Operator algebras - stage for non-commutativity
(Panorama Lectures Series)
III. von Neumann algebras

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- a *trace* if \( \text{tr}(xy) = \text{tr}(yx) \) for all $x, y \in A$;
- *normalised*\(^1\) if $A$ is unital and \( \text{tr}(1) = 1 \);
- *positive* if \( \text{tr}(x^*x) \geq 0 \) $\forall x \in A$;
- *faithful and positive* if $A$ is a $*$-algebra and \( \text{tr}(x^*x) > 0 \) $\forall 0 \neq x \in A$.

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Finite-dimensional $C^*$-algebras

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For example, $M_n(\mathbb{C})$ admits a unique normalised trace ($\text{tr}(x) = \frac{1}{n} \sum_{i=1}^{n} x_{ii}$) which is also faithful and positive.

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**Proposition FDC*: The following conditions on a finite-dimensional unital *-algebra $A$ are equivalent:

1. There exists a unital *-monomorphism $\pi : A \rightarrow M_n(\mathbb{C})$ for some $n$.
2. There exists a faithful positive normalised trace on $A$. 

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For a finite-dimensional \((C)^*\)-algebra \(M\) with faithful positive normalised\(^2\) trace \(\text{tr}\), let us write \(L^2(M, \text{tr}) = \{\hat{x}: x \in M\}\), with \(\langle \hat{x}, \hat{y} \rangle = \text{tr}(y^*x)\), as well as \(\lambda_M, \rho_M: M \rightarrow \mathcal{L}(L^2(M, \text{tr}))\) for the left and right regular representations, i.e., the maps (injective unital \(*\)-homomorphism and \(*\)-antihomomorphism, respectively) defined by

\[
\lambda_M(x)(\hat{y}) = \hat{xy} = \rho_M(y)(\hat{x}).
\]

\(^2\)It is a fact that every finite-dimensional \((C)^*\)-algebra is unital.
The standard form

For a finite-dimensional $(C^*)$-algebra $M$ with faithful positive normalised\(^2\) trace \('tr'\), let us write $L^2(M, tr) = \{\hat{x} : x \in M\}$, with $\langle \hat{x}, \hat{y} \rangle = tr(y^*x)$, as well as $\lambda_M, \rho_M : M \to L(L^2(M, tr))$ for the left and right regular representations, i.e., the maps (injective unital *-homomorphism and *-antihomomorphism, respectively) defined by

$$\lambda_M(x)(\hat{y}) = \hat{x}\hat{y} = \rho_M(y)(\hat{x}) .$$

We shall usually identify $x \in M$ with the operator $\lambda_M(x)$ and thus think of $M$ as (being in standard form and) a subset of $L(L^2(M, tr))$.

The reason for the ‘hats’ is that we wish to distinguish between the operator $x \in L(L^2(M, tr))$ and the vector $\hat{x} \in L^2(M, tr)$.

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Fact: \(\lambda_M(M)' = \rho_M(M)\) and \(\rho_M(M)' = \lambda_M(M)\), where we define the commutant \(S'\) of any set \(S\) of operators on a Hilbert space \(H\) by

\[
S' = \{ x' \in L(H) : xx' = x'x \ \forall x \in S \}.
\]

\(^2\)It is a fact that every finite-dimensional \(C^*-\)algebra is unital.
Suppose $N \subset M$ is a unital inclusion of finite-dimensional $C^*$-algebras and $tr$ is a faithful tracial state on $M$. Then $\hat{N} =: L^2(N, tr|_N)$ sits naturally as a subspace of $\hat{M} =: L^2(M, tr)$. Let us write $e_N$ for the orthogonal projection of $\hat{M}$ onto $\hat{N}$, and $E_N$ for the so-called $tr$-preserving conditional expectation of $M$ onto $N$ defined by

$$E_N(m) = e_N(\hat{m})$$

\footnote{Actually $E_N$ is even $N - N$-bilinear.}
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$$\hat{E}_N(m) = e_N(\hat{m})$$

**Proposition: (CE)**

The map $E_N$ satisfies and is characterised by the following properties:

