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Chapter 1

The Basics of C^*-algebras

1.1 Banach algebras

Definition 1.1.1 A *normed algebra* is a complex algebra A which is a normed space, and the norm satisfies

$$\|ab\| \leq \|a\|\|b\| \text{ for all } a, b \in A.$$

If A (with this norm) is complete, then A is called a *Banach algebra.*

Every closed subalgebra of a Banach algebra is itself a Banach algebra.

Example 1.1.2 Let $\mathbb{C}$ be the complex field. Then $\mathbb{C}$ is a Banach algebra. Let X be a compact Hausdorff space and $C(X)$ the set of continuous functions on X. $C(X)$ is a complex algebra with pointwise operations. With $\|f\| = \sup_{x\in X} |f(x)|$, $C(X)$ is a Banach algebra.

Example 1.1.3 Let M_n be the algebra of $n \times n$ complex matrices. By identifying M_n with $B(\mathbb{C}^n)$, the set of all (bounded) linear maps from the n-dimensional Hilbert space $\mathbb{C}^n$ to $\mathbb{C}^n$, with operator norm, i.e, $\|x\| = \sup_{\xi\in\mathbb{C}^n, \|\xi\|\leq 1} \|x(\xi)\|$, we see that M_n is a Banach algebra.

Example 1.1.4 The set of continuous functions $A(\mathbf{D})$ on the closed unit disk $\mathbf{D}$ in the plane which are analytic on the interior is a closed subalgebra of $C(\mathbf{D})$. Therefore, $A(\mathbf{D})$ is a Banach algebra.

Example 1.1.5 Let X be a Banach space and $B(X)$ be the set of all bounded linear operators on X. If $T, L \in B(X)$, define $TL = T \circ L$. Then $B(X)$ is a complex algebra. With operator norm, $B(X)$ is a Banach algebra.

A *commutative* Banach algebra is a Banach algebra A with the property that $ab = ba$ for all $a, b \in A$ Examples 1.1.2 and 1.1.4 are of commutative Banach algebras while Example 1.1.3 1.1.5 are not commutative.

Definition 1.1.6 In a unital algebra, an element $a \in A$ is called *invertible* if there is an element $b \in A$ such that $ab = ba = 1$. In this case b is unique and written a^{-1}. The set

$$GL(A) = \{a \in A : a \text{ is invertible}\}$$

is a group under multiplication.

We define the *spectrum* of an element a to be the set

$$\mathrm{sp}(a) = sp_A(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin GL(A)\}.$$

Whenever there is no confusion, we will write $\lambda 1$ simply as λ.

The complement of the spectrum is called the *resolvent* and $R(\lambda) = (\lambda - a)^{-1}$ is the *resolvent function.*

Example 1.1.7 Let $A = C(X)$ be as in 1.1.2. Then $\mathrm{sp}(f) = f(X)$ for all $f \in A$. In other words, the spectrum of f is the range of f.

Let $A = M_n$. If $a = (a_{ij}) \in A$, then the reader can check that $\mathrm{sp}(a)$ is the set of eigenvalues of the matrix a.

Proposition 1.1.8 *For any a and b in A,*

$$\mathrm{sp}(ab) \setminus \{0\} = \mathrm{sp}(ba) \setminus \{0\}.$$

Proof. If $\lambda \notin \mathrm{sp}(ab)$ and $\lambda \neq 0$, then there is $c \in A$ such that

$$c(\lambda - ab) = (\lambda - ab)c = 1.$$

Thus $c(ab) = \lambda c - 1 = (ab)c$. So we compute that

$$\begin{aligned} (1 + bca)(\lambda - ba) &= \lambda - ba + \lambda bca - bcaba \\ &= \lambda - ba + b(\lambda - ab)ca = \lambda - ba + ba = \lambda, \end{aligned}$$

which shows that $\lambda^{-1}(1 + bca)$ is the inverse of $\lambda - ba$. Hence $\lambda \notin \mathrm{sp}(ba)$ and $\mathrm{sp}(ba) \setminus \{0\} \subset \mathrm{sp}(ab) \setminus \{0\}$. □

Definition 1.1.9 A Banach algebra A is said to be *unital* if it admits a unit 1 and $\|1\| = 1$. Banach algebras in 1.1.2, 1.1.3 and 1.1.4 are unital.

Lemma 1.1.10 *Let A be a unital Banach algebra and a be an element of A such that $\|1-a\| < 1$. Then $a \in GL(A)$ and*

$$a^{-1} = \sum_{n=0}^{\infty} (1-a)^n.$$

Moreover, $\|a^{-1}\| \le \frac{1}{1-\|1-a\|}$ *and* $\|1-a^{-1}\| \le \frac{\|1-a\|}{1-\|1-a\|}$.

Proof. Since

$$\sum_{n=0}^{\infty} \|(1-a)^n\| \le \sum_{n=0}^{\infty} \|1-a\|^n = \frac{1}{(1-\|1-a\|)} < \infty,$$

the series $\sum_{n=0}^{\infty}(1-a)^n$ is convergent. Let b be its limit in A. Then $\|b\| \le \frac{1}{(1-\|1-a\|)}$ and

$$\|1-b\| \le \sum_{n=1}^{\infty} \|1-a\|^n = \frac{\|1-a\|}{1-\|1-a\|}.$$

One verifies

$$a(\sum_{n=0}^{k}(1-a)^n) = (1-(1-a))(\sum_{n=0}^{k}(1-a)^n) = 1-(1-a)^{k+1}$$

and that it converges to $ab = ba = 1$ as $k \to \infty$. Hence b is the inverse of a.
□

Definition 1.1.11 A function f from an open subset $\Omega \subset \mathbb{C}$ to a Banach algebra is said to be *analytic*, if for any $\lambda_0 \in \Omega$ there is an open neighborhood $O(\lambda_0)$ such that $f(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda-\lambda_0)^n$ converges for every $\lambda \in O(\lambda_0)$. To include the case that $\lambda_0 = \infty$, we say f is analytic at infinity if $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^{-n}$ for all λ in a neighborhood of infinity.

Theorem 1.1.12 *In any unital Banach algebra A, the spectrum of each $a \in A$ is a non-empty compact subset, and the resolvent function is analytic on $\mathbb{C} \setminus \mathrm{sp}(a)$.*

Proof. If $|\lambda| > \|a\|$, then $\|\lambda^{-n}a^n\| \le (\frac{\|a\|}{\lambda})^n$. So the series

$$\sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$$

converges (in norm). Similarly to 1.1.10,

$$(\lambda - a)\sum_{n=0}^{k} \frac{a^n}{\lambda^{n+1}} = 1 - (\frac{a^{k+1}}{\lambda^{k+1}})$$

which converges to 1. This shows that $\sup_{\lambda \in \mathrm{sp}(a)} |\lambda| \le \|a\|$ and $R(\lambda)$ is analytic in $\{\lambda : |\lambda| > \|a\|\}$. Moreover,

$$\lim_{|\lambda|\to\infty} \|R(\lambda)\| \le \lim_{|\lambda|\to\infty} \frac{|\lambda|^{-1}}{1 - (\frac{\|a\|}{\lambda})} = 0. \qquad \text{(e 1.1)}$$

Similarly, if $\lambda_0 - a$ is invertible and $|\lambda - \lambda_0| < \frac{1}{\|(\lambda_0 - a)^{-1}\|}$, then

$$(\lambda - a)^{-1} = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n ((\lambda_0 - a)^{-1})^{n+1}.$$

This also shows that the resolvent is open. Since $\mathrm{sp}(a)$ has been shown to be bounded, $\mathrm{sp}(a)$ is compact. We have also shown that the resolvent function is analytic on the complement of the spectrum.

In particular, $f(R(\lambda))$ is a (scalar) analytic function for every bounded linear functional $f \in A^*$. If $\mathrm{sp}(a)$ were empty, then $f(R(\lambda))$ would be an entire function for every $f \in A^*$. However, (e 1.1) shows that $f(R(\lambda))$ is bounded on the plane. Thus Liouville's theorem implies that $f(R(\lambda))$ is a constant. But (e 1.1) also implies that $f(R(\lambda)) = 0$. So, by the Hahn-Banach theorem, $R(\lambda) = 0$. This is a contradiction. Hence $\mathrm{sp}(a)$ is not empty. □

Corollary 1.1.13 *The only simple commutative unital Banach algebra is* $\mathbb{C}$.

Proof. Suppose that A is a unital commutative Banach algebra and $a \in A$ is not a scalar. Let $\lambda \in \mathrm{sp}(a)$. Set $I = \overline{(a - \lambda)A}$. Then I is clearly a closed ideal of A. No element of the form $(a - \lambda)b$ is invertible in the commutative Banach algebra A. By 1.1.10,

$$\|(a - \lambda)b - 1\| \ge 1.$$

So $1 \notin I$ and I is proper. Therefore, if A is simple, a must be a scalar, whence $A = \mathbb{C}$. □

Lemma 1.1.14 *If p is a polynomial and a is an element of a unital Banach algebra A, then*

$$\mathrm{sp}(p(a)) = p(\mathrm{sp}(a)).$$

Proof. We may assume that p is not constant. If $\lambda \in \mathbb{C}$, there are $c, \beta_1, ..., \beta_n \in \mathbb{C}$ such that

$$p(z) - \lambda = c\prod_{i=1}^{n}(z - \beta_i),$$

and therefore

$$p(a) - \lambda = c\prod_{i=1}^{n}(a - \beta_i).$$

It is clear that $p(a)-\lambda$ is invertible if and only if each $a-\beta_i$ is. It follows that $\lambda \in \mathrm{sp}(p(a))$ if and only if $\lambda = p(\alpha)$ for some $\alpha \in \mathrm{sp}(a)$. Thus $\mathrm{sp}(p(a)) = p(\mathrm{sp}(a))$. □

Definition 1.1.15 Let A be a unital Banach algebra. If $a \in A$, its *spectral radius* is defined to be

$$r(a) = \sup_{\lambda \in \mathrm{sp}(a)} |\lambda|.$$

Theorem 1.1.16 *If a is an element in a unital Banach algebra A, then*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Proof. If $\lambda \in \mathrm{sp}(a)$, then $\lambda^n \in \mathrm{sp}(a^n)$ by 1.1.14, so $|\lambda^n| \le \|a^n\|$. Therefore, $r(a) \le \liminf_{n\to\infty} \|a^n\|^{1/n}$. Let Ω be the open disk in $\mathbb{C}$ with center 0 and radius $1/r(a)$ (or ∞ if $r(a) = 0$). If $\lambda \in \Omega$, then $1 - \lambda a \in GL(A)$. If $f \in A^*$, then $f((1-\lambda a)^{-1})$ is analytic. There are unique complex numbers z_n such that

$$f((1-\lambda a)^{-1}) = \sum_{n=0}^{\infty} z_n\lambda^n \quad (\lambda \in \Omega).$$

However, if $|\lambda| < 1/\|a\| \le 1/r(a)$, then $\|\lambda a\| < 1$, so $(1-\lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$, and therefore, $f((1-\lambda a)^{-1}) = \sum_{n=0}^{\infty} \lambda^n f(a^n)$. It follows that $z_n = f(a^n)$ for all $n \ge 0$. Hence the sequence $\{\lambda^n f(a^n)\}$ converges to zero for each $\lambda \in \Omega$, and therefore is bounded. Since this is true for every $f \in A^*$, by the principle of uniform boundedness, $\{\lambda^n a^n\}$ is a bounded sequence. So we may assume that $|\lambda^n|\|a^n\| \le M$ for all $n \ge 0$ and for some positive number M. Hence

$$\|a^n\|^{1/n} \le \frac{M^{1/n}}{|\lambda|}, \quad n = 0, 1, ...$$

Consequently,

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \le 1/|\lambda|.$$

This implies that if $r(a) < \frac{1}{|\lambda|}$, then $\limsup_{n\to\infty} \|a^n\|^{1/n} \le 1/|\lambda|$. It follows that $\limsup_{n\to\infty} \|a^n\|^{1/n} \le r(a)$. From what we have shown at the beginning of this proof, we obtain

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \le r(a) \le \liminf_{n\to\infty} \|a^n\|^{1/n}.$$

This implies that $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$. □

Example 1.1.17 Let $A = M_3$ and $a = \begin{pmatrix} 1/2 & 1 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$. Then $\|a\| \ge 1$ and $\mathrm{sp}(a) = \{1/2, 1/3\}$. So $r(a) = 1/2$. It follows that $\|a^n\|^{1/n} \to 1/2$.

Let $T \in B(L^2([0,1]))$ be a bounded linear operator defined by

$$T(f) = \int_0^t f(x)dx.$$

The reader can compute that $\|T^n\| \le \frac{1}{n!}$. Hence $r(T) = 0$. Note that $T \ne 0$ (see Exercise 1.11.3).

We now establish the holomorphic functional calculus for elements in Banach algebras (1.1.19). The first application appears in 1.2.9. Later, we will establish continuous functional calculus for commutative C^*-algebras (1.3.5) and Borel functional calculus for normal elements in von Neumann algebras (1.8.5).

Definition 1.1.18 Let x be a fixed element in a unital Banach algebra A. Let f be a holomorphic function in an open neighborhood O_f of $\mathrm{sp}(x)$, and C be a smooth simple closed curve in O_f enclosing $\mathrm{sp}(x)$. We assign the positive orientation to C as in complex analysis. For each $\phi \in A^*$, we consider a continuous function which maps λ to $f(\lambda)\phi((\lambda - x)^{-1})$ on the curve C. Set

$$L(\phi) = \frac{1}{2\pi i}\int_C f(\lambda)\phi((\lambda - x)^{-1})d\lambda.$$

The map $\phi \mapsto L(\phi)$ is a linear functional on A^* and

$$|L(\phi)| \le \frac{l}{2\pi}\|\phi\| \sup\{|f(\lambda)|\|(\lambda - x)^{-1}\| : \lambda \in C\},$$

where l is the length of the curve C. Hence there exists an $F \in A^{**}$ such that $F(\phi) = L(\phi)$.

On the other hand, the function $\lambda \mapsto f(\lambda)(\lambda - x)^{-1}$ is a continuous function from C into A. So the limit of

$$\sum_{i=0}^{n} f(\lambda_i)(\lambda_i - x)^{-1}(\lambda_i - \lambda_{i+1}) \ \ (\text{as } \max_i|\lambda_i - \lambda_{i+1}| \to 0),$$

where $\{\lambda_0, ..., \lambda_n, \lambda_{n+1} = \lambda_0\}$ is a partition of the curve C, converges in norm in A to y. By the continuity of ϕ, we know that $\phi(y) = L(\phi) = F(\phi)$ for all $\phi \in A^*$. Hence $F \in A$. By Cauchy's theorem, F does not depend on the choice of the curve C, but only on the function f. Therefore we may denote F by $f(x)$ and write

$$f(x) = \frac{1}{2\pi i}\int_C f(\lambda)(\lambda - x)^{-1} d\lambda.$$

We denote by $\mathrm{Hol}(\mathrm{sp}(x))$ the algebra of all functions which are holomorphic in a neighborhood of $\mathrm{sp}(x)$.

