Operator algebras - stage for non-commutativity
(Panorama Lectures Series)
III. von Neumann algebras

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IISc, January 29, 2009

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- a *trace* if $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ forall $x, y \in A$;
- *normalised*[1] if A is unital and $\mathrm{tr}(1) = 1$;
- *positive* if $\mathrm{tr}(x^*x) \geq 0 \forall x \in A$;
- *faithful and positive* if A is a *-algebra and $\mathrm{tr}(x^*x) > 0 \ \forall \ 0 \neq x \in A$.

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Proposition FDC*: The following conditions on a finite-dimensional unital *-algebra A are equivalent:

1. There exists a unital *-monomorphism $\pi : A \to M_n(\mathbb{C})$ for some n.
2. There exists a faithful positive normalised trace on A.

□

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The standard form

For a finite-dimensional $(C)^*$-algebra M with faithful positive normalised[2] trace 'tr', let us write $L^2(M, tr) = \{\hat{x} : x \in M\}$, with $\langle \hat{x}, \hat{y} \rangle = \mathrm{tr}(y^* x)$, as well as $\lambda_M, \rho_M : M \to \mathcal{L}(L^2(M, tr))$ for the *left* and *right regular representations*, i.e., the maps (injective unital *-homomorphism and *-antihomomorphism, repectively) defined by

$$\lambda_M(x)(\hat{y}) = \widehat{xy} = \rho_M(y)(\hat{x}) \ .$$

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We shall usually identify $x \in M$ with the operator $\lambda_M(x)$ and thus think of M as (being in *standard form* and) a subset of $\mathcal{L}(L^2(M, tr))$.

The reason for the 'hats' is that we wish to distinguish between the operator $x \in \mathcal{L}(L^2(M, tr))$ and the vector $\hat{x} \in L^2(M, tr)$.

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Fact: $\lambda_M(M)' = \rho_M(M)$ and $\rho_M(M)' = \lambda_M(M)$, where we define the *commutant* S' of any set S of operators on a Hilbert space H by

$$S' = \{x' \in \mathcal{L}(H) : xx' = x'x \ \forall x \in S\})$$

[2] It is a fact that every finite-dimensional C^*-algebra is unital.

Suppose $N \subset M$ is a unital inclusion of finite-dimensional C^*-algebras and tr is a faithful tracial state on M. Then $\hat{N} =: L^2(N, tr|_N)$ sits naturally as a subspace of $\hat{M} =: L^2(M, tr)$. Let us write e_N for the orthogonal projection of $\hat{M}$ onto $\hat{N}$, and E_N for the so-called *tr-preserving conditional expectation of M onto N* defined by

$$\widehat{E_N(m)} = e_N(\hat{m})$$

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Proposition: (CE)
The map E_N satisfies and is characterised by the following properties:

- $tr|_N = tr \circ E$.
- $E_N(nm) = nE_N(m)$, i.e., E_N is N-linear.[3]

[3] Actually E_N is even $N - N$-bilinear.

Write $\mathcal{P}_{min}(Z(M))$ for the set of minimal central projections of a finite-dimensional C^*-algebra. It is a fact that there is a well-defined function $m : \mathcal{P}_{min}(Z(M)) \to \mathbb{N}$, such that $Mq \cong M_{m(q)}(\mathbb{C})\ \forall q \in \mathcal{P}_{min}(Z(M))$; thus the map $M \ni x \stackrel{\pi_q}{\mapsto} xq$ defines an irreducible representation of M; and in fact, $\{\pi_q : q \in \mathcal{P}_{min}(Z(M))\}$ is a complete list, up to unitary equivalence, of pairwise inequivalent irreducible representations of M, and

$$M = \sum_{q \in \mathcal{P}_{min}(Z(M))} Mq \cong \oplus_{q \in \mathcal{P}_{min}(Z(M))} M_{m(q)}(\mathbb{C})$$

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Every trace on the full matrix algebra $M_n(\mathbb{C})$ is a multiple of the usual trace. It follows that any trace ϕ on M is uniquely determined by the function $t_\phi : \mathcal{P}_{min}(Z(M)) \to \mathbb{C}$ defined by $t_\phi(q) = \phi(q_0)$ where q_0 is a minimal projection in Mq. It is clear that ϕ is positive (resp., faithful, resp., normalised) iff $t_\phi(q) \geq 0\ \forall q$ (resp., $t_\phi(q) > 0\ \forall q$, resp. $\sum_{q \in \mathcal{P}_{min}(Z(M))} m(q) t_\phi(q) = 1$).