- $tr|_N = tr \circ E$.
- $E_N(nm) = nE_N(m)$, i.e., $E_N$ is $N$-linear.$^3$

$^3$ Actually $E_N$ is even $N-N$-bilinear.
Write $\mathcal{P}_{\text{min}}(Z(M))$ for the set of minimal central projections of a finite-dimensional $C^*$-algebra. It is a fact that there is a well-defined function $m : \mathcal{P}_{\text{min}}(Z(M)) \to \mathbb{N}$, such that $Mq \cong M_{m(q)}(\mathbb{C}) \ \forall q \in \mathcal{P}_{\text{min}}(Z(M))$; thus the map $M \ni x \mapsto xq$ defines an irreducible representation of $M$; and in fact, 

$\{\pi_q : q \in \mathcal{P}_{\text{min}}(Z(M))\}$ is a complete list, up to unitary equivalence, of pairwise inequivalent irreducible representations of $M$, and

$$M = \sum_{q \in \mathcal{P}_{\text{min}}(Z(M))} Mq \cong \bigoplus_{q \in \mathcal{P}_{\text{min}}(Z(M))} M_{m(q)}(\mathbb{C})$$
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Every trace on the full matrix algebra $M_n(\mathbb{C})$ is a multiple of the usual trace. It follows that any trace $\phi$ on $M$ is uniquely determined by the function $t_\phi : \mathcal{P}_{\text{min}}(Z(M)) \to \mathbb{C}$ defined by $t_\phi(q) = \phi(q_0)$ where $q_0$ is a minimal projection in $Mq$. It is clear that $\phi$ is positive (resp., faithful, resp., normalised) iff $t_\phi(q) \geq 0 \ \forall q$ (resp., $t_\phi(q) > 0 \ \forall q$, resp. $\sum_{q \in \mathcal{P}_{\text{min}}(Z(M))} m(q)t_\phi(q) = 1$).
If $N \subset M$ is a unital $C^*$-subalgebra of $M$, the associated *inclusion matrix* $\Lambda$ is the matrix with rows and columns indexed by $\mathcal{P}_{\min}(Z(N))$ and $\mathcal{P}_{\min}(Z(M))$ respectively, defined by setting $\Lambda_{pq} = \sqrt{\frac{\dim qpMpq}{\dim qpNpq}}$. Alternatively, if we write $\rho_p$ for the irreducible representation of $N$ corresponding to $p$, then $\Lambda_{pq}$ is nothing but the ‘multiplicity with which $\rho_p$ occurs in the irreducible decomposition of $\pi_q|_N$’. This data is sometimes also recorded in a bipartite graph (usually called the *Bratteli diagram* of the inclusion) with even and odd vertices indexed by $\mathcal{P}_{\min}(Z(N))$ and $\mathcal{P}_{\min}(Z(M))$ respectively, with $\Lambda_{pq}$ edges joining the vertices indexed by $p$ and $q$.

\[\footnote{For a finite group $G$, clearly $C^*_{\text{red}}(G) = \mathbb{C}G.}\]

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For example, Bratteli diagram associated to $\mathbb{C}S_2 \subset \mathbb{C}S_3$ is seen to be given by:

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\(\mathbb{C}S_2 \subset \mathbb{C}S_3\)
Proposition (bc): Suppose $N \subset M$ is a unital inclusion of finite dimensional $C^*$-algebras. Let $tr$ be a faithful, unital, positive trace on $M$. Then,

1. The $C^*$ algebra generated by $M$ and $e_N$ in $L(L^2(M, tr))$ is $\rho_M(N)'$.
2. The central support$^5$ of $e_N$ in $\rho_M(N)'$ is 1.
3. $e_Nxe_N = E(x)e_N$ for $x \in M$. (As usual, we identify $m$ with $\lambda_M(m)$.)
4. $N = M \cap \{e_N\}'$.
5. If $\Lambda$ is the inclusion matrix for $N \subset M$ then $\Lambda^t$ is the inclusion matrix for $M \subset \rho_M(N)'$.