Theorem 1.1.19 *Fix an element x in a unital Banach algebra A. The map $f \mapsto f(x)$ from $\mathrm{Hol}(\mathrm{sp}(x))$ into A is a homomorphism which sends the constant function 1 to the identity of A and the identity function on $\mathbb{C}$ to the element x. If $f(z) = \sum_{n=0}^{\infty} c_n z^n$ in a neighborhood of $\mathrm{sp}(x)$, then $f(x) = \sum_{n=0}^{\infty} c_n x^n$.*

Proof. Linearity is clear. Let f and g be functions holomorphic in neighborhoods O_f and O_g of $\mathrm{sp}(x)$, respectively. Set $O = O_f \cap O_g$ and let $C_i, i = 1, 2$ be smooth simple closed curves in O enclosing $\mathrm{sp}(x)$ such that C_1 lies completely inside the curve C_2. Then

$$\begin{aligned} f(x)g(x) &= \left(\frac{1}{2\pi i}\int_{C_1} f(\lambda)(\lambda - x)^{-1} d\lambda\right)\left(\frac{1}{2\pi i}\int_{C_2} g(z)(z - x)^{-1} dz\right) \\ &= -\frac{1}{4\pi^2}\int_{C_1}\left[\int_{C_2} f(\lambda)g(z)(\lambda - x)^{-1}(z - x)^{-1} dz\right] d\lambda \\ &= -\frac{1}{4\pi^2}\int_{C_1}\int_{C_2} f(\lambda)g(z)\frac{[(z - x)^{-1} - (\lambda - x)^{-1}]}{\lambda - z} dz d\lambda \\ &= \frac{1}{2\pi i}\int_{C_1} f(\lambda)(\lambda - x)^{-1}\left(\frac{1}{2\pi i}\int_{C_2}\frac{g(z)}{z - \lambda} dz\right) d\lambda \\ &\quad + \frac{1}{2\pi i}\int_{C_2} g(z)(z - x)^{-1}\left(\frac{1}{2\pi i}\int_{C_1}\frac{f(\lambda)}{\lambda - z} d\lambda\right) dz \end{aligned}$$

$$= \frac{1}{2\pi i}\int_{C_1} f(\lambda)g(\lambda)(\lambda - x)^{-1}d\lambda = (f\cdot g)(x).$$

The second to last equality holds because $\frac{1}{2\pi i}\int_{C_2}\frac{g(z)}{z-\lambda}dz = g(\lambda)$ (by the Cauchy formula) and because $\frac{f(\lambda)}{\lambda - z}$ is holomorphic inside the curve C_2 if $\lambda \in C_1$ (so that $\int_{C_1}\frac{f(\lambda)}{\lambda-z}d\lambda = 0$).

To complete the proof, pick a circle C with center at 0 and large radius. We note that

$$\lambda^{n-1}(1-\lambda^{-1}x)^{-1} = \lambda^{n-1}\sum_{n=0}^{\infty} x^k\lambda^{-k} = \sum_{k=0}^{\infty} x^k\lambda^{n-k-1},$$

where the convergence is in norm and is uniform on C. Therefore

$$\int_C x^k\lambda^{n-k-1}d\lambda = (\int_C \lambda^{n-k-1}d\lambda)x^k = 0\cdot x^k = 0 \qquad \text{(e 1.2)}$$

unless $k = n$, in which case the integral is $2\pi i x^n$. This implies that

$$x^n = \frac{1}{2\pi i}\int_C \lambda^n(\lambda - x)^{-1}d\lambda. \qquad \text{(e 1.3)}$$

Now suppose that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ in a neighborhood of $\mathrm{sp}(x)$. Then it converges in an open disk with center 0. Let C be a circle with center at 0 contained in the open disk. Then the series converges uniformly on C, so from (e 1.3),

$$f(x) = \frac{1}{2\pi i}\int_C f(z)(z-x)^{-1}dz = \sum_{n=0}^{\infty} c_n(\frac{1}{2\pi i}\int_C z^n(z-x)^{-1}dz) = \sum_{n=0}^{\infty} c_n x^n.$$

□

Proposition 1.1.20 *If I is a closed ideal in a Banach algebra, then A/I is a Banach algebra with the quotient norm*

$$\|\bar{a}\| = \|a + I\| = \inf_{b\in I}\|a+b\|.$$

Proof. It is well known that A/I, as a normed space is complete. It remains to show that A/I is a normed algebra. Let $\varepsilon > 0$ and $a, b \in A$. Then, there are $i, j \in I$ such that

$$\|a+i\| < \|a+I\| + \varepsilon \text{ and } \|b+j\| < \|b+I\| + \varepsilon.$$

Hence, for $c = ib + aj + ij \in I$,

$$\|ab + c\| \le \|a+i\|\|b+j\| < (\|a+I\|+\varepsilon)(\|b+I\|+\varepsilon).$$

Thus,

$$\|ab+I\| \leq (\|a+I\|+\varepsilon)(\|b+I\|+\varepsilon).$$

Letting $\varepsilon \to 0$, we obtain

$$\|ab+I\| \leq \|a+I\|\|b+I\|.$$

In other words, A/I is a normed algebra. □

1.2 C^*-algebras

Definition 1.2.1 An algebra A is called a *$*$-algebra* if it is a complex algebra with a conjugate linear involution $*$ which is an anti-isomorphism, i.e., for any $a, b \in A$ and $\alpha \in \mathbb{C}$,

$$(a+b)^* = a^* + b^*, \ (\alpha a)^* = \bar{\alpha}a^*, a^{**} = a \text{ and } (ab)^* = b^*a^*.$$

If $a \in A$, then a^* is called the *adjoint* of a. Let A be a $*$-algebra which is also a normed algebra. A norm on A that satisfies

$$\|a^*a\| = \|a\|^2$$

for all $a \in A$ is called a C^*-norm. If, with this norm, A is complete, then A is called a C^*-algebra.

Since $\|x^*x\| \leq \|x^*\|\|x\|$ we have $\|x\| \leq \|x^*\|$ for all $x \in A$. Thus $\|x^*\| = \|x\|$.

A closed $*$-subalgebra of a C^*-algebra is also a C^*-algebra. Such a $*$-subalgebra will be called a C^*-subalgebra.

Example 1.2.2 (a) Let A be as in 1.1.3 and let $a = (a_{ij}) \in A$. Define $a^* = (\bar{a}_{ji})$. Then A is a C^*-algebra.

(b) Let X be a locally compact Hausdorff space and $C_0(X)$ be the set of all continuous functions vanishing at infinity. Define $f^*(t) = \overline{f(t)}$ (for $t \in X$). Then $C_0(X)$ becomes a $*$-algebra. With $\|f\| = \sup_{t \in X} |f(t)|$, $C_0(X)$ is a C^*-algebra. $C_0(X)$ is unital if and only if X is compact.

Example 1.2.3 Let X be a compact Hausdorff space and $\mathcal{B}(X)$ be the set of all bounded Borel functions on X. With $\|f\| = \sup_{x \in X} |f(x)|$ and $f^*(x) = \overline{f(x)}$, $\mathcal{B}(X)$ becomes a C^*-algebra. For any complex Borel measure

Thus,

$$\|ab + I\| \leq (\|a + I\| + \varepsilon)(\|b + I\| + \varepsilon).$$

Letting $\varepsilon \to 0$, we obtain

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Example 1.2.3 Let X be a compact Hausdorff space and $\mathcal{B}(X)$ be the set of all bounded Borel functions on X. With $\|f\| = \sup_{x \in X} |f(x)|$ and $f^*(x) = \overline{f(x)}$, $\mathcal{B}(X)$ becomes a C^*-algebra. For any complex Borel measure

μ on X, there is a bounded linear functional $F_\mu \in C(X)^*$ such that $F_\mu(f) = \int_X f d\mu$. It is clear that (by considering point-evaluation)

$$\sup_{\|\mu\|\le 1} |F_\mu(f)| \ge \|f\|$$

for any $f \in \mathcal{B}(X)$. On the other hand, if $\|\mu\| \le 1$,

$$|F_\mu(f)| \le \int_X |f| d|\mu| \le \|f\|\|\mu\|.$$

Thus the C^*-algebra $\mathcal{B}(X)$ is the same Banach space when we regard it as a subspace of the second dual of $C(X)$ as described above.

Definition 1.2.4 Let A be a C^*-algebra. An element $x \in A$ is *normal* if $xx^* = x^*x$ and is *self-adjoint* if $x = x^*$. The self-adjoint part of A is denoted by A_{sa}. For each $x \in A$, the element $\frac{1}{2}(x + x^*)$ is in A_{sa}, and is called the real part of x, and the element $-\frac{i}{2}(x - x^*)$ is in A_{sa} and is called the imaginary part of x. It follows that A_{sa} is a closed real subspace of A and each element $x \in A$ has a unique decomposition as $x = y + iz$ with y, $z \in A_{sa}$. An element $p \in A_{sa}$ is called a *projection* if $p^2 = p$.

When A admits a unit, we denote the unit (or identity) by 1_A, or 1 when there is no confusion. If A has a unit, then $A \ne 0$. Since $1_A^* = 1_A$, $\|1_A\| = \|1_A\|^2$ and $\|1_A\| = 1$. Thus a C^*-algebra with unit is a unital Banach algebra. We will call it a *unital* C^*-algebra.

An element u in a unital C^*-algebra is *unitary* if $u^*u = uu^* = 1$. Since $1^* = 1$, $\|u\| = 1$.

One can always unitize a C^*-algebra.

Proposition 1.2.5 *For each C^*-algebra A there is a C^*-algebra $\tilde{A}$ with unit containing A as a closed ideal. If A has no unit, $\tilde{A}/A = \mathbb{C}$.*

Proof. Let $B(A)$ be the set of all bounded linear operators on A. Consider the map $\pi : A \to B(A)$ defined by $\pi(a)b = ab$ for all $a, b \in A$. (This map is often called the left regular representation of A.) It is clear that π is a homomorphism. Since $\|\pi(a)b\| \le \|a\|\|b\|$, $\|\pi(a)\| \le \|a\|$. Also

$$\|a\|^2 = \|aa^*\| = \|\pi(a)a^*\| \le \|\pi(a)\|\|a^*\| = \|\pi(a)\|\|a\|.$$

Hence π is an isometry. Let 1 denote the identity operator on A and let $\tilde{A}$ be the algebra of operators on A of the form $\pi(a) + \lambda \cdot 1$ with $a \in A$ and $\lambda \in \mathbb{C}$. Since $\pi(A)$ is complete and $\tilde{A}/\pi(A) = \mathbb{C}$, $\tilde{A}$ is also complete. Define an involution on $\tilde{A}$ by defining $(\pi(a)+\lambda 1)^* = \pi(a^*)+\bar{\lambda}1$. For each $\varepsilon > 0$, there is $b \in A$ with $\|b\| = 1$ such that

$$\begin{aligned} \|\pi(a)+\lambda 1\|^2 &\leq \varepsilon + \|(a+\lambda)b\|^2 = \varepsilon + \|b^*(a^*+\bar{\lambda})(a+\lambda)b\| \\ &\leq \varepsilon + \|(a^*+\bar{\lambda})(a+\lambda)b\| \leq \varepsilon + \|(\pi(a^*)+\bar{\lambda})(\pi(a)+\lambda)\|. \end{aligned}$$

So $\tilde{A}$ becomes a C^*-algebra. □

Definition 1.2.6 For a non-unital C^*-algebra A, the spectrum of $x \in A$, denoted by $\mathrm{sp}(x)$, is defined to be the spectrum of x in $\tilde{A}$.

Lemma 1.2.7 *If A is a C^*-algebra and $x \in A$ is a normal element, then*

$$\|x\| = r(x).$$

Proof. If $x \in A_{sa}$, then $\|x^2\| = \|x\|^2$ implies that

$$r(x) = \lim_{n\to\infty} \|x^{2^n}\|^{1/2^n} = \|x\|.$$

If in general, x is normal, we obtain

$$\begin{aligned} r(x)^2 &\leq \|x\|^2 = \|x^*x\| = \lim_{n\to\infty} \|(x^*x)^n\|^{1/n} \\ &\leq \lim_{n\to\infty} (\|(x^*)^n\|\|x^n\|)^{1/n} \leq r(x)^2. \end{aligned}$$

Hence $r(x) = \|x\|$. □

Corollary 1.2.8 *There is at most one norm on a $*$-algebra making it a C^*-algebra.*

Proof. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a $*$-algebra A making it a C^*-algebra, then

$$\|a\|_i^2 = \|a^*a\|_i = r(a^*a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(a^*a)\}$$

$(i = 1, 2)$. Thus $\|a\|_1 = \|a\|_2$. □

Lemma 1.2.9 *Let A be a C^*-algebra and $a \in A_{sa}$. Then $\mathrm{sp}(a) \subset \mathbb{R}$. If $u \in A$ is a unitary, then $\mathrm{sp}(u)$ is a subset of the unit circle.*

Proof. Let u be a unitary in a unital C^*-algebra A and $\lambda \in \mathrm{sp}(u)$. Since $\|u\|^2 = \|u^*u\| = \|1\| = 1$, $|\lambda| \leq 1$. However, $\lambda^{-1} \in \mathrm{sp}(u^{-1})$. Since $u^{-1} = u^*$ is also a unitary, we conclude that $|\lambda| = 1$. For $a \in A_{sa}$, by considering $\tilde{A}$ if necessary, we may assume that A is unital. The function $exp(iz) = \sum_{n=0}^{\infty}(iz)^n/n!$ is an entire function. By applying 1.1.19, we see that $u = exp(ia)$ is a unitary (with $u^* = exp(-ia)$). If $\lambda \in \mathrm{sp}(a)$ and $b = \sum_{n=1}^{\infty}(i)^n(a-\lambda)^{n-1}/n!$, then by 1.1.19,

$$exp(ia) - e^{i\lambda} = (exp(i(a-\lambda)) - 1)e^{i\lambda} = (a-\lambda)be^{i\lambda}.$$

Since b commutes with a (by 1.1.19), and $a-\lambda$ is not invertible, $exp(ia)-e^{i\lambda}$ is not invertible. Hence $|e^{i\lambda}| = 1$, and therefore $\lambda \in \mathbb{R}$. In other words, $\mathrm{sp}(a) \subset \mathbb{R}$. □

1.3 Commutative C^*-algebras

The C^*-algebra $C_0(X)$ in 1.2.2 (b) is a commutative C^*-algebra. In this section, we will show that every commutative C^*-algebra has this form.

Definition 1.3.1 A *multiplicative linear functional* on a Banach algebra A is a nonzero homomorphism of A into $\mathbb{C}$. The set of all multiplicative linear functionals on A is called the *maximal ideal space* of A and will be denoted by $\Omega(A)$.

If A is a unital algebra, and I is a proper ideal of A, then an application of Zorn's lemma shows that there is an ideal $M \supset I$ such that A/M is simple and nonzero. Such ideals are called maximal ideal. We leave this to the reader as an exercise.

Theorem 1.3.2 *Let A be a unital commutative Banach algebra.*

(1) *If $\phi \in \Omega(A)$, then $\|\phi\| = 1$.*

(2) *The space $\Omega(A)$ is non-empty, and the map $\phi \mapsto \ker\phi$ defines a bijection from $\Omega(A)$ onto the set of all maximal ideals of A.*

Proof. Suppose that $\phi \in \Omega(A)$ and $a \in A$ such that $\|a\| < 1 = \phi(a)$. Let $b = \sum_{n=1}^{\infty} a^n$. Then $a + ab = b$ and

$$\phi(b) = \phi(a) + \phi(a)\phi(b) = 1 + \phi(b).$$

This is not possible. So $\|\phi\| \le 1$. Since $\phi(1) = 1$, we proved that $\|\phi\| = 1$.

For part (2), let $\phi \in \Omega(A)$. It follows that $M = \ker\phi$ is a closed ideal of codimension 1 in A, and thus is maximal. If $\phi_1, \phi_2 \in \Omega(A)$ and $\ker\phi_1 = \ker\phi_2$, then for each $a \in A$, $a - \phi_2(a) \in \ker\phi_1$. This implies that $\phi_1(a - \phi_2(a)) = 0$ or $\phi_1(a) = \phi_2(a)$. This shows the map is one-to -one.

