Operator algebras - stage for non-commutativity
(Panorama Lectures Series)
II. K-theory for C*-algebras

V.S. Sunder
Institute of Mathematical Sciences
Chennai, India
sunder@imsc.res.in

IISc, January 28, 2009
We begin with a brief re-cap of classical *topological K-theory*, which studies the classification of vector bundles up to so-called *stable equivalence*.
Vector bundles

We begin with a brief re-cap of classical topological $K$-theory, which studies the classification of vector bundles up to so-called stable equivalence.

By a vector bundle of rank $n$ on a compact Hausdorff space $X$ is meant an ordered pair $(E, p)$ consisting of a topological space $E$ and a continuous map $p : E \to X$, which satisfy some requirements which say loosely that:

- for each $x \in X$, the fibre $E_x = \pi^{-1}(x)$ over $x$ has the structure of a vector space of dimension $n$
- the fibres are all ‘tied together in a continuous manner’, the precise formulation being referred to as local triviality.
We begin with a brief re-cap of classical *topological K-theory*, which studies the classification of vector bundles up to so-called *stable equivalence*.

By a **vector bundle of rank** \( n \) on a compact Hausdorff space \( X \) is meant an ordered pair \((E, p)\) consisting of a topological space \( E \) and a continuous map \( p : E \to X \), which satisfy some requirements which say loosely that:

- for each \( x \in X \), the fibre \( E_x = \pi^{-1}(x) \) over \( x \) has the structure of a vector space of dimension \( n \)
- the fibres are all ‘tied together in a continuous manner’, the precise formulation being referred to as *local triviality*.

The prime examples are the tangent bundle \( TM \) and the cotangent bundle \( TM^* \) over a compact manifold. For example,

\[
TS^{n-1} = \{(x, v) \in S^{n-1} \times \mathbb{R}^n : x \cdot v = 0\}
\]
We begin with a brief re-cap of classical topological $K$-theory, which studies the classification of vector bundles up to so-called stable equivalence.

By a **vector bundle of rank** $n$ on a compact Hausdorff space $X$ is meant an ordered pair $(E, p)$ consisting of a topological space $E$ and a continuous map $p : E \to X$, which satisfy some requirements which say loosely that:

- for each $x \in X$, the fibre $E_x = \pi^{-1}(x)$ over $x$ has the structure of a vector space of dimension $n$
- the fibres are all ‘tied together in a continuous manner’, the precise formulation being referred to as local triviality.

The prime examples are the tangent bundle $TM$ and the cotangent bundle $TM^*$ over a compact manifold. For example,

$$TS^{n-1} = \{(x, v) \in S^{n-1} \times \mathbb{R}^n : x \cdot v = 0\}$$

We will, however, be concerned primarily with complex vector bundles here.
If \((E, p)\) is a vector bundle on \(X\), a \textit{section} of \(E\) is a continuous function \(s : X \rightarrow E\) such that \(s(x) \in E_x \forall x \in X\). The set \(\Gamma(E)\) of sections of \(E\) is naturally a vector space - with \[
(\alpha s + \beta t)(x) = \alpha s(x) + \beta t(x),
\]
and with the linear combination on the right interpreted in the vector space \(E_x\). In fact, \(\Gamma(E)\) is naturally a \textit{module} over \(C(X)\) - with \[
(f \cdot s)(x) = f(x)s(x).
\]
If \((E, p)\) is a vector bundle on \(X\), a **section** of \(E\) is a continuous function \(s : X \to E\) such that \(s(x) \in E_x \quad \forall x \in X\). The set \(\Gamma(E)\) of sections of \(E\) is naturally a vector space - with

\[
(\alpha s + \beta t)(x) = \alpha s(x) + \beta t(x),
\]

and with the linear combination on the right interpreted in the vector space \(E_x\). In fact, \(\Gamma(E)\) is naturally a **module** over \(C(X)\) - with

\[
(f \cdot s)(x) = f(x)s(x).
\]

**Theorem:** *(Serre-Swan theorem: )*  
If \((E, p)\) is a vector bundle over a compact Hausdorff space \(X\), then \(\Gamma(X)\) is a **finitely generated projective module** over \(C(X)\) (i.e., there exist finitely many elements \(s_1, \ldots, s_n \in \Gamma(X)\) such that \(\Gamma(E) = \sum_{i=1}^{n} C(X) \cdot s_i\)).
If \((E, p)\) is a vector bundle on \(X\), a \textit{section} of \(E\) is a continuous function \(s : X \to E\) such that \(s(x) \in E_x \forall x \in X\). The set \(\Gamma(E)\) of sections of \(E\) is naturally a vector space - with

\[(\alpha s + \beta t)(x) = \alpha s(x) + \beta t(x),\]

and with the linear combination on the right interpreted in the vector space \(E_x\).

