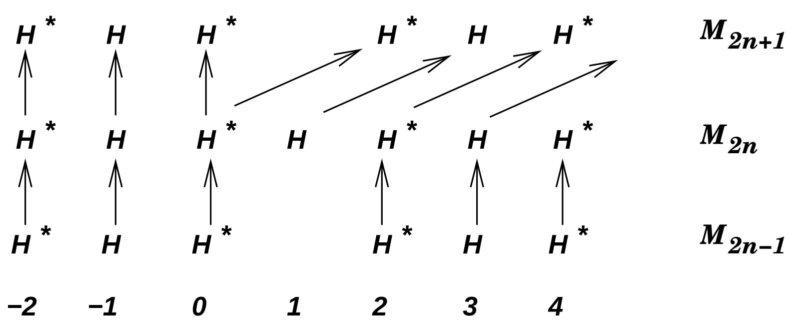
for the associated basic construction tower.

The crucial step in getting a handle on the above basic construction tower is contained in the following Lemma.

LEMMA 4.1. *1.*

$$\mathcal{M}_n \cong \begin{cases} (A_{(-\infty,0]} \cup A_{[2,\infty)})'' & \textit{if } n \textit{ is odd} \\ A_{(-\infty,\infty)} & \textit{if } n \textit{ is odd} \end{cases}, \tag{4.4}$$

2. The inclusions in the tower are best seen via the above identifications and the following illustrative diagram:



This diagram is intended to signify that, once the $\mathcal{M}_k$'s have been identified as in (4.4), the inclusion of $\mathcal{M}_{2n-1}$ into $\mathcal{M}_{2n}$ is the natural one, while that of $\mathcal{M}_{2n}$ into $\mathcal{M}_{2n+1}$ is given - in the notation of (3.3) - as follows:

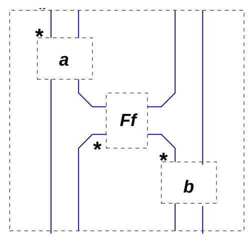
$$\left(\cdots x^{(-1)} \rtimes f^{(0)} \rtimes y^{(1)} \rtimes g^{(2)} \rtimes z^{(3)} \rtimes k^{(4)} \rtimes \cdots\right) \mapsto$$
$$\left(\cdots x^{(-1)} \rtimes (f_1)^{(0)} \rtimes 1_H^{(1)} \rtimes (f_2)^{(2)} \rtimes y^{(3)} \rtimes g^{(4)} \rtimes z^{(5)} \rtimes k^{(6)} \rtimes \cdots\right)$$

3. Furthermore, with respect to these identifications, the Jones projections turn out to be given by

$$\mathcal{M}_n \ni \widetilde{e_n} \leftrightarrow \begin{cases} \phi^{(2)} & \textit{if } n \textit{ is odd} \\ h^{(1)} & \textit{if } n \textit{ is even} \end{cases}$$

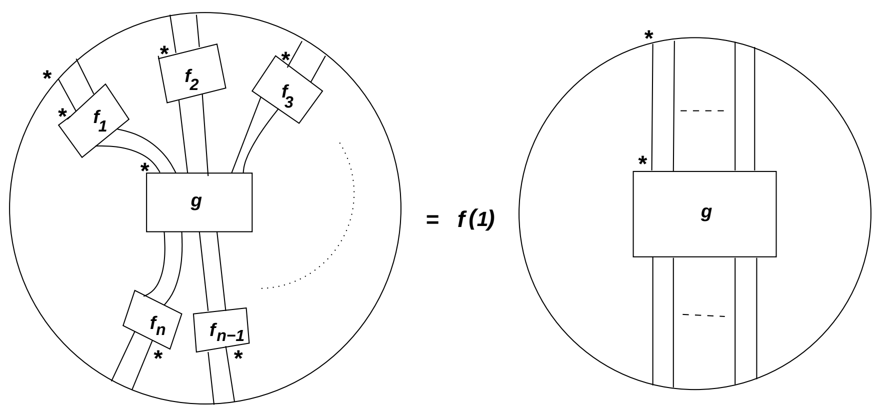
5 The planar algebra $\mathcal{P}(H)$

With $N = M^H \subset M$ as in §3, it should be noted that the isomorphism $A_{[1,3]} \cong P_4$ is the map which sends $a \rtimes f \rtimes b$ to the labelled tangle given below, where $F : H^* \to H$ is the 'Fourier transform':



We are now in a position to describe the planar algebra $\mathcal{P}(H)$.

THEOREM 5.1. *$\mathcal{P}(H)$ may be identified with the planar subalgebra of $P(H^{*op})$, with $\mathcal{P}_n(H)$ consisting of those elements $g \in \mathcal{P}_n(H^{*op})$ which satisfy*

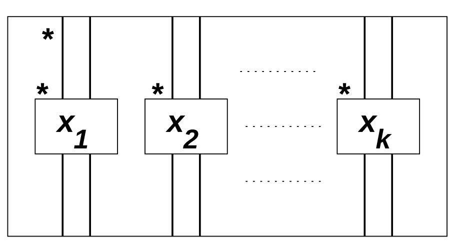


for all $f \in P_2(H^{*op}) = H^{*op}$. (Recall our 'Sweedler-like notation', whereby $\Delta_n(f) = f_1 \otimes \cdots \otimes f_n$, with Δ_n denoting iterated comultiplication.)

It follows that

$$\mathcal{P}_{2k}(H) = P_{2k}(H^{*op}) \cap \Delta_k(H^{*op})' ,$$

where a k-fold decomposable tensor $\otimes x_i$ is thought of as the $2k$-box below:



(Here, of course, we write H^{*op} to denote the dual H^* viewed as a Hopf algebra when equipped with the 'opposite multiplication'.

Corollary 5.2. *If H^* is commutative - equivalently, if $H = \mathbb{C}G$ for some finite group G - then the subfactors $M \subset M_1$ and $\mathcal{N} \subset \mathcal{M}$ are isomorphic.*

(We know of no proof of this fact which does not go through planar algebras.)

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