Conversely, if M is a maximal ideal, then $\mathrm{dist}(M, 1) \ge 1$ because the open unit ball with center at 1 consists of only invertible elements (by that $u = exp(ia)$ is a unitary (with $u^* = exp(-ia)$). If $\lambda \in \mathrm{sp}(a)$ and $b = \sum_{n=1}^{\infty}(i)^n(a-\lambda)^{n-1}/n!$, then by 1.1.19,

$$exp(ia) - e^{i\lambda} = (exp(i(a-\lambda)) - 1)e^{i\lambda} = (a-\lambda)be^{i\lambda}.$$

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Conversely, if M is a maximal ideal, then $\mathrm{dist}(M, 1) \ge 1$ because the open unit ball with center at 1 consists of only invertible elements (by

1.1.10). It follows that the closure of M still does not contain 1. It is easy to see that the closure is an ideal. So it is a proper ideal. We conclude that M itself is closed. So the quotient A/M is a simple commutative Banach algebra. By Lemma 1.1.13 $A/M = \mathbb{C}$. So this quotient map ϕ gives a continuous homomorphism from $A \to \mathbb{C}$ with $\ker\phi = M$. The map is therefore bijective.

To see that $\Omega(A)$ is non-empty, we may assume that $A \neq \mathbb{C}$, otherwise the identification of A with $\mathbb{C}$ gives a nonzero homomorphism. So A is not simple (by 1.1.13). Let I be a proper ideal of A. Since A has an identity, there is a maximal proper ideal of A containing I (see 1.3.1). □

If A is a unital commutative Banach algebra, it follows from 1.3.2 that $\Omega(A)$ is a subset of the (closed) unit ball of A^*. So $\Omega(A)$ with the relative weak *-topology becomes a topological space.

Theorem 1.3.3 *If A is a unital commutative Banach algebra, then $\Omega(A)$ is a compact Hausdorff space.*

Proof. It is easy to check that $\Omega(A)$ is weak* closed in the closed unit ball of A^*. Since the unit ball of A^* is weak* compact (Banach-Alaoglu theorem), $\Omega(A)$ is weak* compact. □

Definition 1.3.4 Suppose that A is a unital commutative Banach algebra and $\Omega(A)$ is its maximal ideal space. If $a \in A$, we define a function $\check{a}$ by

$$\check{a}(t) = t(a) \quad (\, t \in \Omega(A) \,).$$

The set $\{\phi \in \Omega(A) : |\phi(a)| \geq \alpha\}$ is weak* closed (for every $\alpha > 0$) in the closed unit ball of A^*. Thus it is weak* compact by the Banach-Alaoglu theorem. Hence $\check{a} \in C(\Omega(A))$. Thus we define a map $\Gamma : a \to \check{a}$ from A into $C(\Omega(A))$. This map is called the Gelfand transform. It is clear that Γ is a homomorphism.

Assume A is a non-unital C^*-algebra and $\tilde{A}$ is its unitization. If $\phi : A \to \mathbb{C}$ is a nonzero homomorphism, then $\tilde{\phi}(a + \lambda) = \phi(a) + \lambda$ ($a \in A$ and $\lambda \in \mathbb{C}$) is a homomorphism from $\tilde{A}$ into $\mathbb{C}$. Thus $\Omega(A)$ is a subset of $\Omega(\tilde{A})$. In fact $\Omega(\tilde{A}) = \Omega(A) \cup \{\pi\}$, where π is determined by $\tilde{A}/A = \mathbb{C}$, i.e., $\ker\pi = A$. Since $\Omega(\tilde{A})$ is a compact Hausdorff space, we conclude that $\Omega(A)$ is a locally compact Hausdorff space. The restriction of the Gelfand transform of $\tilde{A}$ on A maps A into $C_0(\Omega(A))$.

The following is the Gelfand theorem for commutative C^*-algebras. Note that if A is unital, $\Omega(A)$ is compact and $C_0(\Omega(A)) = C(\Omega(A))$.

Theorem 1.3.5 *Suppose that A is a commutative C^*-algebra. Then the Gelfand transform $a \mapsto \check{a}$ is a $*$-preserving isometry from A onto $C_0(\Omega(A))$.*

Proof. Suppose first that A is unital. Let $t \in \Omega(A)$ and $a \in A$ then $t(a) - a \in \ker t$. Since $\ker t$ is a maximal ideal, $t(a) \in \mathrm{sp}(a)$. Therefore, if $a \in A_{sa}$, then $t(a) \in \mathbb{R}$ by 1.2.9. By writing $a = (1/2)(a+a^*) + i[(1/2i)(a-a^*)]$, we conclude that $t(a^*) = \overline{t(a)}$ for each $a \in A$. This shows that $\Gamma(a) = \check{a}$ is a $*$-preserving homomorphism from A into $C(\Omega)$.

If a is invertible, then $\Gamma(a^{-1}) = \Gamma(a)^{-1}$, since Γ is a homomorphism. Conversely, if a is not invertible, then the ideal $I = \overline{aA}$ is proper and thus is contained in a maximal ideal M (we still assume that A is unital). If $\phi \in \Omega(A)$ and $\ker\phi = M$, then $\phi(a) = 0$. Thus $\check{a}$ is not invertible in $C(\Omega(A))$. This implies that $\check{a}$ and a have the same spectrum and this coincides with the range of $\check{a}$ (see 1.1.7). Since the norm in $C(\Omega(A))$ is the supremum norm and the range of $\check{a}$ is $\mathrm{sp}(a)$, we conclude that $\|\check{a}\|_\infty = r(a) = \|a\|$, since a is normal (see 1.2.7). This says that Γ is a $*$-preserving isometry from A into $C(\Omega(A))$. Since points in $\Omega(A)$ are multiplicative linear functionals which can be only distinguished by elements in A, $\{\check{a} : a \in A\}$ separates points in $\Omega(A)$. It follows from the Stone-Weierstrass Theorem that Γ is surjective.

For the non-unital case, from the above, $\Gamma : \tilde{A} \to C(\Omega(\tilde{A}))$ is a surjective $*$-preserving isometry. Since $\tilde{A}/A = \mathbb{C}$, $\Gamma(A)$ is a maximal ideal. Note as in 1.3.4, we may write $\Omega(\tilde{A}) = \Omega(A) \cup \{\pi\}$, where π is determined by the quotient map $\tilde{A} \to \tilde{A}/A = \mathbb{C}$. Then $C_0(\Omega(A))$ is the (maximal) ideal (of $C(\Omega(\tilde{A}))$) of continuous functions vanishing at π, which is identified naturally with $C_0(\Omega(A))$. □

Let S be a subset of a C^*-algebra A. Then the C^*-subalgebra of A generated by S, denoted by $C^*(S)$, is the smallest C^*-subalgebra of A containing S.

Corollary 1.3.6 *If a is a normal element in a unital C^*-algebra A, then there is an isometric $*$-isomorphism from $C^*(a)$ to $C_0(\mathrm{sp}(a) \setminus \{0\})$ which sends a to the identity function on $\mathrm{sp}(a)$.*

Proof. Since a is normal, $C^*(a)$ is commutative. Note that $\phi \in \Omega(C^*(a))$ is determined by $\phi(a) = \lambda$. Thus the map from $\Omega(C^*(a))$ into $\mathbb{C}$ taking ϕ to $\phi(a)$ is a homeomorphism onto $\check{a}(\Omega(A))$. From the Gelfand transform, $\check{a}(\Omega(A)) = \mathrm{sp}(a)$. This map identifies $\check{a}$ with the identity function z on $\mathrm{sp}(a)$

as desired. It follows from 1.3.5 that this map is an isometric $*$-isomorphism
. □

Definition 1.3.7 If a is a normal element of A and $f \in C_0(\mathrm{sp}(a) \setminus \{0\})$ we denote by $f(a)$ the element of A corresponding to f via the isomorphism given in 1.3.6. Corollary 1.3.6 is known as the *continuous functional calculus* for normal elements.

The following corollary is often called *the spectral mapping theorem.*

Corollary 1.3.8 *Let a be a normal element in a unital C^*-algebra A and $f \in C(\mathrm{sp}(a))$. Then*

$$f(\mathrm{sp}(a)) = \mathrm{sp}(f(a)).$$

Proof. This follows immediately from the above corollary. □

Theorem 1.3.9 *Let $A = C(X)$ and $B = C(Y)$, where X and Y are two compact Hausdorff spaces. Then $A \cong B$ if and only if X and Y are homeomorphic.*

Proof. Let $\phi : A \to B$ be an isomorphism. Define $\tau : Y \to X$ by $\tau(y)(f) = y(\phi(f))$. Here we identify X with the maximal ideal space (multiplicative states) of A and Y with the maximal ideal space of B. The continuity of ϕ implies that τ is continuous. Since ϕ is onto, there is $f \in A$ such that $\tau(y_1) \neq \tau(y_2)$ if $y_1 \neq y_2$. Thus τ is injective. Since Y is a compact Hausdorff space, $F = \tau(Y)$ is a closed subset of X. Let

$$I = \{f \in C(X) : f|_F = 0\}.$$

Then $\ker\phi \supset I$. Thus $F = X$. In other words τ is onto, whence τ is a homeomorphism.

Conversely if $\tau : Y \to X$ is a homeomorphism, define $\phi(f)(y) = f(\tau(y))$. To verify that ϕ is an isomorphism one can simply reverse the above argument, and we leave this to the reader. □

Remark 1.3.10 The Gelfand representation theorem for commutative C^*-algebras is fundamentally important. Even in a non-commutative C^*-algebra A, we often obtain useful information of A via the study of certain commutative C^*-subalgebras of A. So Theorem 1.3.5 certainly plays an important role in studying non-commutative C^*-algebras as well. Theorem 1.3.9 shows that to study commutative C^*-algebras, it is equivalent to study their maximal ideal spaces. Therefore the commutative C^*-algebra theory is the theory of topology. Much of general C^*-algebra theory can be described as non-commutative topology.

1.4 Positive cones

Example 1.4.1 Let H be a Hilbert space and let $B(H)$ be the space of all bounded operators from H to H. If $T_1, T_2 \in B(H)$, we define $(T_1T_2)(v) = T_1(T_2(v))$ for $v \in H$. With the operator norm $\|T\| = \sup_{\|v\|=1} \|T(v)\|$, $B(H)$ is a Banach algebra. If $T \in B(H)$, define T^* by $\langle Tx, y\rangle = \langle x, T^*y\rangle$ $(x, y \in H)$, where $\langle \cdot, \cdot \rangle$ is the inner product on H. Then one easily checks

$$\|T^*T\| = \sup_{\|x\|=1=\|y\|} |\langle T^*Tx, y\rangle| = \sup_{\|x\|=1=\|y\|} |\langle Tx, Ty\rangle| = \|T\|^2.$$

Thus $B(H)$ is a C^*-algebra.

Every closed $*$-subalgebra of $B(H)$ is a C^*-algebra. Later in this chapter we will show that every C^*-algebra is a closed $*$-subalgebra of $B(H)$ for some Hilbert space.

Example 1.4.2 Let H be a Hilbert space. An operator $T \in B(H)$ is said to be *compact* if it maps bounded sets to precompact subsets. If the range of T is finite dimensional, then T is compact. Denote by $\mathcal{K}(H)$ the set of all compact operators in $B(H)$. $\mathcal{K}(H)$ is clearly a closed subspace of $B(H)$. If $T \in \mathcal{K}(H)$, $L \in B(H)$, then TL, $LT \in \mathcal{K}(H)$. Therefore $\mathcal{K}(H)$ is a closed ideal of $B(H)$. Moreover if $T \in \mathcal{K}(H)$, then $T^* \in \mathcal{K}(H)$. Therefore $\mathcal{K}$ is a C^*-subalgebra of $B(H)$. When H is separable infinite dimensional Hilbert space, we will use $\mathcal{K}$ for $\mathcal{K}(H)$. $\mathcal{K}$ is a very important example of a (simple) C^*-algebra.

Example 1.4.3 To show that any C^*-algebra is a C^*-subalgebra of $B(H)$ for some Hilbert space H, we need to study the order structure of C^*-algebras. Recall that an operator $T \in B(H)$ is called *positive* if $\langle Tv, v\rangle \geq 0$ for all $v \in H$. In the case that H is finite dimensional, T is positive if and only if it is self-adjoint and all eigenvalues are nonnegative.

If $A = C(X)$, an element $f \in A$ is positive if $f(t) \geq 0$ for all $t \in X$.

We now introduce the following definition.

Definition 1.4.4 An element a in a C^*-algebra A is *positive* if $a \in A_{sa}$ and $\mathrm{sp}(a) \subset \mathbb{R}_+$. We write $a \geq 0$ if a is positive. The set of all positive elements in A will be denoted by A_+.

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A projection p is always positive, since $p^2 = p$ and $p \in A_{sa}$ implies that $\mathrm{sp}(p) = \{0, 1\}$, by 1.3.6.

Lemma 1.4.5 *Let A be a C^*-algebra and let $a \in A$. Then the following are equivalent:*

(i) $a \geq 0$;

(ii) $a = b^2$ *for some* $b \in A_{sa}$;

(iii) $a = a^*$ *and* $\|t - a\| \leq t$ *for any* $t \geq \|a\|$;

(iv) $a = a^*$ *and* $\|t - a\| \leq t$ *for some* $t \geq \|a\|$.

Proof. (i) $\Rightarrow$ (ii): In $C^*(a)$, by 1.3.6, we set $b = a^{1/2}$.

(ii) $\Rightarrow$ (i): Consider the C^*-subalgebra $C^*(b)$. Then $a \in C^*(b)$ and, by 1.3.6, $C^*(b) = C_0(\mathrm{sp}(b))$. We see that $a = a^*$ since $b \in A_{sa}$.

(i) $\Rightarrow$ (iii): Since $t - a$ is normal, from 1.3.5 we have (with $t \geq \|a\|$),

$$\|t - a\| = \sup\{|t - \lambda| : \lambda \in \mathrm{sp}(a)\} \leq t.$$

(iii) $\Rightarrow$ (iv) is immediate.

(iv) $\Rightarrow$ (i): If $\lambda \in \mathrm{sp}(a)$ then $t - \lambda \in \mathrm{sp}(t - a)$. Thus

$$|t - \lambda| \leq \|t - a\| \leq t.$$

Therefore $\lambda > 0$ since $\lambda \leq t$. □

Definition 1.4.6 Let $a \in A_{sa}$. Then $a^2 \in A_+$. Set $|a| = (a^2)^{1/2}$ (by 1.3.6). Then $a_+ = (1/2)(|a| + a)$ and $a_- = (1/2)(|a| - a)$. By the Gelfand transform, $|a|$, a_+ and a_- are positive. Moreover, by 1.3.6, $a_+ a_- = 0$ and $a = a_+ - a_-$.

Corollary 1.4.7 *If A is a unital C^*-algebra, then A is the linear span of unitaries.*

Proof. Let $a \in A_{sa}$ and $\|a\| \leq 1$. Then $1 - a^2 \in A_+$. Put

$$u = a + i(1 - a^2)^{1/2} \text{ and } v = a - i(1 - a^2)^{1/2}.$$

Then $u^* = v$ and $u^* u = u u^* = 1$. So u and v are unitaries. On the other hand, we have $a = (1/2)(u + v)$. □

Theorem 1.4.8 *The set A_+ is a closed cone ($a + b \in A_+$, if $a, b \in A_+$ and $A_+ \cap A_- = \{0\}$) and $a \in A_+$ if and only if $a = x^* x$ for some $x \in A$.*

Proof. It follows from (iii) of 1.4.5 that A_+ is closed and, if $a \in A_+$, then $\lambda a \in A_+$ for any $\lambda \in \mathbb{R}_+$. To see that A_+ is a cone, take a and b in A_+. By

(iii) of 1.4.5,

$$\begin{aligned}\|(\|a\|+\|b\|)-(a+b)\| &= \|(\|a\|-a)+(\|b\|-b)\| \\ &\leq \|\|a\|-a\|+\|\|b\|-b\| \leq \|a\|+\|b\|.\end{aligned}$$

Since $\|a\|+\|b\| \geq \|a+b\|$, from the above and (iv) of 1.4.5, $a+b \in A_+$.