In fact, \(\Gamma(E)\) is naturally a \textit{module} over \(C(X)\) - with

\[(f \cdot s)(x) = f(x)s(x).\]

\textbf{Theorem: (Serre-Swan theorem: )}

If \((E, p)\) is a vector bundle over a compact Hausdorff space \(X\), then \(\Gamma(X)\) is a \textit{finitely generated projective module} over \(C(X)\) (i.e., there exist finitely many elements \(s_1, \ldots, s_n \in \Gamma(X)\) such that \(\Gamma(E) = \sum_{i=1}^{n} C(X) \cdot s_i)\).

Further, \textbf{every} finitely generated projective module over \(C(X)\) is of this form.
Notice next that if $A$ is any unital $C^*$-algebra, so is $M_n(A)$ (in a natural way); the algebraic operations are the natural ones, while the norm may be obtained thus: if $A \hookrightarrow \mathcal{L}(\mathcal{H})$, then $M_n(A) \hookrightarrow M_n(\mathcal{L}(\mathcal{H})) \cong \mathcal{L}(\mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H})$. We shall identify $M_n(A)$ with the ‘northwest corner’ of $M_{n+1}(A)$ via $x \sim \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$.
Notice next that if $A$ is any unital $C^*$-algebra, so is $M_n(A)$ (in a natural way); the algebraic operations are the natural ones, while the norm may be obtained thus: if $A \hookrightarrow \mathcal{L}(\mathcal{H})$, then $M_n(A) \hookrightarrow M_n(\mathcal{L}(\mathcal{H})) \cong \mathcal{L}(\mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus n \text{ terms} \mathcal{H})$. We shall identify $M_n(A)$ with the ‘northwest corner’ of $M_{n+1}(A)$ via $x \sim \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$.

Write $\mathcal{P}_n(A) = \mathcal{P}(M_n(A))$, and $\mathcal{U}_n(A) = \mathcal{U}(M_n(A))$ where $\mathcal{P}(B)$ (resp., $\mathcal{U}(B)$) denotes the set $\{ p \in B : p = p^2 = p^* \}$, (resp., $\{ u \in B : u^* u = uu^* = 1 \}$) of projections (resp., unitary elements) in any $C^*$-algebra $B$. 
Notice next that if $A$ is any unital $C^*$-algebra, so is $M_n(A)$ (in a natural way); the algebraic operations are the natural ones, while the norm may be obtained thus: if $A \hookrightarrow \mathcal{L}(\mathcal{H})$, then $M_n(A) \hookrightarrow M_n(\mathcal{L}(\mathcal{H})) \cong \mathcal{L}(\mathcal{H} \oplus \mathcal{H} \oplus^{n \text{ terms}} \mathcal{H})$. We shall identify $M_n(A)$ with the ‘northwest corner’ of $M_{n+1}(A)$ via $x \sim \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$.

Write $P_n(A) = P(M_n(A))$, and $U_n(A) = U(M_n(A))$ where $P(B)$ (resp., $U(B)$) denotes the set $\{ p \in B : p = p^2 = p^* \}$, (resp., $\{ u \in B : u^* u = uu^* = 1 \}$) of projections (resp., unitary elements) in any $C^*$-algebra $B$.