$^5$The central support of a projection is the smallest central projection which dominates it.
The basic construction

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This *basic construction* - i.e., the passage from $N \subset M$ to $M \subset \rho_M(N)'$ - extends almost verbatim from inclusions of finite-dimensional $C^*$-algebras to one good infinite-dimensional case, that of the so-called *finite-depth subfactors* which we shall discuss in the next lecture!

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We now proceed to infinite dimensions.

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> 2. various aspects of the quantum mechanical formalism

**Definition 1:** A von Neumann algebra is the commutant of a unitary group representation (say \( \pi \) of \( G \)): i.e.,

\[
M = \{ x \in \mathcal{L}(\mathcal{H}) : x\pi(g) = \pi(g)x \ \forall g \in G \}
\]

Note that \( \mathcal{L}(\mathcal{H}) \) is a \( C^* \)-algebra w.r.t. the ‘operator norm’

\[
\|x\| = \text{sup}\{\|x\xi\| : \xi \in \mathcal{H}, \|\xi\| = 1\}
\]

and ‘Hilbert space adjoint’.
Definitions: (a) $S' = \{ x' \in \mathcal{L} (\mathcal{H}) : xx' = x'x \ \forall x \in S \}$, for $S \subset \mathcal{L} (\mathcal{H})$

(b) SOT on $\mathcal{L} (\mathcal{H})$: $x_n \rightarrow x \iff \| x_n \xi - x \xi \| \rightarrow 0 \ \forall \xi$ (i.e., $x_n \xi \rightarrow x \xi$ strongly $\forall \xi$)

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**von Neumann’s double commutant theorem (DCT):**

Let \( M \) be a unital self-adjoint subalgebra of \( \mathcal{L}(\mathcal{H}) \). TFAE:

(i) \( M \) is SOT-closed

(ii) \( M \) is WOT-closed

(iii) \( M = M'' = (M')' \)
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The equivalence of definitions 1 and 2 is a consequence of the spectral theorem and the fact that any norm-closed unital *-subalgebra $A$ of $\mathcal{L}(\mathcal{H})$ is linearly spanned by the set $\mathcal{U}(A) = \{ u \in A : u^*u = uu^* = 1 \}$ of its unitary elements.
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A von Neumann algebra is closed under all ‘canonical constructions’: for instance, if $x \to \{ 1_E(x) : E \in \mathcal{B}_C \}$ is the spectral measure associated with a normal operator $x$, then $x \in M \iff 1_E(x) \in M \ \forall \ E \in \mathcal{B}_C$. 
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(*Reason: \( \Rightarrow \): Since \( 1_E(uxu^*) = u1_E(x)u^* \) for all unitary \( u \) (the spectral measure is a canonical construction),

\[
x \in M, u' \in \mathcal{U}(M') \quad \Rightarrow \quad u'1_E(x)u'^* = 1_E(u'xu'^*) \\
\Rightarrow \quad 1_E(x) \in (\mathcal{U}(M'))' = M
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Definition 2: A von Neumann algebra is an $M$ as in DCT above.

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$$x \in M, u' \in \mathcal{U}(M') \Rightarrow u'1_E(x)u'^* = 1_E(u'xu'^*)$$

$$\Rightarrow 1_E(x) \in (\mathcal{U}(M'))' = M$$

$\Leftarrow$: Uniform approximability of bounded measurable functions implies (by the spectral theorem) that

$$M = [\mathcal{P}(M)] = (\text{span } \mathcal{P}(M))^\perp \quad (*)$$

where $\mathcal{P}(M) = \{p \in M : p = p^2 = p^*\}$ is the set of projections in $M$. 