Now suppose that $a = x^*x$ for some $x \in A$. Then $a^* = a$. Set $a = a_+ - a_-$ as in 1.4.6. We have

$$-(xa_-^{1/2})^*(xa_-^{1/2}) = -a_-^{1/2}x^*xa_-^{1/2} = -a_-^{1/2}(a_+ - a_-)a_-^{1/2} = a_-^2 \in A_+.$$

Put $xa_-^{1/2} = b + ic$ with b, $c \in A_{sa}$. Then

$$(xa_-^{1/2})(xa_-^{1/2})^* = 2(b^2+c^2) + [-(xa_-^{1/2})^*(xa_-^{1/2})] \in A_+$$

since A_+ is a cone. It follows from 1.1.8 that $\mathrm{sp}(a_-^2) \subset \mathbb{R}_+ \cap \mathbb{R}_- = \{0\}$. Therefore $a_- = 0$ and $a \geq 0$. □

Definition 1.4.9 In A_{sa}, we write $b \leq a$ if $a - b \geq 0$. Since $A_{sa} = A_+ - A_+$ and $A_+ \cap (-A_+) = \{0\}$, A_{sa} becomes a partially ordered real vector space.

Theorem 1.4.10 *Let A be a C^*-algebra.*
(1) $A_+ = \{a^*a : a \in A\}$.
(2) *If a, $b \in A_{sa}$ and $c \in A$, then $a \leq b$ implies that $c^*ac \leq c^*bc$.*
(3) *If $0 \leq a \leq b$, $\|a\| \leq \|b\|$.*
(4) *If A is unital and a, $b \in A_+$ are invertible, then $a \leq b$ implies that $0 \leq b^{-1} \leq a^{-1}$.*

Proof. Condition (1) follows from 1.4.8 and the fact that positive elements have positive square roots. To see (2), we note that $c^*bc - c^*ac = c^*(b-a)c$. So (2) follows from (1). For (3), we may assume that A is unital. The inequality $b \leq \|b\| \cdot 1$ follows from 1.3.5 by considering $C^*(b, 1)$. Hence $a \leq \|b\| \cdot 1$. We then apply 1.3.5 to $C^*(a, 1)$ to obtain $\|a\| \leq \|b\|$.

To see (4), we note that if $b \geq 0$ and invertible, then $b^{-1} \geq 0$ (by 1.3.6). It follows from (2) that $b^{-1/2}ab^{-1/2} \leq b^{-1/2}bb^{-1/2} = 1$. Hence $\|a^{1/2}b^{-1/2}\| \leq 1$ (by (3)). Thus $\|a^{1/2}b^{-1}a^{1/2}\| \leq 1$ which implies that $a^{1/2}b^{-1}a^{1/2} \leq 1$. By (2), we have

$$b^{-1} = (a^{-1/2})(a^{1/2}b^{-1}a^{1/2})(a^{-1/2}) \leq (a^{-1/2})1(a^{-1/2}) = a^{-1}.$$

□

Theorem 1.4.11 *Let A be a C^*-algebra. If a, $b \in A_+$ such that $a \leq b$, then $a^\alpha \leq b^\alpha$ for any $0 \leq \alpha \leq 1$.*

Proof. By considering $\tilde{A}$, we may assume that A is unital. Fix $0 \le a \le b$ in A_+. Let $C = \{\alpha \in \mathbb{R}_+ : a^\alpha \le b^\alpha\}$. Then $0, 1 \in C$. Since the function $f_n(t) = t^{\alpha_n} \to t^\alpha$ if $\alpha_n \to \alpha$ for $\alpha_n \ge 0$ uniformly on $[0, K]$ for any $K > 0$, $x^{\alpha_n} \to x^\alpha$ for any positive element $x \in A_+$ if $\alpha_n \to \alpha$ (and $\alpha_n \ge 0$). Since A_+ is closed, if $\alpha_n \in C$ and $\alpha_n \to \alpha$, then $b^\alpha - a^\alpha \in A_+$. This shows that C is closed. To prove the theorem, it suffices to show that C is convex. We first consider the case that both a and b are invertible. Let α, $\beta \in C$. Then

$$b^{-\alpha/2} a^\alpha b^{-\alpha/2} \le 1 \text{ and } b^{-\beta/2} a^\beta b^{-\beta/2} \le 1.$$

Hence, by 1.4.10, $\|b^{-\alpha/2} a^\alpha b^{-\alpha/2}\| \le 1$ and $\|b^{-\beta/2} a^\beta b^{-\beta/2}\| \le 1$. So $\|a^{\alpha/2} b^{-\alpha/2}\|^2 \le 1$ and $\|a^{\beta/2} b^{-\beta/2}\|^2 \le 1$. Therefore (using the fact that $\mathrm{sp}(xy) = \mathrm{sp}(yx)$)

$$\begin{aligned} 1 &\ge \|(b^{-\beta/2} a^{\beta/2})(a^{\alpha/2} b^{-\alpha/2})\| = \|b^{-\beta/2}[a^{(\alpha+\beta)/2} b^{-\alpha/2}]\| \\ &\ge r(b^{-\beta/2}[a^{(\alpha+\beta)/2} b^{-\alpha/2}]) = r(a^{(\alpha+\beta)/2} b^{-(\alpha+\beta)/2}) \\ &= r([a^{(\alpha+\beta)/2} b^{-(\alpha+\beta)/4}] b^{-(\alpha+\beta)/4}) = r(b^{-(\alpha+\beta)/4} a^{(\alpha+\beta)/2} b^{-(\alpha+\beta)/4}) \\ &= \|b^{-(\alpha+\beta)/4} a^{(\alpha+\beta)/2} b^{-(\alpha+\beta)/4}\| \end{aligned}$$

Therefore we have $b^{-(\alpha+\beta)/4} a^{(\alpha+\beta)/2} b^{-(\alpha+\beta)/4} \le 1$. Hence $a^{(\alpha+\beta)/2} \le b^{(\alpha+\beta)/2}$. Therefore C contains $(1/2)(\alpha+\beta)$. Consequently, $C \supset [0, 1]$. In general, for any $\varepsilon > 0$, we have

$$(b+\varepsilon)^\alpha - (a+\varepsilon)^\alpha \ge 0 \quad (0 \le \alpha \le 1).$$

Since A_+ is closed, letting $\varepsilon \to 0$, we obtain $a^\alpha \le b^\alpha$. □

Example 1.4.12 It is not true that $0 \le a \le b$ implies $a^2 \le b^2$ in general non-commutative C^*-algebras. For example, let $A = M_2$ and let

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } q = 1/2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then both p and q are projections and $p \le p + q$. But $p^2 = p \not\le (p+q)^2 = p + q + pq + qp$, since

$$q + pq + qp = 1/2 \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

is not positive.

1.5 Approximate identities, hereditary C^*-subalgebras and quotients

To deal with non-unital C^*-algebras, and avoid the troubles caused by the absence of unit, one can often embed A into $\tilde{A}$. However, more often, one has to work in the original non-unital C^*-algebra. Therefore the notion of approximate identity is essential.

Example 1.5.1 Let H be a Hilbert space with an orthonormal basis $\{v_n\}_{n=1}^{\infty}$. The C^*-algebra $\mathcal{K}$, the set of all compact operators on H, is a non-unital C^*-algebra. Let p_n be the projection (from H) onto $\mathrm{span}\{v_1, ..., v_n\}$. Then $\{p_n\}$ is an increasing sequence of projections in $\mathcal{K}$. The reader easily checks that, for any $x \in \mathcal{K}$,

$$\|p_n x - x\| \to 0 \text{ and } \|x - xp_n\| \to 0 \quad \text{as } n \to \infty.$$

The sequence $\{p_n\}$ plays a role which is similar to that of identity in a unital C^*-algebra.

Let X be a non-compact but σ-compact and locally compact Hausdorff space X. Then $C_0(X) = A$ is non-unital. Moreover, $X = \cup_{n=1}^{\infty} X_n$, where each X_n is compact and X_{n+1} contains a neighborhood of X_n. It is easy to produce an increasing sequence of positive functions $f_n \in C_0(X)$ such that $0 \le f_n \le 1$, $f_n(t) = 1$ on X_n and $f_n(t) = 0$ if $t \notin X_{n+1}$. One checks that, for any $g \in C_0(X)$,

$$\|gf_n - g\| \to 0 \quad \text{as } n \to \infty.$$

Definition 1.5.2 An *approximate identity* for a C^*-algebra A is an increasing net $\{e_\lambda\}_{\lambda \in \Lambda}$ of positive elements in the closed unit ball of A such that

$$a = \lim_{\lambda} a e_\lambda.$$

Equivalently, $a = \lim_\lambda e_\lambda a$ for all $a \in A$.

Lemma 1.5.3 *Let A be a C^*-algebra and denote by Λ the set of all elements $a \in A_+$ with $\|a\| < 1$. Then Λ is upwards-directed; i.e., if $a, b \in \Lambda$, then there exists some $c \in \Lambda$ such that $a, b \le c$.*

Proof. Suppose that $a \in A_+$. Then $1 + a \in GL(\tilde{A})$, and $a(1+a)^{-1} = 1 - (1+a)^{-1}$. We claim that

$$a, b \in A_+ \text{ and } a \le b \to a(1+a)^{-1} \le b(1+b)^{-1}. \qquad \text{(e 5.4)}$$

In fact, if $a \le b$, then $1+a \le 1+b$ which implies that $(1+b)^{-1} \le (1+a)^{-1}$ by 1.4.10 (4). Consequently $1-(1+a)^{-1} \le 1-(1+b)^{-1}$. This is the same as

$$a(1+a)^{-1} \le b(1+b)^{-1}.$$

So the claim is proved.

We note also (by 1.3.6) that if $a \in A_+$, then $a(1+a)^{-1} \in \Lambda$. Now suppose that $a, b \in \Lambda$. Put

$$x = a(1-a)^{-1},\ y = b(1-b)^{-1} \text{ and } c = (x+y)(1+x+y)^{-1}.$$

Since $x+y \in A_+$, $c \in \Lambda$. We note that $(1+x)^{-1} = 1-a$ and $x(1+x)^{-1} = a$. So, by e 5.4, $a = x(1+x)^{-1} \le c$, since $x \le x+y$. Similarly, $b \le c$. This proves the lemma. □

The following positive continuous functions will be used throughout this book.

Definition 1.5.4 Let

$$f_\varepsilon(t) = \begin{cases} 1 & \text{if } \varepsilon \le t \\ \text{linear} & \text{if } \varepsilon/2 \le t < \varepsilon \\ 0 & \text{if } 0 \le t \le \varepsilon/2 \end{cases} \qquad \text{(e 5.5)}$$

so that $f_\varepsilon \in C_0((0,K])$ for any $K > 0$, and $0 \le f_\varepsilon \le 1$. Note that if $\alpha \ge \beta > 0$, then $f_\beta \ge f_\alpha$. Also $f_{\frac{1}{2n}} f_{\frac{1}{n}} = f_{\frac{1}{n}}$.

Let $a \in A_+$. Then it follows from 1.3.6 that $f_\varepsilon(a)a \to a$ if $\varepsilon \to 0$.

Theorem 1.5.5 *Every C^*-algebra A admits an approximate identity. Indeed, if Λ is the upwards-directed set of all $a \in A_+$ with $\|a\| < 1$ and $e_\lambda = \lambda$ for all $\lambda \in \Lambda$, then $\{e_\lambda\}_{\lambda\in\Lambda}$ forms an approximate identity for A.*

Proof. From 1.5.3 $\{e_\lambda\}$ is an increasing net in the closed unit ball of A. We need to show that $\lim_\lambda ae_\lambda = a$ for all $a \in A$. Since Λ spans A, it suffices to assume that $a \in A_+$. Since $\{\|a(1-e_\lambda)a\|\}$ is decreasing, it suffices to show that there are $u_n \in \{e_\lambda\}$ such that $\|a(1-u_n)a\| \to 0$ (as $n \to \infty$). Note that $u_n = (1-\frac{1}{n^2})f_{\frac{1}{n}}(a) \in \Lambda$. We have $t(1-(1-\frac{1}{n^2})f_{\frac{1}{n}}(t))t \to 0$ as $n \to \infty$ uniformly on $[0,1]$. It follows from 1.3.6 that

$$\|a(1-u_n)a\| \to 0 \quad \text{as } n \to \infty.$$

□

Corollary 1.5.6 *If A is separable, then A admits a countable approximate identity.*

Proof. Fix a dense sequence $\{x_n\}$ of A. Let $\{e_\lambda\}$ be an approximate identity for A. Choose $e_k \in \{e_\lambda\}$ such that

$$\|x_i e_k - x_i\| < 1/k \quad (1 \le i \le k)$$

and $e_{k+1} \ge e_k$, $k = 1, 2, ...,$. Thus $\{e_k\}$ forms an approximate identity for A. □

Definition 1.5.7 A C^*-algebra is said to be *σ-unital* if A admits a countable approximate identity.

The above corollary (1.5.6) shows that every separable C^*-algebra is σ-unital.

Definition 1.5.8 A C^*-subalgebra B of A is said to be *hereditary* if for any $a \in A_+$ and $b \in B_+$ the inequality $a \le b$ implies that $a \in B$. Obviously, 0 and A are hereditary C^*-subalgebras of A. Any intersection of hereditary C^*-subalgebras is hereditary. Let $S \subset A$. The hereditary C^*-subalgebra *generated* by S is the smallest hereditary C^*-subalgebra of A containing S. Such a hereditary C^*-subalgebra is denoted by $\mathrm{Her}(S)$.

Lemma 1.5.9 *Let A be a C^*-algebra and $a \in A_+$. Then $\overline{aAa}$ is the hereditary C^*-subalgebra generated by a.*

Proof. Let $B = \overline{aAa}$. It is easy to see that aAa is a $*$-subalgebra. It follows that B is a C^*-subalgebra.

Claim (1): If $c \in B_+$ then $\overline{cAc} \subset B$. In fact, if $ax_na \to c$, then, for any $y \in A$, $ax_nayax_na \to cyc$. Therefore $cAc \subset B$. This proves the claim.

Claim (2): Let $e_n = f_{1/n}(a)$. Then $\{e_n\}$ forms an approximate identity for B. In fact by 1.5.4, $\|e_na - a\| \to 0$ as $n \to \infty$. Suppose $x = aba$ for some $b \in A$. Then we have $\|e_nx - x\| \to 0$ and $\|xe_n - x\| \to 0$ as $n \to \infty$. Since aAa is dense in B, we conclude that $\{e_n\}$ is an approximate identity for B.

Since $ae_na \to a^2$, $a^2 \in B$. Hence $a \in B$ and $C^*(a) \subset B$. In particular, from Claim (1), $f_{1/n}(a)Af_{1/n}(a) \subset B$.

Now suppose that $0 \le c \le b$ with $b \in B_+$. We will show that $c \in B$. By 1.4.11, $c^{1/2} \le b^{1/2}$. Thus

$$\|(1-e_n)c(1-e_n)\| \le \|(1-e_n)b(1-e_n)\| \le \|(1-e_n)b\| \to 0.$$

Therefore $\|(1-e_n)c^{1/2}\| \to 0$, or $e_n c^{1/2} \to c^{1/2}$. Hence $e_n c e_n \to c$. By Claim (1), $e_n c e_n \in B$. Therefore $c \in B$. So B is hereditary. We have shown that $a \in \overline{aAa}$. It follows that $\mathrm{Her}(a) = \overline{aAa}$. □

Theorem 1.5.10 *A C^*-algebra A is σ-unital if and only if there is $a \in A_+$ such that* $\mathrm{Her}(a) = A$.

Proof. We have shown (from Claim (2) in the proof of 1.5.9) that A is σ-unital if $A = \mathrm{Her}(a)$ for some $a \in A_+$.