Regard $P_n(A)$ (resp., $U_n(A)$) as being included in $P_{n+1}(A)$ (resp., $U_{n+1}(A)$) via the identification

$$P_n(A) \ni p \sim \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \in P_{n+1}(A),$$

(resp. $u \sim \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}$) and write $P_\infty(A)$, $M_\infty(A)$ and $U_\infty(A)$ for the indicated increasing union.
A finitely generated projective module over $A$ is of the form

$$V_p = \{ \xi \in M_{1 \times n}(A) : \xi = \xi p \},$$

for some $p \in \mathcal{P}_n(A)$, and some positive integer $n$ - where of course the $A$ action on $V_p$ is given by

$$(a \cdot \xi)_i = a\xi_i.$$
Towards defining $K_0(A)$

A finitely generated projective module over $A$ is of the form

$$V_p = \{ \xi \in M_{1 \times n}(A) : \xi = \xi p \},$$

for some $p \in P_n(A)$, and some positive integer $n$ - where of course the $A$ action on $V_p$ is given by

$$(a \cdot \xi)_i = a \xi_i.$$

It is not hard to see that if $p, q \in P_\infty(A)$, then a linear map $x : V_p \to V_q$ is $A$-linear if and only if there exists a matrix $X = ((x_{ij})) \in M_\infty(A)$ such that

$$x \cdot v = v \cdot X \text{ and } X = pxq$$

where we think of elements of $V_p$ and $V_q$ as row vectors. (This assertion is an instance of the thesis 'what commutes with all left-multiplications must be a right-multiplication', many instances of which we will keep running into.) In particular, modules $V_p$ and $V_q$ are isomorphic iff there exists a $u \in M_\infty(A)$ such that $u^* u = p$ and $uu^* = q$; write $p \sim q$ when this happens.
Towards defining $K_0(A)$

A finitely generated projective module over $A$ is of the form

$$V_p = \{ \xi \in M_{1 \times n}(A) : \xi = \xi p \},$$

for some $p \in P_n(A)$, and some positive integer $n$ - where of course the $A$ action on $V_p$ is given by

$$(a \cdot \xi)_i = a\xi_i.$$

It is not hard to see that if $p, q \in P_\infty(A)$, then a linear map $x : V_p \to V_q$ is $A$-linear if and only if there exists a matrix $X = ((x_{ij})) \in M_\infty(A)$ such that

$$x \cdot v = v \cdot X \quad \text{and} \quad X = pxq$$

where we think of elements of $V_p$ and $V_q$ as row vectors. (This assertion is an instance of the thesis ‘what commutes with all left-multiplications must be a right-multiplication’, many instances of which we will keep running into.) In particular, modules $V_p$ and $V_q$ are isomorphic iff there exists a $u \in M_\infty(A)$ such that $u^*u = p$ and $uu^* = q$; write $p \sim q$ when this happens.

**Proposition** The set $K_0(A) = P_\infty(A)/\sim$ is an abelian monoid (=semigroup with identity) with respect to addition defined by

$$[p] + [q] = [p \oplus q],$$

the identity element being $[0]$. 
If $S$ is an abelian semigroup, the set $\{a - b : a, b \in S\}$ of formal differences in $S$ - with the convention that $a - b = c - d$ iff $a + d + f = c + b + f$ for some $f \in S$ - turns out to be an abelian group, called the **Grothendieck group** of $S$. 
If $S$ is an abelian semigroup, the set $\{a - b : a, b \in S\}$ of formal differences in $S$ - with the convention that $a - b = c - d$ iff $a + d + f = c + b + f$ for some $f \in S$ - turns out to be an abelian group, called the Grothendieck group of $S$.

**Definition:** If $A$ is a unital $C^*$-algebra, then

(i) $K_0(A)$ is defined to be the Grothendieck group of $K_0(A)$:

(ii) $K_1(A)$ is defined to be the quotient of the group $\mathcal{U}_\infty(A)$ by the normal subgroup $\mathcal{U}_\infty(A)^{(0)}$ (defined by the connected component of its identity element).
It turns out that $K_1(A)$ is also an abelian group, with the group law being given in two equivalent ways, thus: if $u \in \mathcal{U}_m(A), \ v \in \mathcal{U}_k(A)$, then

$$[uv] = [u][v] = \left[ \begin{array}{cc} u & 0 \\ 0 & 1_k \end{array} \right] \left[ \begin{array}{cc} 1_m & 0 \\ 0 & v \end{array} \right] = [u \oplus v]$$

where we write $1_\ell$ for the identity in $M_\ell(A)$.
It turns out that $K_1(A)$ is also an abelian group, with the group law being given in two equivalent ways, thus: if $u \in \mathcal{U}_m(A), v \in \mathcal{U}_k(A)$, then

$$[uv] = [u][v] = \left[ \begin{array}{cc} u & 0 \\ 0 & 1_k \end{array} \right] \left[ \begin{array}{cc} 1_m & 0 \\ 0 & v \end{array} \right] = [u \oplus v]$$

where we write $1_\ell$ for the identity in $M_\ell(A)$.