Suppose $M = \pi(G)'$ as before. Then

$$p \leftrightarrow \text{ran } p$$

establishes a bijection

$$\mathcal{P}(M) \leftrightarrow G\text{-stable subspaces}$$

So, for instance, eqn. (*) shows that

$$(\pi(G))'' = \mathcal{L}(\mathcal{H}) \iff M = \mathbb{C} \iff \pi \text{ is irreducible}$$
Murray-von Neumann equivalence

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Under the correspondence, of sub-reps of $\pi$ to $\mathcal{P}(M)$, (unitary) equivalence of sub-repreps of $\pi$ translates to **Murray-von Neumann equivalence** on $\mathcal{P}(M)$:

$$p \sim_M q \iff \exists u \in M \text{ such that } u^*u = p, \quad uu^* = q$$
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More generally, define

$$p \preceq_M q \iff \exists p_0 \in \mathcal{P}(M) \text{ such that } p \sim_M p_0 \leq q$$
Proposition:

The following conditions are equivalent:

1. Either $p \preceq_M q$ or $q \preceq_M p$, $\forall p, q \in P(M)$.
2. $M$ has trivial center: $Z(M) = M \cap M' = \mathbb{C}$

Such an $M$ is called a factor.
Proposition:

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Such an \( M \) is called a \textbf{factor}.

If \( M = \pi(G)' \), with \( G \) finite, then \( M \) is a factor iff \( \pi \) is isotypical.

In general, any von Neumann algebra is a \textbf{direct integral} of factors.
Call a projection $p \in \mathcal{P}(M)$ infinite rel $M$ if $\exists p_0 \neq p \in \mathcal{P}(M)$ such that $p \sim_M p_0 \leq p$; otherwise, call $p$ finite (rel $M$).
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Say $M$ is **finite** if $1$ is finite.

A factor $M$ is said to be of:

1. **type I** if there is a minimal non-zero projection in $M$.

2. **type II** if it contains non-zero finite projections, but no minimal non-zero projection.

3. **type III** if it contains no non-zero finite projection.
Definition 3: $M$ is a von Neumann algebra if

- $M$ is a $C^*$-algebra (i.e., a Banach $*$-algebra satisfying $\| xx^* \| = \| x \|^2 \ \forall \ x$)

- $M$ is a dual Banach space: i.e., $\exists$ a Banach space $M_*$ such that $M \cong M_*$ as a Banach space.
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**Example:** $M = L^\infty(\Omega, \mathcal{B}, \mu)$. Can also view it as acting on $L^2(\Omega, \mathcal{B}, \mu)$ as multiplication operators. (In fact, every commutative von Neumann algebra is isomorphic to an $L^\infty(\Omega, \mathcal{B}, \mu)$.)
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Fact: The predual $M_*$ of $M$ is unique up to isometric isomorphism. So, by Alaoglu - $\exists$ a canonical locally convex (weak-*) top. on $M$ w.r.t. which the unit ball of $M$ is compact. This is called the $\sigma$-weak topology on $M$. 
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**Gelfand-Naimark theorem**: Any von Neumann algebra is isomorphic to a vN-subalgebra of some $L(H)$. (So the abstract and concrete (= tied down to Hilbert space) definitions are equivalent.)
We give a brief idea of the proof of the Gelfand Naimark theorem in the von Neumann algebra context. We assume, for simplicity, that all our von Neumann algebras have separable preduals. It is a fact that such an $M$ admits a faithful normal state $\phi$ on $M$. 

For such a $\phi$, we construct the standard form of $M$. As in the finite-dimensional case, let us write $\hat{M} = \{\hat{x} : x \in M\}$. The hypothesis on $\phi$ guarantees that the equation $\langle \hat{x}, \hat{y} \rangle = \phi(y^* x)$ defines a genuine inner product on $\hat{M}$. Let $L^2(M, \phi)$ denote the Hilbert space completion of $\hat{M}$. A little $C^*$-trickery shows that the mapping $\hat{y} \mapsto c_y$ extends to a (necessarily) unique bounded operator $\lambda_M(x)$ on $L^2(M, \phi)$. It is fairly routine to then verify that $\lambda_M$ is a normal isomorphism onto its image.

The same trickery shows why there is a difficulty in establishing a similar assertion regarding $\rho_M$ and why things go through smoothly when $\phi$ is a trace—which situation is what we will be addressing the next two lectures. The full story of how one makes do with non-tracial states involves the celebrated and technically slightly complicated Tomita Takesaki theory, which we shall say nothing more about.
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References