For the converse, let $\{e_n\}$ be an approximate identity for A. Put $a = \sum_{n=1}^{\infty}(1/2^n)e_n$ and $B = \mathrm{Her}(a)$. Since $e_n \le 2^n a$, by 1.5.9, $e_n \in B$ for all n. As in the proof of 1.5.9, $e_n b e_n \in B$ for any $b \in A_+$. Since $e_n b^{1/2} \to b^{1/2}$, $e_n b e_n \to b$. Therefore $b \in B$. It follows that $A = B$. □

Corollary 1.5.11 *If A is a σ-unital C^*-algebra, then A admits a countable approximate identity $\{e_n\}$ satisfying*

$$e_n e_{n+1} = e_n = e_{n+1} e_n \quad n = 1, 2, \ldots.$$

Proof. Let $A = \overline{aAa}$ for some $a \in A_+$. Set $e_n = f_{1/2^n}(a)$. Then $\{e_n\}$ is an approximate identity. By 1.5.4, $f_{1/2^n} f_{1/2^{(n+1)}} = f_{1/2^n}$ for all n. □

Theorem 1.5.12 *If I is an ideal in a C^*-algebra A, then I is a hereditary C^*-subalgebra of A. If $\{e_\lambda\}$ is an approximate identity for I, then for each $a \in A$,*

$$\|\bar{a}\| = \inf_{b \in I} \|a + b\| = \lim_{\lambda} \|a - e_\lambda a\| = \lim_{\lambda} \|a - a e_\lambda\|.$$

Proof. Let $B = I^* \cap I$. Then B is a C^*-subalgebra of A. Let $\{e_\lambda\}$ be an approximate identity for B. Note that $e_\lambda \in B \subset I$. If $a \in I$, $\lim_\lambda a^*a(1 - e_\lambda) = 0$. Hence

$$\lim_{\lambda} \|a - a e_\lambda\|^2 = \lim_{\lambda} \|(1 - e_\lambda)a^*a(1 - e_\lambda)\| \le \lim_{\lambda} \|a^*a(1 - e_\lambda)\| = 0.$$

Therefore $a = \lim_\lambda a e_\lambda$. Hence $a^* = \lim_\lambda e_\lambda a^*$. Since $e_\lambda a^* \in I$, we conclude that $a^* \in I$. Therefore I is C^*-subalgebra.

To see it is hereditary, let $0 \le c \le a$, where $c \in A_+$ and $a \in I_+$. By 1.5.9 $c \in \mathrm{Her}(a) = \overline{aAa}$. But $\overline{aAa} \subset I$. Hence I is hereditary.

Now suppose that $\{e_\lambda\}$ is an approximate identity for I.

By definition, $\|\bar{a}\| = \inf_{b \in I} \|a + b\|$ for all $a \in A$. Fix $a \in A$. Put $\alpha = \inf_{b \in I} \|a + b\|$ and $\beta = \lim_\lambda \|a(1 - e_\lambda)a^*\|$ (it exists since the net is decreasing). We have

$$\alpha^2 \le \|a(1-e_\lambda)\|^2 = \|a(1-e_\lambda)^2 a^*\| \le \|a(1-e_\lambda)a^*\| \to \beta.$$

Hence $\alpha^2 \le \beta$. Thus it suffices to show that $\alpha^2 \ge \beta$. To do this, let $\varepsilon > 0$ and take $b \in I$ such that $\alpha + \varepsilon \ge \|a+b\|$. Then

$$(\alpha+\varepsilon)^2 \ge \|a+b\|\|1-e_\lambda\|\|a^*+b^*\| \ge \|(a+b)(1-e_\lambda)(a^*+b^*)\| \to \beta$$

since both $\|b(1-e_\lambda)\|$ and $\|b^*(1-e_\lambda)\|$ tend to zero. Thus we have

$$(\alpha+\varepsilon)^2 \ge \beta$$

for every $\varepsilon > 0$. Hence $\alpha^2 \ge \beta$. □

Corollary 1.5.13 *If I is a closed ideal of A, then A/I equipped with its natural operation is a C^*-algebra. (In particular, $\bar{x}^* = \overline{x^*}$.)*

Proof. It follows from 1.1.20 that it suffices to show that $\|\bar{x}^*\bar{x}\| = \|\bar{x}\|^2$. Let $\{e_\lambda\}$ be an approximate identity for I. Then,

$$\begin{aligned}\|\bar{x}^*\bar{x}\| &= \lim_\lambda \|x^*x(1-e_\lambda)\| \ge \lim_\lambda \|(1-e_\lambda)x^*x(1-e_\lambda)\| \\ &= \lim_\lambda \|x(1-e_\lambda)\|^2 = \|\bar{x}\|^2.\end{aligned}$$

□

Definition 1.5.14 From now on, an *ideal* of a C^*-algebra is *always* closed, and a homomorphism from a C^*-algebra to another is *always* a $*$-homomorphism.

Theorem 1.5.15 *Each homomorphism $h : A \to B$ is norm decreasing, and $h(A)$ is always a C^*-subalgebra of B. If h is injective, then it is an isometry.*

Proof. If $a \in A_{sa}$, then $\mathrm{sp}(h(a)) \setminus \{0\} \subset \mathrm{sp}(a) \setminus \{0\}$. Since $\|a\| = r(a)$ by 1.2.7, we conclude that for each $a \in A$,

$$\|h(a)\|^2 = \|h(a^*a)\| \le \|a^*a\| = \|a\|^2.$$

Thus h is norm decreasing.

Now assume that h is injective. To show that h is an isometry, we need to show that $\|h(a)\|^2 = \|a\|^2$, i.e., $\|h(a^*a)\| = \|a^*a\|$. Thus it suffices to show that $\|h(x)\| = \|x\|$ for all $x \in A_+$. Fix such an x and by restricting to $C^*(x)$, we may assume that A is commutative. Also, by considering $\tilde{A}$ and extending h to $\tilde{h} : \tilde{A} \to \tilde{B}$ if necessary, we may assume that A and B

are unital. Thus, by 1.3.5, we may assume that $A = C(S)$, where S is a compact Hausdorff space. Since $h(A)$ is commutative, by taking its closure, we may also assume that $B = C(T)$ for some compact Hausdorff space T. Define $h^* : T \to S$ by $h^*(t) = t \circ h$. As in 1.3.9, since h is injective, $h^*(T) = S$. Therefore, for each $g \in C(S)$,

$$\|g\| = \sup_{s \in S} |g(s)| = \sup_{t \in T} |g(h^*(t))| = \|h(g)\|.$$

Thus h is an isometry.

To complete the proof, we note the $I = \ker h$ is an ideal of A (closed). So h induces an isomorphism from A/I into B. Therefore A/I is isometrically $*$-isomorphic to $h(A)$. It follows that $h(A)$ is a C^*-subalgebra. □

Definition 1.5.16 From now on, an *isomorphism* h from a C^*-algebra A to another C^*-algebra B is *always* a $*$-isomorphism. Furthermore, it is an isometric $*$-isomorphism. If A is isomorphic to B, we write $A \cong B$.

Corollary 1.5.17 *Let I be a closed ideal of a C^*-algebra A and let B be a C^*-subalgebra of A. Then $B + I$ is a C^*-subalgebra of A. Moreover, $B/(B \cap I) \cong (B + I)/I$.*

Proof. Clearly $B + I$ is a $*$-subalgebra of A containing I and B. Let $\pi : A \to A/I$ be the quotient map. By 1.5.15, $\pi(B)$ is closed in A/I, whence the preimage of $\pi(B)$ is closed, i.e., $B + I$ is closed.

Note that the map $b + B \cap I \mapsto b + I$ is a $*$-isomorphism from $B/(B \cap I)$ to $(B + I)/I$. Thus, by 1.5.15, it is a C^*-isomorphism. □

1.6 Positive linear functionals and a Gelfand-Naimark theorem

Definition 1.6.1 A linear map $\phi : A \to B$ between C^*-algebras is said to be *self-adjoint* if $\phi(A_{sa}) \subset B_{sa}$, and *positive* if $\phi(A_+) \subset B_+$. It follows that if ϕ is positive then ϕ is self-adjoint.

Every homomorphism $h : A \to B$ is positive.

If $B = \mathbb{C}$, then a positive linear map $\phi : A \to \mathbb{C}$ is called a *positive linear functional.* If, in addition, ϕ is bounded and $\|\phi\| = 1$, then ϕ is called a *state* on A. If ϕ is a linear functional, we write $\phi \geq 0$ if it is positive.

are unital. Thus, by 1.3.5, we may assume that $A = C(S)$, where S is a compact Hausdorff space. Since $h(A)$ is commutative, by taking its closure, we may also assume that $B = C(T)$ for some compact Hausdorff space T. Define $h^* : T \to S$ by $h^*(t) = t \circ h$. As in 1.3.9, since h is injective, $h^*(T) = S$. Therefore, for each $g \in C(S)$,

$$\|g\| = \sup_{s \in S} |g(s)| = \sup_{t \in T} |g(h^*(t))| = \|h(g)\|.$$

Thus h is an isometry.

To complete the proof, we note the $I = \ker h$ is an ideal of A (closed). So h induces an isomorphism from A/I into B. Therefore A/I is isometrically $*$-isomorphic to $h(A)$. It follows that $h(A)$ is a C^*-subalgebra. □

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Proof. Clearly $B + I$ is a $*$-subalgebra of A containing I and B. Let $\pi : A \to A/I$ be the quotient map. By 1.5.15, $\pi(B)$ is closed in A/I, whence the preimage of $\pi(B)$ is closed, i.e., $B + I$ is closed.

Note that the map $b + B \cap I \mapsto b + I$ is a $*$-isomorphism from $B/(B \cap I)$ to $(B + I)/I$. Thus, by 1.5.15, it is a C^*-isomorphism. □

1.6 Positive linear functionals and a Gelfand-Naimark theorem

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Every homomorphism $h : A \to B$ is positive.

If $B = \mathbb{C}$, then a positive linear map $\phi : A \to \mathbb{C}$ is called a *positive linear functional*. If, in addition, ϕ is bounded and $\|\phi\| = 1$, then ϕ is called a *state* on A. If ϕ is a linear functional, we write $\phi \geq 0$ if it is positive.

A positive linear functional $\tau : A \to \mathbb{C}$ is called a *trace* if $\phi(u^*au) = \phi(a)$ for all $a \in A$ and all unitaries $u \in \tilde{A}$. A trace which is also a state is called a *tracial state.*

Example 1.6.2 Let $A = C(X)$, where X is a compact Hausdorff space, and let μ be a positive Borel finite measure. Then the linear functional τ defined by

$$\tau(f) = \int f d\mu$$

is positive. Since A is commutative, τ is also a trace.

Example 1.6.3 Let $A = M_n$. Define a linear functional Tr on A by

$$Tr((a_{ij})) = \sum_{i=1}^{n} a_{ii}.$$

This is a trace. It is called the standard trace on M_n. The normalized trace tr on A is defined to be $tr = (1/n)Tr$. It is a tracial state.

Example 1.6.4 Let A be a C^*-subalgebra of $B(H)$, where H is a Hilbert space. Let $v \in H$ be a nonzero vector. Define $f(a) = \langle a(v), v\rangle$ for $a \in A$. Then f is positive. To see this, note that for $a \geq 0$, there is $x \in A$ such that $a = x^*x$. Thus $f(a) = \langle x^*x(v), v\rangle = \langle x(v), x(v)\rangle = \|x(v)\| \geq 0$. This functional is, in general, not a trace.

Lemma 1.6.5 *Every positive linear functional on a C^*-algebra A is bounded.*

Proof. Let ϕ be a positive linear functional on A. If ϕ is not bounded, then there is a sequence $\{x_n\} \subset A$ with $\|x_k\| \leq 1$ such that $|\phi(x_k)| \to \infty$. Since $\phi(A_{sa}) \subset \mathbb{R}$, by considering the real part and imaginary part of x_k, without loss of generality, we may assume that $x_k \in A_{sa}$. By considering $x_k = (x_k)_+ - (x_k)_-$, we may further assume that $x_k \in A_+$. By passing to a subsequence if necessary, we may assume that $\phi(x_k) \geq 2^k$, $k = 1, 2, ...,$. Set $x = \sum_{k=1}^{\infty} \frac{x_k}{2^k}$. Then $x \in A_+$. For any n,

$$n \leq \sum_{k=1}^{\infty} 2^{-k}\phi(x_k) = \phi(\sum_{k=1}^{\infty} \frac{x_k}{2^k}) \leq \phi(x).$$

This is impossible. Thus ϕ must be bounded. □

The following is the Cauchy-Schwarz inequality.

Lemma 1.6.6 *If ϕ is a positive linear functional on a C^*-algebra A, then for all a, $b \in A$*

$$|\phi(b^*a)|^2 \le \phi(b^*b)\phi(a^*a). \tag{e 6.6}$$

Proof. For each complex number λ, we have $\phi((\lambda a+b)^*(\lambda a+b)) \ge 0$. We may assume that $\phi(b^*a) \ne 0$. With $\lambda = \frac{t|\phi(a^*b)|}{\phi(b^*a)}$ and $t \in \mathbb{R}$, this gives

$$t^2\phi(a^*a) + 2t|\phi(b^*a)| + \phi(b^*b) \ge 0$$

($\overline{\phi(b^*a)} = \phi(a^*b)$). If $\phi(a^*a) = 0$, the above becomes

$$2t|\phi(b^*a)| + \phi(b^*b) \ge 0$$

for all $t \in \mathbb{R}$. By taking negative t with large $|t|$, we conclude that $\phi(b^*a) = 0$. The inequality (e 6.6) holds in this case.

If $\phi(a^*a) \ne 0$, choose $t = -\frac{|\phi(b^*a)|}{\phi(a^*a)}$. We obtain

$$\frac{|\phi(b^*a)|^2}{\phi(a^*a)} - 2\frac{|\phi(b^*a)|^2}{\phi(a^*a)} + \phi(b^*b) \ge 0.$$

Thus (e 6.6) follows. □

Theorem 1.6.7 *An element ϕ in A^* is positive if and only if* $\lim_\lambda \phi(e_\lambda) = \|\phi\|$ *for some approximate identity $\{e_\lambda\}$ in A.*

Proof. If ϕ is a positive linear functional on A and $\{e_\lambda\}$ is any approximate identity for A, then $\phi(e_\lambda)$ is increasing. Hence it has a limit, say l. Then $l \le \|\phi\|$. For each $a \in A$ with $\|a\| \le 1$, by 1.6.6,

$$|\phi(e_\lambda a)|^2 \le \phi(e_\lambda^2)\phi(a^*a) \le \phi(e_\lambda)\|\phi\| \le l\|\phi\|.$$

Since ϕ is continuous (1.6.5), we obtain $|\phi(a)|^2 \le l\|\phi\|$, whence $\|\phi\|^2 \le l\|\phi\|$ and $l = \|\phi\|$.

To prove the converse, suppose that $\{\phi(e_\lambda)\}$ converges to $\|\phi\|$. We first show that $\phi(A_{sa}) \subset \mathbb{R}$. Let $a \in A_{sa}$ with $\|a\| \le 1$ and write $\phi(a) = \alpha + i\beta$ with α, $\beta \in \mathbb{R}$. By multiplying by -1 if necessary, we may assume that $\beta \ge 0$. Choose e_λ so that $\|e_\lambda a - a e_\lambda\| < 1/n$. Then

$$\|ne_\lambda - ia\|^2 = \|n^2 e_\lambda + a^2 - in(ae_\lambda - e_\lambda a)\| \le n^2 + 2.$$

On the other hand

$$\lim_\lambda |\phi(ne_\lambda - ia)|^2 = (n\|\phi\| + \beta)^2 + \alpha^2.$$

Combining these two inequalities, we obtain

$$(n\|\phi\| + \beta)^2 + \alpha^2 \le (n^2 + 2)\|\phi\|^2$$

for all n. Thus

$$2n\|\phi\|\beta + \beta^2 + \alpha^2 \le 2\|\phi\|^2.$$

for all n. Hence $\beta = 0$. So $\phi(A_{sa}) \subset \mathbb{R}$.