Some fundamental properties of the $K$-groups, which we shall briefly discuss below, are:

- Functoriality
- Normalisation
- Stability
- Homotopy invariance
**Functoriality:** $K_i, i = 0, 1$ define covariant functors from the category of $C^*$-algebras to abelian groups; i.e., if $\phi \in \text{Hom}(A, B)$ is a morphism of $C^*$-algebras, there exist group homomorphisms $K_i(\phi) = \phi_* : K_i(A) \to K_i(B)$ satisfying the usual functoriality requirements - of being well-behaved with respect to compositions and identity morphisms: i.e.,

$$K_i(\phi \circ \psi) = K_i(\phi) \circ K_i(\psi), \ K_i(id_A) = id_{K_i(A)}.$$
Some basic properties (contd.)

**Functoriality:** $K_i, i = 0, 1$ define covariant functors from the category of $C^*$-algebras to abelian groups; i.e., if $\phi \in \text{Hom}(A, B)$ is a morphism of $C^*$-algebras, there exist group homomorphisms $K_i(\phi) = \phi_* : K_i(A) \rightarrow K_i(B)$ satisfying the usual functoriality requirements - of being well-behaved with respect to compositions and identity morphisms: i.e.,

$$K_i(\phi \circ \psi) = K_i(\phi) \circ K_i(\psi), \ K_i(id_A) = id_{K_i(A)}.$$

**Normalisation:**

$$K_0(\mathbb{C}) = \mathbb{Z}, \ K_1(\mathbb{C}) = \{0\}.$$
**Functoriality:** $K_i, i = 0, 1$ define covariant functors from the category of $C^*$-algebras to abelian groups; i.e., if $\phi \in Hom(A, B)$ is a morphism of $C^*$-algebras, there exist group homomorphisms $K_i(\phi) = \phi_* : K_i(A) \rightarrow K_i(B)$ satisfying the usual functoriality requirements - of being well-behaved with respect to compositions and identity morphisms: i.e.,

$$K_i(\phi \circ \psi) = K_i(\phi) \circ K_i(\psi), \ K_i(id_A) = id_{K_i(A)}.$$

**Normalisation:**

$$K_0(\mathbb{C}) = \mathbb{Z}, \ K_1(\mathbb{C}) = \{0\}.$$

**Stability:** If $\phi : A \rightarrow M_n(A)$ is defined by $\phi(a) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, then $\phi_*$ is an isomorphism.
Functoriality: $K_i, i = 0, 1$ define covariant functors from the category of $C^*$-algebras to abelian groups; i.e., if $\phi \in \text{Hom}(A, B)$ is a morphism of $C^*$-algebras, there exist group homomorphisms $K_i(\phi) = \phi_* : K_i(A) \to K_i(B)$ satisfying the usual functoriality requirements - of being well-behaved with respect to compositions and identity morphisms: i.e.,

$$K_i(\phi \circ \psi) = K_i(\phi) \circ K_i(\psi), \ K_i(id_A) = id_{K_i(A)}.$$ 

Normalisation:

$$K_0(\mathbb{C}) = \mathbb{Z}, \ K_1(\mathbb{C}) = \{0\}.$$  

Stability: If $\phi : A \to M_n(A)$ is defined by $\phi(a) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, then $\phi_*$ is an isomorphism.

Homotopy invariance: If $\{\phi_t : t \in [0, 1]\}$ is a continuously varying family of homomorphisms from $A$ into $B$ (or equivalently, if there exists a homomorphism $A \ni a \mapsto (t \mapsto \phi_t(a)) \in C([0, 1], B)$), then $(\phi_0)_* = (\phi_1)_*$. 
**Use of homotopy of invariance**

*Example:* If $X$ is a contractible space, then $K_i(C(X)) = K_i(\mathbb{C})$. 
**Example:** If \( X \) is a contractible space, then \( K_i(C(X)) = K_i(\mathbb{C}) \).