If $N \subset M$ is a unital C^*-subalgebra of M, the associated *inclusion matrix* Λ is the matrix with rows and columns indexed by $\mathcal{P}_{min}(Z(N))$ and $\mathcal{P}_{min}(Z(M))$ respectively, defined by setting $\Lambda_{pq} = \sqrt{\frac{dim\ \ qpMqp}{dim\ \ qpNqp}}$. Alternatively, if we write ρ_p for the irreducible representation of N corresponding to p, then Λ_{pq} is nothing but the 'multiplicity with which ρ_p occurs in the irreducible decomposition of $\pi_q|_N$'. This data is sometimes also recorded in a bipartite graph (usually called the *Bratteli diagram* of the inclusion) with even and odd vertices indexed by $\mathcal{P}_{min}(Z(N)$ and $\mathcal{P}_{min}(Z(M))$ repectively, with Λ_{pq} edges joining the vertices indexed by p and q.

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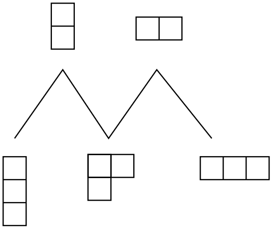
For example, Bratteli diagram associated to $\mathbb{C}S_2 \subset \mathbb{C}S_3$[4] is seen to be given by:



[4] For a finite group G, clearly $C^*_{red}(G) = \mathbb{C}G$.

Propostion (bc): Suppose $N \subset M$ is a unital inclusion of finite dimensional C^*- algebras. Let tr be a faithful, unital, positive trace on M. Then,

1. The C^* algebra generated by M and e_N in $\mathcal{L}(L^2(M, tr))$ is $\rho_M(N)'$.
2. The central support[5] of e_N in $\rho_M(N)'$ is 1.
3. $e_N x e_N = E(x) e_N$ for $x \in M$. (As usual, we identify m with $\lambda_M(m)$.)
4. $N = M \cap \{e_N\}'$.
5. If Λ is the inclusion matrix for $N \subset M$ then Λ^t is the inclusion matrix for $M \subset \rho_M(N)'$.

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This *basic construction* - i.e., the passage from $N \subset M$ to $M \subset \rho_M(N)'$ - extends almost verbatim from inclusions of finite-dimensional C^*-algebras to one good infinite-dimensional case, that of the so-called *finite-depth subfactors* which we shall discuss in the next lecture!

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We now proceed to infinite dimensions.

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Definition 1: A von Neumann algebra is the commutant of a unitary group representation (say π of G): i.e.,

$$M = \{x \in \mathcal{L}(\mathcal{H}) : x\pi(g) = \pi(g)x \ \forall g \in G\}$$

Note that $\mathcal{L}(\mathcal{H})$ is a C^*-algebra w.r.t. the 'operator norm' $\|x\| = \sup\{\|x\xi\| : \xi \in \mathcal{H}, \|\xi\| = 1\}$ and 'Hilbert space adjoint'.

Definitions: (a) $S' = \{x' \in \mathcal{L}(\mathcal{H}) : xx' = x'x \ \forall x \in S\}$, for $S \subset \mathcal{L}(\mathcal{H})$

(b) SOT on $\mathcal{L}(\mathcal{H})$: $x_n \to x \Leftrightarrow \|x_n\xi - x\xi\| \to 0 \ \forall \xi$ (i.e., $x_n\xi \to x\xi$ strongly $\forall \xi$)

(c) WOT on $\mathcal{L}(\mathcal{H})$: $x_n \to x \Leftrightarrow \langle x_n\xi - x\xi, \eta\rangle \to 0 \forall \xi, \eta$ (i.e., $x_n\xi \to x\xi$ weakly $\forall \xi$)

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von Neumann's double commutant theorem (DCT):

Let M be a unital self-adjoint subalgebra of $\mathcal{L}(\mathcal{H})$. TFAE:

(i) M is SOT-closed

(ii) M is WOT-closed

(iii) $M = M'' = (M')'$ $\square$

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A von Neumann algebra is closed under all 'canonical constructions': for instance, if $x \rightarrow \{1_E(x) : E \in \mathcal{B}_{\mathbb{C}}\}$ is the spectral measure associated with a normal operator x, then $x \in M \Leftrightarrow 1_E(x) \in M \ \forall \ E \in \mathcal{B}_{\mathbb{C}}$.

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(*Reason:* $\Rightarrow$: Since $1_E(uxu^*) = u1_E(x)u^*$ for all unitary u (the spectral measure is a canonical construction),

$$\begin{aligned} x \in M, u' \in \mathcal{U}(M') \quad &\Rightarrow \quad u'1_E(x)u'^* = 1_E(u'xu'^*) \\ &\Rightarrow \quad 1_E(x) \in \left(\mathcal{U}(M')\right)' = M \quad) \end{aligned}$$

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$\Leftarrow$: Uniform approximability of bounded measurable functions implies (by the spectral theorem) that

$$M = [\mathcal{P}(M)] = (span\ \mathcal{P}(M))^- \quad (*),$$

where $\mathcal{P}(M) = \{p \in M : p = p^2 = p^*\}$ is the set of projections in M.