Now if $a \in A_+$ with $\|a\| \le 1$, then $e_\lambda - a \in A_{sa}$ and $e_\lambda - a \le e_\lambda \le 1$. Therefore

$$\phi(e_\lambda - a) \le \|\phi\|.$$

Taking the limit, we obtain $\|\phi\| - \phi(a) \le \|\phi\|$. It follows that $\phi(a) \ge 0$ which implies that $\phi \ge 0$. □

Remark 1.6.8 We actually proved that $\lim_\lambda \phi(e_\lambda) = \|\phi\|$ for any approximate identity $\{e_\lambda\}$ of A. If A is unital then Theorem 1.6.7 also implies that $\phi \ge 0$ if and only if $\phi(1) = \|\phi\|$.

Lemma 1.6.9 *Let $\tilde{A}$ be the C^*-algebra obtained by adjoining a unit to the C^*-algebra A. For each positive linear functional ϕ on A define an extension $\tilde{\phi}$ on $\tilde{A}$ by setting $\tilde{\phi}(1) = \|\phi\|$. Then $\tilde{\phi}$ is positive on $\tilde{A}$ and $\|\tilde{\phi}\| = \|\phi\|$.*

Proof. Let $\{e_\lambda\}$ be an approximate identity for A. For each $a \in A$ and $\alpha \in \mathbb{C}$, by 1.4.10 (3),

$$\begin{aligned}\limsup_\lambda \|\alpha e_\lambda + a\|^2 &= \limsup_\lambda \||\alpha|^2 e_\lambda^2 + \bar{\alpha} e_\lambda a + \alpha a^* e_\lambda + a^* a\| \\ &\le \limsup_\lambda \||\alpha|^2 1 + \bar{\alpha} e_\lambda a + \alpha a^* e_\lambda + a^* a\| = \|\alpha 1 + a\|^2.\end{aligned}$$

It follows from 1.6.7 that

$$|\tilde{\phi}(\alpha + a)| = \lim_\lambda |\phi(\alpha e_\lambda + a)| \le \|\alpha 1 + a\|\|\phi\|.$$

Hence $\|\tilde{\phi}\| = \|\phi\|$. Since $\tilde{\phi}(1) = \|\tilde{\phi}\|$, by 1.6.7, $\tilde{\phi} \ge 0$. □

Proposition 1.6.10 *Let B be a C^*-subalgebra of a C^*-algebra A. For each positive linear functional ϕ on B there is a positive linear functional $\tilde{\phi}$ on A such that $\tilde{\phi}|_B = \phi$ and $\|\tilde{\phi}\| = \|\phi\|$. If furthermore B is hereditary, then the extension is unique.*

Proof. We may assume that A has a unit by replacing A by $\tilde{A}$ if necessary. Extend ϕ to $C^*(B, 1_A) \cong \tilde{B}$ by setting $\phi(1) = \|\phi\|$. It follows from 1.6.9 that so extended ϕ is positive and preserves the norm. Therefore, without loss of generality, we may assume that A has a unit 1 and $1 \in B$.

By the Hahn-Banach theorem, there is a linear functional $\tilde{\phi}$ such that $\tilde{\phi}|_B = \phi$ and $\|\tilde{\phi}\| = \|\phi\|$. Since $\tilde{\phi}(1) = \|\tilde{\phi}\|$, by 1.6.7, $\tilde{\phi} \geq 0$.

To prove the second part, assume that B is hereditary and let $\{e_\lambda\}$ be an approximate identity for B. Suppose that ψ is a positive linear functional on A with $\psi|_B = \phi$ and $\|\psi\| = \|\phi\|$. Then $\|\psi\| = \lim_\lambda \phi(e_\lambda)$ by 1.6.7. So $\lim_\lambda \psi(1_A - e_\lambda) = 0$. It follows that $\psi((1_A - e_\lambda)^2) \leq \psi(1_A - e_\lambda) \to 0$. By 1.6.6, for any $c \in A$,

$$|\psi((1 - e_\lambda)ac)|^2 \leq \psi((1_A - e_\lambda)^2)\psi(c^*a^*ac) \to 0.$$

Therefore

$$\psi(a) = \lim_\lambda \psi(e_\lambda a e_\lambda).$$

Since $e_\lambda \tilde{A} e_\lambda \subset B$, we have

$$\psi(a) = \lim_\lambda \psi(e_\lambda a e_\lambda) = \lim_\lambda \phi(e_\lambda a e_\lambda)$$

for every $a \in A$. Thus $\psi = \tilde{\phi}$. □

Corollary 1.6.11 *Let $a \in A$ be a normal element. Then there is a state ϕ on A such that $|\phi(a)| = \|a\|$.*

Proof. Let $B = C^*(a)$. Since $r(a) = \|a\|$, by 1.2.7, there is $\lambda \in \mathrm{sp}(a)$ such that $|\lambda| = \|a\|$. Identify B with $C_0(\mathrm{sp}(a))$, by 1.3.6, we define a state ϕ_1 on B by $\phi_1(g) = g(\lambda)$ for $g \in C_0(\mathrm{sp}(a))$. In particular, $|\phi_1(a)| = \|a\|$. By 1.6.10, there is a state ϕ on A such that $\phi|_B = \phi_1$. □

Definition 1.6.12 Let A be a C^*-algebra and $f \in A^*$. Define $f^*(a) = \overline{f(a^*)}$. Denote by f_{sa} and f_{im} the self-adjoint linear functionals $(1/2)(f+f^*)$ and $(1/2i)(f - f^*)$, respectively. We also use $\mathrm{Re}f$ for the real part of f, i.e., $\mathrm{Re}f(a) = (1/2)[f(a) + \overline{f(a)}]$ (for $a \in A$). $\mathrm{Re}f$ is a real linear functional on A (regard A as a real Banach space). One should note that f_{sa} and $\mathrm{Re}f$

are different. It is standard that $\|f\| = \|\mathrm{Re}f\|$. This is usually used in the proof of the Hahn-Banach theorem (to pass from the real version to the complex version). It will be used later in these notes.

We have the following non-commutative Jordan decomposition theorem.

Proposition 1.6.13 *Let A be a C^*-algebra and $f \in A^*$. Then f is a linear combination of states. More precisely, we have the following decomposition:*

$$f = f_{sa} + if_{im}, f_{sa} = (f_{sa})_+ - (f_{sa})_- \text{ and } f_{im} = (f_{im})_+ - (f_{im})_-,$$

where $\|f_{sa}\| \le \|f\|, \|f_{im}\| \le \|f\|$, $(f_{sa})_+$, $(f_{sa})_-$, $(f_{im})_+$ and $(f_{im})_-$ are positive and

$$\|f_{sa}\| = \|(f_{sa})_+\| + \|(f_{sa})_-\| \text{ and } \|f_{im}\| = \|(f_{im})_+\| + \|(f_{im})_-\|.$$

Proof. It is clear that f_{sa} and f_{im} are self-adjoint and $\|f_{sa}\|$, $\|f_{im}\| \le \|f\|$ are obvious. For the rest of the proof, we may assume that f is self-adjoint. Let Ω be the set of all positive linear functionals g with $\|g\| \le 1$. With the weak*-topology, Ω is compact. View A as a closed subspace of $C(\Omega)$. Note that $A_+ \subset C(\Omega)$. Let $f \in A^*$. Then f can be extended to a bounded linear functional $\tilde{f}$ on $C(\Omega)$ with the same norm. Therefore there is a complex Radon measure μ on Ω such that

$$\tilde{f}(x) = \int_\Omega x d\mu \quad (x \in C(\Omega)).$$

Let $\mu = \nu_1 + i\nu_2$, where ν_1 and ν_2 are signed measures. So ν_j $(j = 1, 2)$ gives self-adjoint bounded linear functionals on A which will be denoted by f_j $(j = 1, 2)$. Since f is self-adjoint, $(f_2)|_A = 0$. So f_1 extends f (necessarily they have the same norm). Therefore we may assume that $\tilde{f}$ is self-adjoint. So we may assume that μ is a signed measure. Then by Jordan decomposition, we have positive measures μ_j $(j = 1, 2)$ such that $\mu = \mu_1 - \mu_2$ and $\|\mu\| = \|\mu_1\| + \|\mu_2\|$. Each μ_j gives a positive linear functional f_j on A $(j = 1, 2)$. We have the following

$$\|f\| = \|\tilde{f}|_A\| \le \|(f_1)|_A\| + \|(f_2)|_A\| \le \|f_1\| + \|f_2\| = \|\tilde{f}\| = \|f\|.$$

Thus $\|f\| = \|(f_1)|_A\| + \|(f_2)|_A\|$. □

Definition 1.6.14 A *representation* of a C^*-algebra A is a pair (H, π), where H is a Hilbert space and $\pi : A \to B(H)$ is a homomorphism. We say

(H, π) is *faithful* if π is injective. A *cyclic representation* is a representation (H, π) with a vector $v \in H$ such that $\pi(A)v$ is dense in H; and the vector v is called cyclic.

Theorem 1.6.15 *For each positive linear functional ϕ on a C^*-algebra A there is a cyclic representation (H_ϕ, π_ϕ) of A with a cyclic vector v_ϕ such that $\langle \pi_\phi(a)v_\phi, v_\phi\rangle = \phi(a)$ for all $a \in A$.*

Proof. Define the *left kernel* of ϕ as the set

$$N_\phi = \{a \in A : \phi(a^*a) = 0\}.$$

Since ϕ is continuous, N_ϕ is a closed subspace. It follows from the Cauchy-Schwarz inequality (1.6.6) that N_ϕ is a closed left ideal. On A/N_ϕ define

$$\langle \bar{a}, \bar{b}\rangle = \phi(b^*a). \tag{e 6.7}$$

It is easy to see that the above is well defined on $A/N_\phi \times A/N_\phi$. By the Cauchy-Schwarz inequality (1.6.6) again, A/N_ϕ becomes an inner product space. Let H_ϕ be the completion. Define $\pi_\phi(a)(\bar{x}) = \overline{ax}$ for all a, $x \in A$. It is well-defined. In fact, if $y \in A$ such that $\bar{y} = \bar{x}$, then $a(y - x) \in N_\phi$, since N_ϕ is a left ideal. Thus $\pi_\phi(a)$ is a linear map on A/N_ϕ. Furthermore,

$$\|\pi_\phi(a)\bar{x}\|^2 = \|\overline{ax}\|^2 = \phi(x^*a^*ax) \le \|a\|^2\phi(x^*x) = \|a\|^2\|\bar{x}\|^2$$

(the inequality holds because $a^*a \le \|a\|^2$ and ϕ is positive). Thus π_ϕ can be uniquely extended to an operator (still denoted by $\pi_\phi(a)$) on H_ϕ such that $\|\pi_\phi(a)\| \le \|a\|$. Also,

$$\langle \pi_\phi(a)(\bar{x}), \bar{y}\rangle = \phi(y^*ax) = \langle \bar{x}, \pi_\phi(a^*)(\bar{y})\rangle$$

which shows that $\pi_\phi(a)^* = \pi_\phi(a^*)$. So $\pi_\phi : A \to B(H_\phi)$ gives a representation.

To complete the proof, let $\{e_\lambda\}$ be an approximate identity for A. Then for $\lambda < \mu$,

$$\|\bar{e}_\mu - \bar{e}_\lambda\|^2 = \phi((e_\mu - e_\lambda)^2) \le \phi(e_\mu - e_\lambda).$$

Since $\phi(e_\lambda) \to \|\phi\|$, the net $\{\bar{e}_\lambda\}$ is convergent with a limit $v_\phi \in H_\phi$. For each $a \in A$ we have

$$\pi_\phi(a)(v_\phi) = \lim_\lambda \pi_\phi(a)(\bar{e}_\lambda) = \lim_\lambda \overline{ae_\lambda} = \bar{a},$$

since $a \to \bar{a}$ is continuous. Hence v_ϕ is cyclic. Moreover, since

$$\langle \pi_\phi(a^*a)(v_\phi), v_\phi \rangle = \langle \bar{a}, \bar{a} \rangle = \phi(a^*a),$$

we have $\langle \pi_\phi(a)(v_\phi), v_\phi \rangle = \phi(a)$ for all $a \in A_+$ (by 1.4.8) and by linearity for all $a \in A$. □

Definition 1.6.16 Let $(H_\lambda, \pi_\lambda)_{\lambda \in \Lambda}$ be a family of representations. Let $H = \oplus_\lambda H_\lambda$ be the Hilbert space direct sum. Define $\pi(a)(\{v_\lambda\}) = \{\pi_\lambda(a)(v_\lambda)\}$. One verifies that $\pi : A \to B(H)$ gives a representation of A. This representation is called the direct sum of $\{\pi_\lambda\}$.

Let S be the state space of A. By 1.6.15, for each $t \in S$, there is a representation π_t of A. The direct sum π_U of $\{\pi_t\}_{t \in S}$ is called the *universal representation* of A.

Theorem 1.6.17 (Gelfand-Naimark) *If A is a C^*-algebra, then it has a faithful representation. In other words, every C^*-algebra is isometrically $*$-isomorphic to a C^*-subalgebra of $B(H)$ for some Hilbert space H.*

Proof. We will show that the universal representation π_U is faithful. Let $a \in A$ be nonzero with $\|a\| \le 1$. It follows from 1.6.11 that there is $\tau \in S$ such that $|\tau((a^*a)^2)| = \|(a^*a)^2\| = \|a\|^4$. Then

$$\begin{aligned} \|\pi_U(a)\|^2 &\ge \|\pi_\tau(a)\|^2 \ge \|\pi_\tau(a)(\overline{(a^*a)^{1/2}})\|^2 \\ &= \tau((a^*a)^{1/2}(a^*a)(a^*a)^{1/2}) = \tau((a^*a)^2) = \|a\|^4. \end{aligned}$$

Thus π_U is injective. □

1.7 Von Neumann algebras

Definition 1.7.1 Let H be a Hilbert space and $B(H)$ be the C^*-algebra of all bounded operators on H. The *strong* (operator) topology on $B(H)$ is the locally convex space topology associated with the family of semi-norms of the form $x \mapsto \|x(\xi)\|$, $x \in B(H)$ and $\xi \in H$. In other words, a net $\{x_\lambda\}$ converges strongly to x and only if $\{x_\lambda(\xi)\}$ converges to $x(\xi)$ for all $\xi \in H$.

The *weak* (operator) topology on $B(H)$ is the locally convex space topology associated with the family of semi-norms of the form $x \mapsto |\langle x(\xi), \eta \rangle|$, $x \in B(H)$ and $\xi, \eta \in H$. In other words, a net $\{x_\lambda\}$ converges weakly to $x \in B(H)$ if and only if $\langle (x_\lambda - x)(\xi), \eta \rangle \to 0$ for all $\xi, \eta \in H$.

since $a \to \bar{a}$ is continuous. Hence v_ϕ is cyclic. Moreover, since

$$\langle \pi_\phi(a^*a)(v_\phi), v_\phi \rangle = \langle \bar{a}, \bar{a} \rangle = \phi(a^*a),$$

we have $\langle \pi_\phi(a)(v_\phi), v_\phi \rangle = \phi(a)$ for all $a \in A_+$ (by 1.4.8) and by linearity for all $a \in A$. □

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$$\begin{aligned} \|\pi_U(a)\|^2 &\ge \|\pi_\tau(a)\|^2 \ge \|\pi_\tau(a)(\overline{(a^*a)^{1/2}})\|^2 \\ &= \tau((a^*a)^{1/2}(a^*a)(a^*a)^{1/2}) = \tau((a^*a)^2) = \|a\|^4. \end{aligned}$$

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Example 1.7.2 Let H be an infinite dimensional Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Define $a_n(\xi) = \langle \xi, e_n \rangle e_1$ for $\xi \in H$ $(n = 1, 2, ...)$. Then $a_n \in B(H)$. We have
$\lim_{n\to\infty} \|a_n(\xi)\| = \lim_{n\to\infty} |\langle \xi, e_n \rangle| = 0$. Thus a_n strongly converges to zero. However, $\|a_n\| = 1$ since $\|a_n(e_n)\| = 1$ for all n. Thus the strong topology is weaker than the norm topology.