**Proof:** Let \( \{h_t : t \in [0, 1]\} \) be a homotopy with \( h_1 = id_X \) and \( h_0(x) = x_0 \in X \quad \forall x \in X \). Consider \( \phi_t(= h_t^*) : C(X) \to C(X) \) defined by \( \phi_t(f) = f \circ h_t \). Then \( \phi_1 = id_{C(X)} \) while \( \phi_0(f) \) is the constant function identically equal to \( f(x_0) \). So, if \( j \) denotes the inclusion map \( j : \mathbb{C} \to C(X) \), and if we write \( f(x_0) = ev_0(f) \), we have commutative diagrams of maps:

\[
\begin{array}{ccc}
C(X) & \xrightarrow{\phi_0} & C(X) \\
\downarrow ev_0 & & \downarrow ev_0 \\
\mathbb{C} & \xrightarrow{id} & \mathbb{C}
\end{array}
\]

and

\[
\begin{array}{ccc}
K_i(C(X)) & \xrightarrow{(\phi_0)^*} & K_i(C(X)) \\
(\downarrow (ev_0)^*) & & \downarrow (ev_0)^* \\
K_i(\mathbb{C}) & \xrightarrow{id} & K_i(\mathbb{C})
\end{array}
\]

Since \( \phi_0^* = \phi_i^* = id^* \), the second diagram shows that \( j^* \) is an isomorphism with inverse \( ev_0^* \).
Before proceeding further, we need to discuss non-unital $C^*$-algebras. (This corresponds to studying vector bundles over locally compact non-compact spaces.) If $A$ is any $C^*$-algebra - with or without identity - then $\tilde{A} = A \times \mathbb{C}$ becomes a unital $C^*$-algebra thus:

$$(x, \lambda) \cdot (y, \mu) = (xy + \lambda y + \mu x, \lambda \mu)$$

$$\|(x, \lambda)\| = \sup \{ \|xa + \lambda a\| : a \in A, \|a\| = 1 \}$$

(Addition and involution are componentwise, and $(0, 1)$ is the identity.) Further $\epsilon : \tilde{A} \to \mathbb{C}$ defined by $\epsilon(x, \lambda) = \lambda$ is a homomorphism of unital $C^*$-algebras, with $\ker(\epsilon) = A$; thus $A$ is an ideal of co-dimension 1 in $\tilde{A}$. 

---

*V.S. Sunder IMSc, Chennai*  
*Operator algebras - stage for non-commutativity (Panorama Lectures Series)*
Non-unital $C^*$-algebras

Before proceeding further, we need to discuss non-unital $C^*$-algebras. (This corresponds to studying vector bundles over locally compact non-compact spaces.) If $A$ is any $C^*$-algebra - with or without identity - then $\tilde{A} = A \times \mathbb{C}$ becomes a unital $C^*$-algebra thus:

$$(x, \lambda) \cdot (y, \mu) = (xy + \lambda y + \mu x, \lambda \mu)$$

$$\|(x, \lambda)\| = \sup\{\|xa + \lambda a\| : a \in A, \|a\| = 1\}$$

(Addition and involution are componentwise, and $(0, 1)$ is the identity.) Further $\epsilon : \tilde{A} \to \mathbb{C}$ defined by $\epsilon(x, \lambda) = \lambda$ is a homomorphism of unital $C^*$-algebras, with $\ker(\epsilon) = A$; thus $A$ is an ideal of co-dimension 1 in $\tilde{A}$.

Example: In case $A = C_0(X)$ is the algebra of continuous functions on a locally compact space $X$ which ‘vanish at infinity’, the ‘unitisation’ $\tilde{A}$ can be identified with $C(\hat{X})$, where $\hat{X} = (X \cup \{\infty\})$ is the one-point compactification of $X$, and $\epsilon(f) = f(\infty)$.)
Non-unital $C^*$-algebras

Before proceeding further, we need to discuss non-unital $C^*$-algebras. (This corresponds to studying vector bundles over locally compact non-compact spaces.) If $A$ is any $C^*$-algebra - with or without identity - then $\tilde{A} = A \times \mathbb{C}$ becomes a unital $C^*$-algebra thus:

$$(x, \lambda) \cdot (y, \mu) = (xy + \lambda y + \mu x, \lambda \mu)$$

$$\| (x, \lambda) \| = \sup \{ \| xa + \lambda a \| : a \in A, \| a \| = 1 \}$$

(Addition and involution are componentwise, and $(0, 1)$ is the identity.) Further $\epsilon : \tilde{A} \to \mathbb{C}$ defined by $\epsilon(x, \lambda) = \lambda$ is a homomorphism of unital $C^*$-algebras, with $\ker(\epsilon) = A$; thus $A$ is an ideal of co-dimension 1 in $\tilde{A}$.