Suppose $M = \pi(G)'$ as before. Then

$$p \leftrightarrow ran\ p$$

establishes a bijection

$$\mathcal{P}(M) \leftrightarrow G\text{-stable subspaces}$$

So, for instance, eqn. (*) shows that

$$(\pi(G))'' = \mathcal{L}(\mathcal{H}) \Leftrightarrow M = \mathbb{C} \Leftrightarrow \pi \text{ is irreducible}$$

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Under the correspondence, of sub-reps of π to $\mathcal{P}(M)$, (unitary) equivalence of sub-repreps of π translates to **Murray-von Neumann equivalence** on $\mathcal{P}(M)$:

$$p \sim_M q \Leftrightarrow \exists u \in M \text{ such that } u^*u = p,\ uu^* = q$$

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More generally, define

$$p \preceq_M q \Leftrightarrow \exists p_0 \in \mathcal{P}(M) \text{ such that } p \sim_M p_0 \leq q$$

Proposition:

The following conditions are equivalent:

1. Either $p \preceq_M q$ or $q \preceq_M p, \ \forall p, q \in \mathcal{P}(M)$.
2. M has trivial center: $Z(M) = M \cap M' = \mathbb{C}$

Such an M is called a **factor**. □

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If $M = \pi(G)'$, with G finite, then M is a factor iff π is isotypical.

In general, any von Neumann algebra is a **direct integral** of factors.

Call a projection $p \in \mathcal{P}(M)$ **infinite rel** M if $\exists p_0 \neq p \in \mathcal{P}(M)$ such that $p \sim_M p_0 \leq p$; otherwise, call p **finite** (rel M).

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Murray von-Neumann classification of factors

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A factor M is said to be of:

1. **type** *I* if there is a minimal non-zero projection in M.
2. **type** *II* if it contains non-zero finite projections, but no minimal non-zero projection.
3. **type** *III* if it contains no non-zero finite projection.

Definition 3: M is a von Neumann algebra if

- M is a C^*-algebra (i.e., a Banach *-algebra satisfying $\|x^*x\| = \|x\|^2 \; \forall \; x$)
- M is a dual Banach space: i.e., $\exists$ a Banach space M_* such that $M \cong M_*^*$ as a Banach space.

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Example: $M = L^\infty(\Omega, \mathcal{B}, \mu)$. Can also view it as acting on $L^2(\Omega, \mathcal{B}, \mu)$ as multiplication operators. (In fact, every commutative von Neumann algebra is isomorphic to an $L^\infty(\Omega, \mathcal{B}, \mu)$.)

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Fact: The predual M_* of M is unique up to isometric isomorphism. So, - by Alaoglu - $\exists$ a canonical locally convex (weak-*) top. on M w.r.t. which the unit ball of M is compact. This is called **the σ-weak topology** on M.

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The algebra $\mathcal{L}(\mathcal{H})$, for any Hilbert space $\mathcal{H}$, is a von Neumann algebra - with pre-dual being the space $\mathcal{L}_*(\mathcal{H})$ of trace-class operators.

Any σ-weakly closed $*$-subalgebra of a von Neumann algebra is a von Neumann algebra, i.e., it is a von Neumann subalgebra.

Gelfand-Naimark theorem: Any von Neumann algebra is isomorphic to a vN-subalgebra of some $\mathcal{L}(\mathcal{H})$. (So the abstract and concrete ($=$ tied down to Hilbert space) definitions are equivalent.)

We give a brief idea of the proof of the Gelfand Naimark theorem in the von Neumann algebra context. We assume, for simplicity, that all our von Neumann algebras have separable preduals. It is a fact that such an M admits a faithful normal state ϕ on M.

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For such a ϕ, we construct the standard form of M. As in the finite-dimensional case, let us write $\hat{M} = \{\hat{x} : x \in M\}$. The hypothesis on ϕ guaranteess that the equation $\langle \hat{x}, \hat{y} \rangle = \phi(y^* x)$ defines a genuine inner product on $\hat{M}$. Let $L^2(M, \phi)$ denote the Hilbert space completion of $\hat{M}$. A little C^* trickery shows that the mapping $\hat{y} \mapsto \widehat{xy}$ extends to a (necessarily) unique bounded operator $\lambda_M(x)$ on $L^2(M, \phi)$. It is fairly routine to then verify that λ_M is a normal isomorphism onto its image.

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The same trickery shows why there is a difficulty in establishing a similar assertion regarding ρ_M and why things go through smoothly when ϕ is a trace - which situation is what we will be addressing the next two lectures. The full story of how one makes do with non-tracial states involves the celebrated and technically slightly complicted **Tomita Takesaki theory**, which we shall say nothing more about.

References

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