Define $b_n(\xi) = \langle \xi, e_1 \rangle e_n$ $(n = 1, ...)$. Then, for any ξ, $\eta \in H$, $|\langle b_n(\xi), \eta \rangle| = |\langle \xi, e_1 \rangle \langle e_n, \eta \rangle| \to 0$, as $n \to \infty$. So b_n converges weakly to zero. However, $\|b_n(e_1)\| = \|e_n\| = 1$. So b_n does not converge to zero in the strong topology.

The reader may want to take a look at exercises (1.11.18-1.11.22) for some additional information about the weak and strong (operator) topologies.

Proposition 1.7.3 *Let $\{a_\lambda\}$ be an increasing net of positive operators in $B(H)$ which is also bounded above. Then $\{a_\lambda\}$ converges strongly to a positive operator $a \in B(H)$.*

Proof. Suppose that there is $M > 0$ such that $\|a_\lambda\| \le M$. For any $\xi \in H$, $\{\langle a_\lambda(\xi), \xi \rangle\}$ is a bounded increasing sequence. So it is convergent. Using the polarization identity

$$\langle a_\lambda(\xi), \eta \rangle = (1/4) \sum_{k=0}^{3} i^k \langle a_\lambda(\xi + i^k \eta), \xi + i^k \eta \rangle,$$

we see that $\{\langle a_\lambda)(\xi), \eta \rangle\}$ is convergent for all $\xi, \eta \in H$. Denote by $L(\xi, \eta)$ the limit. The map $(\xi, \eta) \mapsto L(\xi, \eta)$ is linear in the first variable and conjugate linear in the second. We also have

$$|L(\xi, \eta)| = \lim_{\lambda} |\langle a_\lambda(\xi), \eta \rangle| \le M\|\xi\|\|\eta\|$$

for all ξ, $\eta \in H$. By the Riesz representation theorem, there is $a \in B(H)$ such that $\langle a(\xi), \eta \rangle = L(\xi, \eta)$ for all $\xi, \eta \in H$. Clearly $\|a\| \le M$ and $a_\lambda \le a$. Moreover,

$$\begin{aligned} \|a(\xi) - a_\lambda(\xi)\|^2 &= \|(a - a_\lambda)^{1/2}(a - a_\lambda)^{1/2}(\xi)\|^2 \\ &\le \|a - a_\lambda\|\|(a - a_\lambda)^{1/2}(\xi)\|^2 \le 2M\langle (a - a_\lambda)(\xi), \xi \rangle, \end{aligned}$$

and $\langle (a - a_\lambda)(\xi), \xi \rangle \to 0$, so $a(\xi) = \lim_\lambda a_\lambda(\xi)$. Thus a_λ converges strongly to a. □

Definition 1.7.4 If H is a Hilbert space, we write $H^{(n)}$ for the orthogonal sum of n copies of H. If $a \in M_n(B(H))$, we define $\phi(a) \in B(H^{(n)})$ by setting

$$\phi(a)(x_1, ..., x_n) = (\sum_{j=1}^{n} a_{1j}(x_j), ..., \sum_{j=1}^{n} a_{nj}(x_j))$$

for all $(x_1, ..., x_n) \in H^{(n)}$.

It is easy to verify that the map

$$\phi : M_n(B(H)) \to B(H^{(n)}), a \mapsto \phi(a),$$

is a $*$-isomorphism. We call ϕ the canonical $*$-isomorphism of $M_n(B(H))$ onto $B(H^{(n)})$, and use it to identify these two algebras. We define a norm on $M_n(B(H))$ making it a C^*-algebra by setting $\|a\| = \|\phi(a)\|$. The following inequalities for $a \in M_n(B(H))$ are easily verified:

$$\|a_{ij}\| \leq \|a\| \leq \sum_{k,l=1}^{n} \|a_{kl}\| \;\; (i, j = 1, ..., n) \tag{e 7.8}$$

For each $i \leq n$ let P_i be the projection of $H^{(n)}$ onto the ith copy of H. Each element $x \in B(H^{(n)})$ has a representation $(a_{ij})_{1 \leq i,j \leq n}$ with $a_{ij} \in B(H)$. We define an *amplification* ρ (of multiplicity n) from $B(H) \to B(H^{(n)})$ by setting $\rho(a) = \text{diag}(a, ..., a)$ (where a repeats n times).

Lemma 1.7.5 *Let ϕ be a linear functional on $B(H)$. Then the following are equivalent:*

(i) $\phi(a) = \sum_{k=1}^{n} \langle a(\xi_k), \eta_k \rangle$ *for some* $\xi_1, ..., \xi_n, \eta_1, ..., \eta_n \in H$ *and for all* $a \in B(H)$;

(ii) ϕ *is weakly continuous;*

(iii) ϕ *is strongly continuous.*

Proof. It is obvious that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). To prove (iii) $\Rightarrow$ (i), suppose that ϕ is strongly continuous. Therefore, there exist vectors $\xi_1, ..., \xi_n$ and $\delta > 0$ such that $|\phi(a)| \leq 1$, whenever $\max_k\{\|a(\xi_k)\|\} \leq \delta$ for all $a \in B(H)$. For any $a \in B(H)$, put $b = \frac{\delta a}{(\sum_{k=1}^{n} \|a\xi_k\|^2)^{1/2}}$. Then $|\phi(b)| \leq 1$. Thus

$$|\phi(a)| \leq (1/\delta)(\sum_{k=1}^{n} \|a\xi_k\|^2)^{1/2}. \tag{e 7.9}$$

for all $a \in B(H)$. With notation as in 1.7.4, we define $\xi = \xi_1 \oplus \cdots \oplus \xi_n$ in $H^{(n)}$. On the vector subspace $V = \{\rho(a)\xi : a \in B(H)\}$, define

$$\psi(\rho(a)\xi) = \phi(a).$$

From the definition (in 1.7.4), ψ is a linear functional (on the span) and $|\psi(\rho(a)\xi)| \leq (1/\delta)\|\rho(a)\xi\|$. So it is a bounded linear functional. It extends to a bounded linear functional on H_0, the closure of V. By the Riesz representation theorem, there is a vector $\eta = \eta_1 \oplus \cdots \oplus \eta_n \in H_0 \subset H^{(n)}$ such that

$$\phi(a) = \langle \rho(a)(\xi), \eta \rangle = \sum_{k=1}^{n} \langle a\xi_k, \eta_k \rangle.$$

□

Corollary 1.7.6 *Each strongly closed convex set in $B(H)$ is weakly closed.*

We leave this to the reader for an exercise (1.11.23).

Definition 1.7.7 For each subset $M \subset B(H)$, let M' denote the *commutant* of M, i.e.,

$$M' = \{a \in B(H) : ab = ba \text{ for all } b \in M\}.$$

It is easy to verify that M' is weakly closed. If M is self-adjoint, then M' is a C^*-algebra. We will write M'' for $(M')'$.

The following is von Neumann's double commutant theorem.

Theorem 1.7.8 *Let M be a C^*-subalgebra of $B(H)$ containing the identity. The following are equivalent:*
(i) $M = M''$.
(ii) M is weakly closed.
(iii) M is strongly closed.

Proof. The implication (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) follows from 1.7.7 and 1.7.6. We will show (iii) $\Rightarrow$ (i). Fix $\xi \in H$ let P be the projection on the closure of $\{a\xi : a \in M\}$. Note that $P\xi = \xi$ since $1 \in M$. Since $PaP = aP$ for all $a \in M$, $Pa^* = Pa^*P$ for all $a \in M$. Therefore $P \in M'$. Let $x \in M''$. Then $Px = xP$. Hence $x\xi \in PH$. Thus for any $\varepsilon > 0$ there is an $a \in M$ with $\|(x - a)\xi\| < \varepsilon$.

To show that x is in the strong closure of M, take finitely many vectors $\xi_1, ..., \xi_n \in H$. Set $\xi = \xi_1 \oplus \cdots \oplus \xi_n \in H^{(n)}$. It is easily checked that

$$\rho(M)' = \{a \in B(H^{(n)}) : a_{ij} \in M'\}.$$

Therefore $\rho(x) \in \rho(M)''$. From what we have proved in the first part of the proof, we obtain $a \in M$ such that

$$\sum_{k=1}^{n} \|(x-a)\xi_k\|^2 = \|(\rho(x) - \rho(a))\xi\|^2 < \varepsilon^2.$$

It follows that x is in the strong closure of M. Thus $x \in M$. In other words, $M'' \subset M$. Since clearly $M \subset M''$, $M = M''$. □

Definition 1.7.9 A weakly closed C^*-subalgebra $M \subset B(H)$ is called a *von Neumann* algebra. In other words, a C^*-subalgebra $M \subset B(H)$ is a von Neumann algebra if $M = M''$ and $1 \in M$.

Definition 1.7.10 Let A be a C^*-algebra and $\pi_U : A \to B(H_U)$ be the universal representation. Then $(\pi_U(A))''$ is called the *enveloping von Neumann algebra* (or universal weak closure) of A and will be denoted by A''. A representation $\pi : A \to B(H)$ is said to be *non-degenerate* if $\{\pi(a)H : a \in A\}$ is dense in H.

Proposition 1.7.11 *Every non-degenerate representation is a direct sum of cyclic representations*

The proof is an exercise (see 1.11.14).

Lemma 1.7.12 *Let $f \in A^*$ be self-adjoint. Then there are vectors $\xi, \eta \in H_U$ with $\|\xi\|^2, \|\eta\|^2 \le \|f\|$ such that $f(a) = \langle \pi_U(a)(\xi), \eta\rangle$ for all $a \in A$.*

Proof. To save notation, we may assume that $\|f\| = 1$. By 1.6.13, there are positive linear functionals f_j, $j = 1, 2$, such that $f = f_1 - f_2$ and $\|f\| = \|f_1\| + \|f_2\|$. Each f_j is a positive scalar multiple of a state on A. Therefore there are mutually orthogonal vectors $\xi_j \in H_U$ $(j = 1, 2)$ such that $\|\xi_j\|^2 = \|f_j\|$ and $f_j(a) = \langle \pi_U(a)(\xi_j), \xi_j\rangle$, $j = 1, 2$. Set $\xi = \xi_1 \oplus \xi_2$ and $\eta = \xi_1 \oplus (-\xi_2)$. Then $f(a) = \langle \pi_U(a)\xi, \eta\rangle$ for all $a \in A$. Furthermore,

$$\|\eta\|^2 = \|\xi\|^2 = \|\xi_1\|^2 + \|\xi_2\|^2 \le \|f\|.$$

□

Theorem 1.7.13 *The enveloping von Neumann algebra A'' of a C^*-algebra A is isometrically isomorphic, as a Banach space, to the second dual A^{**} of A which is the identity map on A.*

Proof. Let $j : A \to A^{**}$ be the usual embedding. Let (H_U, π_U) be the universal representation of A. For each $f \in A^*$, by 1.6.13, there are $\xi_f, \eta_f \in H_U$ such that $f(a) = \langle \pi_U(a)\xi_f, \eta_f \rangle$. Define $J : A'' \to A^{**}$ by $J(x)(f) = \langle x(\xi_f), \eta_f \rangle$ for each state f. It follows from 1.7.12 that $J(x)$ defines an element in A^{**}. In fact (using 1.7.12), we have

$$\|J(x)\| = \sup_{\|f\| \le 1, f \in A^*} |f(x)| \le \sup_{\|\xi\|, \|\eta\| \le 2} |\langle x(\xi), \eta \rangle| \le 4\|x\|.$$

On the other hand, if $\|x\| \le 1$, and $\xi \in H_U$ with $\|\xi\| \le 1$, $\langle \pi_U(a)\xi, x(\xi) \rangle$ for $a \in A$ defines a linear functional $\phi_{\xi, x(\xi)}$ on A with $\|\phi_{\xi, x(\xi)}\| \le \|x(\xi)\|$. So

$$|J(x)(\phi_{\xi, x(\xi)})| = |\langle x(\xi), x(\xi) \rangle| = \|x(\xi)\| \|x(\xi)\|.$$

Hence $\|J(x)\| \ge \|x(\xi)\|$ for all $\xi \in H_U$ with $\|\xi\| \le 1$. This implies that $\|J(x)\| \ge \|x\|$.

Suppose that $x \in A''_{sa}$ and $\{a_\lambda\} \subset A$ such that $\pi_U(a_\lambda)$ converges to x weakly in A''. By replacing a_λ by $(1/2)(a_\lambda + a_\lambda^*)$ (see Exercise (1.11.18)), we may assume that a_λ are self-adjoint. This implies that, for any self-adjoint $\phi \in A^*$, $J(x)(\phi)$ is real. Fix $f \in A^*$ with $\|f\| \le 1$, $x \in A''_{sa}$, and assume that $|f(x)| = e^{i\theta} f(x)$. Define $F = e^{i\theta} f$. Note that $1 \ge \|F\| = \|\mathrm{Re} F\|$. We have

$$|f(x)| = F(x) = \mathrm{Re} F(x) = F_{sa}(x) = |\langle x(\xi), \eta \rangle|$$

for some $\xi, \eta \in H_U$ with $\|\xi\|, \|\eta\| \le 1$ (by 1.7.12). Thus

$$|f(x)| = |\langle x(\xi), \eta \rangle| \le \|x\|.$$

This implies that $\|J(x)\| = \|x\|$ for all $x \in A''_{sa}$.

In general, note that $M_2(A'') = (M_2(A))''$. Let $J_2 : (M_2(A))'' \to M_2(A)^{**}$ be as above. Fix $f \in A^*$ and $x \in A''$. There are $\xi_f, \eta_f \in H_U$ such that $J(x)(f) = \langle x(\xi_f), \eta_f \rangle$.

In $H^{(2)}$, set $\xi' = 0 \oplus \xi_f$ and $\eta' = \eta_f \oplus 0$. Define a linear functional ϕ on Q, where

$$Q = \{(a_{ij}) \in M_2(A) : a_{11} = a_{22} = 0\},$$

by $\phi(z) = \langle z(\xi'), \eta' \rangle$ for $z \in Q$. So $\|\phi\| = \|f\|$. Extend ϕ further to $M_2(A)$ such that $\|\phi\| = \|f\|$. By 1.6.13 and 1.7.12, there are $\xi_1, \eta_1 \in H_U$ such that $J_2(z)(\phi) = \langle z(\xi_1), \eta_1 \rangle$ for all $z \in M_2(A)''$. Suppose that $z = (z_{ij}) \in (M_2(A))''$ such that $z_{11} = z_{22} = 0$. Since z is in the weak (operator) closure of $\pi_U(Q)$, $J_2(z)(\phi) = \langle z(\xi'), \eta' \rangle = f(z_{12})$. Put $b = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}$. Then $b \in M_2(A'')_{sa}$. Therefore, from what we have shown, $\|J_2(b)\| = \|b\| = \|x\|$. So

$$|J(x)(f)| = |J_2(b)(\phi)| \le \|b\|\|\phi\| = \|x\|\|f\|.$$

This implies that $\|J(x)\| \le \|x\|$. Therefore J is an isometry. Since A is weakly dense in A'', $J(A)$ is dense in $J(A'')$ in the weak*-topology as a subset of $(A^*)^*$. However, $j(A)$ is dense in $(A^*)^*$ in the weak*-topology. Therefore, since $J(a) = j(a)$ for $a \in A$, $J(A'') = (A^*)^* = A^{**}$. □

The following corollary follows from the above theorem and the uniform boundedness theorem.