**Example:** In case $A = C_0(X)$ is the algebra of continuous functions on a locally compact space $X$ which ‘vanish at infinity’, the ‘unitisation’ $\tilde{A}$ can be identified with $C(\hat{X})$, where $\hat{X} = (X \cup \{\infty\})$ is the one-point compactification of $X$, and $\epsilon(f) = f(\infty)$.

For a possibly non-unital $A$, define

$$K_i(A) = \ker K_i(\epsilon).$$
Six term exact sequence: If

\[ 0 \rightarrow J \xrightarrow{j} A \xrightarrow{\pi} B \rightarrow 0 \]

is a short exact sequence of $C^*$-algebras, then there exists an associated six term exact sequence of $K$-groups

\[
\begin{array}{cccc}
K_0(J) & j_* & K_0(A) & \pi_* & K_0(B) \\
\partial_1 & \uparrow & & \downarrow \partial_0 & \\
K_1(B) & \pi_* & K_1(A) & i_* & K_1(J)
\end{array}
\]

where the two connecting homomorphisms $\partial_i$ are ‘natural’.

It is worth noting the special case when the short exact sequence splits - i.e., when there exists a $\ast$-homomorphism $s: B \rightarrow A$ such that $\pi \circ s = \text{id}_B$; in this case, also $\pi_*$ is surjective, whence both connecting maps must be the zero maps, so the six-term sequence above splits into two short exact sequences

\[ 0 \rightarrow K_i(J) \xrightarrow{j_*} K_i(A) \xrightarrow{\pi_*} K_i(B) \rightarrow 0 \]
Six term exact sequence: If

$$0 \to J \overset{j}{\to} A \overset{\pi}{\to} B \to 0$$

is a short exact sequence of $C^*$-algebras, then there exists an associated six term exact sequence of $K$-groups

$$\begin{array}{cccc}
K_0(J) & \overset{j_*}{\to} & K_0(A) & \overset{\pi_*}{\to} & K_0(B) \\
\partial_1 & \uparrow & & \downarrow \partial_0 \\
K_1(B) & \overset{\pi_*}{\leftarrow} & K_1(A) & \overset{j_*}{\leftarrow} & K_1(J)
\end{array}$$

where the two connecting homomorphisms $\partial_i$ are ‘natural’.

It is worth noting the special case when the short exact sequence splits - i.e., when there exists a $*$-homomorphism $s : B \to A$ such that $\pi \circ s = id_B$; in this case, also $\pi_*$ is surjective, whence both connecting maps must be the zero maps, so the six-term sequence above splits into two short exact sequences

$$0 \to K_i(J) \overset{j_*}{\to} K_i(A) \overset{\pi_*}{\to} K_i(B) \to 0$$
Example: Consider the short exact sequence

\[ 0 \rightarrow C_0((0, 1]) \xrightarrow{j} C([0, 1]) \xrightarrow{ev_0} \mathbb{C} \rightarrow 0 \]

Since \( K_i(ev_0) : K_i(C([0, 1])) \cong K_i(\mathbb{C}) \) it follows from the six term exact sequence that

\[ K_i(C_0((0, 1])) = 0. \]
**Example:** Consider the short exact sequence

\[ 0 \to C_0((0,1]) \xrightarrow{j} C([0,1]) \xrightarrow{ev_0} \mathbb{C} \to 0 \]

Since \( K_i(ev_0) : K_i(C([0,1])) \cong K_i(\mathbb{C}) \) it follows from the six term exact sequence that

\[ K_i(C_0((0,1])) = 0. \]

Next, the six term exact sequence for the short exact sequence

\[ 0 \to C_0((0,1]) \xrightarrow{j} C((0,1]) \xrightarrow{ev_1} \mathbb{C} \to 0 \]

is seen to be

\[
\begin{array}{cccccc}
K_0(C_0((0,1])) & j_* & 0 & (ev_1)_* & K_0(\mathbb{C}) \\
\partial_1 \uparrow & & & \downarrow \partial_0 & \\
K_1(\mathbb{C}) & (ev_1)_* & 0 & j_* & K_1(C_0((0,1]))
\end{array}
\]

so \( K_i(C_0(\mathbb{R})) \cong K_i(C_0((0,1])) = K_{i+1}(\mathbb{C}) \mod 2. \)
**Example:** Consider the short exact sequence

\[ 0 \to C_0((0, 1)) \xrightarrow{j} C([0, 1]) \xrightarrow{ev_0} \mathbb{C} \to 0 \]