Corollary 1.7.14 *Let $\{a_\lambda\}$ be a net in a C^*-algebra A''. If $\{a_\lambda\}$ converges in the weak operator topology in A'', then $\{\|a_\lambda\|\}$ is bounded.*

Remark 1.7.15 The weak operator topology for A as a subalgebra of A'' acting on H_U is called the *σ-weak topology*. So the weak operator topology for A'' is sometimes called σ-weak topology.

1.8 Enveloping von Neumann algebras and the spectral theorem

Lemma 1.8.1 *Let (π_1, H_1) and (π_2, H_2) be two cyclic representations of a C^*-algebra A with cyclic vectors ξ_1 and ξ_2 ($\|\xi_1\| = \|\xi_2\|$). Then there exists an isometry $u : H_1 \to H_2$ such that $\pi_1 = u^*\pi_2 u$ with $u(\xi_1) = \xi_2$ if and only if $\langle \pi_1(a)\xi_1, \xi_1 \rangle = \langle \pi_2(a)\xi_2, \xi_2 \rangle$ for all $a \in A$.*

Proof. If $u\xi_1 = \xi_2$ then

$$\langle \pi_1(a)\xi_1, \xi_1 \rangle = \langle u^*\pi_2(a)u\xi_1, \xi_1 \rangle = \langle \pi_2(a)\xi_2, \xi_2 \rangle$$

for all $a \in A$.

Conversely, define a linear map u from $\pi_1(A)\xi_1$ onto $\pi_2(A)\xi_2$ by $u(\pi_1(a)\xi_1) = \pi_2(a)\xi_2$. Since

$$\|u(\pi_1(a)\xi_1)\|^2 = \langle \pi_2(a^*a)\xi_2, \xi_2 \rangle = \langle \pi_1(a^*a)\xi_1, \xi_1 \rangle = \|\pi_1(a)\xi_1\|^2,$$

by $\phi(z) = \langle z(\xi'), \eta' \rangle$ for $z \in Q$. So $\|\phi\| = \|f\|$. Extend ϕ further to $M_2(A)$ such that $\|\phi\| = \|f\|$. By 1.6.13 and 1.7.12, there are $\xi_1, \eta_1 \in H_U$ such that $J_2(z)(\phi) = \langle z(\xi_1), \eta_1 \rangle$ for all $z \in M_2(A)''$. Suppose that $z = (z_{ij}) \in (M_2(A))''$ such that $z_{11} = z_{22} = 0$. Since z is in the weak (operator) closure of $\pi_U(Q)$, $J_2(z)(\phi) = \langle z(\xi'), \eta' \rangle = f(z_{12})$. Put $b = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}$. Then $b \in M_2(A'')_{sa}$. Therefore, from what we have shown, $\|J_2(b)\| = \|b\| = \|x\|$. So

$$|J(x)(f)| = |J_2(b)(\phi)| \le \|b\|\|\phi\| = \|x\|\|f\|.$$

This implies that $\|J(x)\| \le \|x\|$. Therefore J is an isometry. Since A is weakly dense in A'', $J(A)$ is dense in $J(A'')$ in the weak*-topology as a subset of $(A^*)^*$. However, $j(A)$ is dense in $(A^*)^*$ in the weak*-topology. Therefore, since $J(a) = j(a)$ for $a \in A$, $J(A'') = (A^*)^* = A^{**}$. □

The following corollary follows from the above theorem and the uniform boundedness theorem.

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Proof. If $u\xi_1 = \xi_2$ then

$$\langle \pi_1(a)\xi_1, \xi_1 \rangle = \langle u^*\pi_2(a)u\xi_1, \xi_1 \rangle = \langle \pi_2(a)\xi_2, \xi_2 \rangle$$

for all $a \in A$.

Conversely, define a linear map u from $\pi_1(A)\xi_1$ onto $\pi_2(A)\xi_2$ by $u(\pi_1(a)\xi_1) = \pi_2(a)\xi_2$. Since

$$\|u(\pi_1(a)\xi_1)\|^2 = \langle \pi_2(a^*a)\xi_2, \xi_2 \rangle = \langle \pi_1(a^*a)\xi_1, \xi_1 \rangle = \|\pi_1(a)\xi_1\|^2,$$

we see that u extends to an isometry from the closure of $\pi_1(A)\xi_1$ onto the closure of $\pi_2(A)\xi_2$. Since ξ_1 and ξ_2 are cyclic vectors, u is an isometry from H_1 onto H_2. We have

$$u\pi_1(a)\pi_1(b)\xi_1 = \pi_2(ab)\xi_2 = \pi_2(a)u\pi_1(b)\xi_1$$

for all a, $b \in A$. Thus $u\pi_1(a) = \pi_2(a)u$ since $\{\pi_1(b)\xi_1 : b \in A\}$ is dense in H_1. The lemma follows. □

Theorem 1.8.2 *Let A be a C^*-algebra and $\pi : A \to B(H)$ be a non-degenerate representation. Then there is a unique $\pi'' : A'' \to \pi(A)''$ such that $\pi''|_A = \pi$ and it is (σ-) weak-weak continuous, i.e., if $\{x_\lambda\}$ is a weak convergent net in the von Neumann algebra A'' then $\{\pi''(x_\lambda)\}$ is a weak convergent net in $B(H)$.*

Proof. First assume that (π, H) is cyclic with cyclic vector ξ_1 and $\|\xi_1\| = 1$. Then $\phi(a) = \langle \pi_1(a)\xi_1, \xi_1\rangle$ is a state. It follows from 1.8.1 that we may assume that $(\pi, H) = (\pi_\phi, H_\phi)$. Let p_ϕ be the projection of H_U onto H_ϕ. Then $p_\phi\pi_U(a) = \pi_U(a)p_\phi = \pi_\phi(a)$ for all $a \in A$. Thus $p_\phi \in A'$. It is clear that $x \mapsto p_\phi x$ is weak-weak continuous from A'' into $\pi(A)''$ which extends π (from A to $\pi(A)$).

By 1.7.11, in general, $\pi = \oplus_\lambda \pi_\lambda$, where each π_λ is a cyclic representation. Therefore, there are states ϕ_λ of A and an isometry $U : H \to \oplus_\lambda H_{\phi_\lambda}$ such that $U^*\pi U = \oplus_\lambda \pi_{\phi_\lambda}$. So we may assume that $\pi = \oplus_\lambda \pi_{\phi_\lambda}$. Let p be the projection from H_U onto $\oplus_\lambda H_{\phi_\lambda}$. Then the map $x \mapsto px$ is weak-weak continuous and extends π. □

Definition 1.8.3 Let X be a compact Hausdorff space. Let $\mathcal{B}(X)$ be the set of all bounded Borel functions on X. $\mathcal{B}(X)$ is a C^*-algebra containing $C(X)$ as a C^*-subalgebra. It follows from 1.2.3 that $\mathcal{B}(X)$ is a subspace in $C(X)^{**}$. By 1.7.13, $C(X)'' = C(X)^{**}$. So $\mathcal{B}(X)$ is a C^*-subalgebra of $C(X)''$. We say $\{f_\lambda\}$ converges weakly to f in $\mathcal{B}(X)$, if they do so as elements of the von Neumann algebra $C(X)''$. If $\{f_\lambda\}$ is bounded and converges pointwise everywhere on X, then, by Lebesgue's Dominated Convergence Theorem, $\{f_\lambda\}$ converges weakly.

Corollary 1.8.4 *Let $\phi : C(X) \to B(H)$ be a unital homomorphism. Then there is a unique homomorphism $\tilde{\phi} : \mathcal{B}(X) \to B(H)$ such that $\tilde{\phi}|_{C(X)} = \phi$ and $\{f_\lambda\}$ converges weakly to f in $\mathcal{B}(X)$ implies that $\{\tilde{\phi}(f_\lambda)\}$ converges weakly to $\tilde{\phi}(f)$ in $B(H)$.*

Definition 1.8.5 Let $a \in B(H)$ be a normal element. Then $A = C^*(1, a)$ is a C^*-subalgebra. Denote by $j : A \to B(H)$ the embedding. Then by 1.8.4, let $\tilde{j} : \mathcal{B}(X) \to B(H)$ be the extension, where $X = \mathrm{sp}(a)$. We will use $f(a)$ for $\tilde{j}(f)$, $f \in \mathcal{B}(X)$. Thus the following is called the *Borel functional calculus.*

Corollary 1.8.6 *Let M be a von Neumann algebra and $a \in M$ a normal element. Then there is a weak-weak continuous homomorphism from $\mathcal{B}(\mathrm{sp}(a)) \to M$ which maps $z \to a$.*

Definition 1.8.7 Let X be a compact Hausdorff space and H be a Hilbert space. A *spectral measure* E relative to (X, H) is a map from the Borel sets of X to the set of projections in $B(H)$ such that

(1) $E(\emptyset) = 0$, $E(X) = 1$;

(2) $E(S_1 \cap S_2) = E(S_1)E(S_2)$ for all Borel sets S_1, S_2 of X;

(3) for all $\xi, \eta \in H$, the function $E_{\xi,\eta} : S \mapsto \langle E(S)\xi, \eta\rangle$ is a regular Borel complex measure.

For any simple function $g = \sum_{k=1}^{n} \alpha_k \chi_{S_k}$,

$$\int_X g(\lambda) dE_\lambda = \sum_{k=1}^{n} \alpha_k E(S_k)$$

is an operator in $B(H)$ (S_k are Borel sets). Let $f \in \mathcal{B}(X)$. There is a sequence $\{g_n\}$ of simple functions such that $\|g_n - f\| \to 0$ as $n \to \infty$. Since $\mu(S) = \langle E(S)\xi, \eta\rangle$ defines a regular Borel complex measure, $\int_X g_n dE_\lambda$ converges weakly to an element (in $B(H)$). Denote by $\int_X f dE_\lambda$ the weak limit. Clearly this limit does not depend on the choice of g_n.

We have the following corollary otherwise known as the *spectral theorem.*

Corollary 1.8.8 *Let a be a normal operator on a Hilbert space H. Then there is a unique spectral measure E relative to $(\mathrm{sp}(a), H)$ such that*

$$a = \int_{\mathrm{sp}(a)} \lambda dE_\lambda.$$

Proof. By the Borel functional calculus, the embedding $j : C^*(1, a) \to B(H)$ extends to a weak-weak continuous homomorphism $\tilde{j} : \mathcal{B}(X) \to B(H)$. For each Borel subset of $\mathrm{sp}(a)$, $\tilde{j}(\chi_S) = E(S)$ is a projection. It is easy to verify that $\{E(S) : S \text{ Borel}\}$ forms a spectral measure. Since the identity function on $\mathrm{sp}(a)$ is Borel, it follows from the Borel functional calculus that

$$a = \int_{\mathrm{sp}(a)} \lambda dE_\lambda.$$

We leave it to the reader to check the uniqueness of E. □

Recall the range projection p of an operator $a \in B(H)$ is the projection on the closure of $\{a(\xi) : \xi \in H\}$.

Proposition 1.8.9 *If M is a von Neumann algebra, then it contains the range projection of every element in M.*

Proof. Let $a \in M$. It is clear that we may assume that $\|a\| \leq 1$. Since $(aa^*)^{1/2}$ and a have the same range projection, we may assume that $a \geq 0$. Note that $\{a^{1/n}\}$ is increasing and bounded. It follows from 1.7.3 that $\{a^{1/n}\}$ converges strongly to a positive element, say $q \in M$. Let $f = \chi_{(0,\|a\|]}$. Then, by 1.8.6, $a^{1/n}$ converges weakly to $f(a)$, which is a projection. Therefore $q = f(a)$. Since $a^{1/n} \in C^*(a)$ and the polynomials are dense in $C(\mathrm{sp}(a))$, $\overline{q(H)} \subset \overline{a(H)}$. One the other hand, $qa = f(a)a = a$. Hence $\overline{a(H)} \subset \overline{q(H)}$. Therefore q is the range projection of a. □

Definition 1.8.10 An operator u on H is called a *partial isometry* if u^*u is a projection. Since

$$(uu^*)^3 = u(u^*u)(u^*u)u^* = (uu^*)^2.$$

$\mathrm{sp}(uu^*) = \{0, 1\}$ by the spectral mapping theorem. Therefore, uu^* is also a projection.

Proposition 1.8.11 *Each element x in a von Neumann algebra M has a polar decomposition: there is a unique partial isometry $u \in M$ such that u^*u is the range projection of $|x|$ and $x = u|x|$.*

Proof. Set $u_n = x((1/n) + |x|)^{-1}$ and denote by p the range projection of $|x|$. Since $x = xp$, we have $u_n = u_np$. We compute that

$$\begin{aligned}
&(u_n - u_m)^*(u_n - u_m) \\
&= [(\frac{1}{n} + |x|)^{-1} - (\frac{1}{m} + |x|)^{-1})]x^*x[(\frac{1}{n} + |x|)^{-1} - (\frac{1}{m} + |x|)^{-1}] \\
&= [(\frac{1}{n} + |x|)^{-1} - (\frac{1}{m} + |x|)^{-1})]^2|x|^2. \qquad \text{(e 8.10)}
\end{aligned}$$

This converges weakly to zero (as $n, m \to \infty$) by the Borel functional calculus. So, for any $\xi \in H$,

$$\|(u_n - u_m)(\xi)\|^2 = |\langle (u_n - u_m)^*(u_n - u_m)(\xi), \xi\rangle| \to 0$$

as $n, m \to \infty$. This implies that $\{u_n\}$ converges strongly to an element $u \in M$ with $up = u$. As in (e 8.10),

$$(u_n|x| - u_m|x|)^*(u_n|x| - u_m|x|) = [(\frac{1}{n} + |x|)^{-1} - (\frac{1}{m} + |x|)^{-1})]^2|x|^4.$$

By the continuous functional calculus (see 1.3.6), the above converges in norm (as $n, m \to \infty$). On the other hand, by the Borel functional calculus, $((1/n)+|x|)^{-1}|x|$ converges weakly to p. Thus $\{u_n|x|\}$ converges in norm to x. Hence $x = u|x|$. Since $pu^*up = (u^*)(u) = u^*u$, $p(u^*u) = (u^*u)p = u^*u$. On the other hand, $x^*x = |x|u^*u|x|$. For any $\xi, \eta \in H$,

$$\langle (u^*u - p)(|x|(\xi)), |x|\eta\rangle = \langle |x|(u^*u - p)(|x|(\xi)), \eta\rangle = 0.$$

Regarding $u^*u - p$ as an operator on $p(H)$, the above implies that $u^*u = p$.

To see the decomposition is unique, let $x = v|x|$ and $v^*v = p$. Then $v|x| = u|x|$, or $(v - u)|x| = 0$. Therefore, $(v - u)p = 0$. This implies that $v = u$. □

1.9 Examples of C^*-algebras

In this section we give some examples of C^*-algebras. More will be presented later.

We first give more information about the C^*-algebras $\mathcal{K}$ and $B(H)$. If H is a Hilbert space, an operator $x \in B(H)$ is said to have finite-rank, if the range of x is a finite dimensional subspace. Denote by $F(H)$ the set of finite-rank operators on H. It is easy to check that $F(H)$ is a $*$-subalgebra of $B(H)$ and is an ideal (not necessary closed) of $B(H)$.

Clearly every operator in $F(H)$ is compact. It is also easy to see that $F(H)$ is a linear span of rank-one projections.

Lemma 1.9.1 *If H is a Hilbert space and $K(H)$ is the C^*-algebra of all compact operators on H, then $F(H)$ is dense in $K(H)$.*

Proof. Since $\overline{F(H)}$ and $K(H)$ are both self-adjoint, it suffices to show that every self-adjoint element $x \in K(H)$ is in $\overline{F(H)}$. We may even further