Since \( K_i(ev_0) : K_i(C([0, 1])) \cong K_i(\mathbb{C}) \) it follows from the six term exact sequence that

\[ K_i(C_0((0, 1))) = 0. \]

Next, the six term exact sequence for the short exact sequence

\[ 0 \to C_0((0, 1)) \xrightarrow{j} C((0, 1]) \xrightarrow{ev_1} \mathbb{C} \to 0 \]

is seen to be

\[
\begin{array}{cccc}
K_0(C_0((0, 1))) & \xrightarrow{j_*} & 0 & \xrightarrow{(ev_1)_*} \\
\partial_1 & \uparrow & \downarrow & \\
K_1(\mathbb{C}) & \xleftarrow{(ev_1)_*} & 0 & \xleftarrow{j_*} \\
\end{array}
\]

so \( K_i(C_0(\mathbb{R})) \cong K_i(C_0((0, 1))) = K_{i+1}(\mathbb{C}) \) (mod 2).

Similar reasoning, applied to \( C_0(\mathbb{R}; A) \), essentially yields the **Bott periodicity theorem:**

\[ K_i(C_0(\mathbb{R}; A)) = K_{i+1}(A) \mod 2. \]
The $K$ groups for spheres

Applied inductively to $A = C_0(\mathbb{R}^n)$, we conclude that

$$K_i(C_0(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z} & \text{if } (n - i) \text{ is even} \\ 0 & \text{otherwise} \end{cases}.$$
The $K$ groups for spheres

Applied inductively to $A = C_0(\mathbb{R}^n)$, we conclude that

$$K_i(C_0(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z} & \text{if } (n - i) \text{ is even} \\ 0 & \text{otherwise} \end{cases}.$$  

The short exact sequence

$$0 \to C_0(\mathbb{R}^n) \xrightarrow{j} C(S^n) \xrightarrow{ev} \mathbb{C} \to 0$$

is split by the inclusion morphism $\eta : \mathbb{C} \to C(S^n)$, so that we have a short exact sequence

$$0 \to K_i(C_0(\mathbb{R}^n)) \xrightarrow{j_*} K_i(C(S^n)) \xrightarrow{\pi_*} K_i(\mathbb{C}) \to 0$$

which also splits and we may deduce that

$$K_i(C(S^n)) \cong K_i(C_0(\mathbb{R}^n)) \oplus K_i(\mathbb{C}).$$
The virtue of traces

The simplest non-abelian $C^*$-algebras are the $M_n(\mathbb{C})$’s, and we may conclude from the ‘stability’ of $K$-groups that

$$K_i(M_n(\mathbb{C})) \cong K_i(\mathbb{C}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n = 1 \end{cases}.$$
The virtue of traces

The simplest non-abelian $C^*$-algebras are the $M_n(\mathbb{C})$'s, and we may conclude from the ‘stability’ of $K$-groups that

$$K_i(M_n(\mathbb{C})) \cong K_i(\mathbb{C}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n = 1 \end{cases}.$$ 

We shall give another proof that $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$. Consider the map $	au : M_\infty(M_n(\mathbb{C})) \to \mathbb{C}$ by

$$\tau((x_{ij})) = \sum_i Tr(x_{ii}),$$

where $Tr$ denotes the usual trace (= sum of diagonal entries) on the matrix algebra.
The virtue of traces

The simplest non-abelian $C^*$-algebras are the $M_n(\mathbb{C})$’s, and we may conclude from the ‘stability’ of $K$-groups that

$$K_i(M_n(\mathbb{C})) \cong K_i(\mathbb{C}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n = 1 \end{cases}.$$ 

We shall give another proof that $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$. Consider the map

$$\tau : M_\infty(M_n(\mathbb{C})) \to \mathbb{C}$$

by

$$\tau((x_{ij})) = \sum_i Tr(x_{ii}),$$

where $Tr$ denotes the usual trace (≡ sum of diagonal entries) on the matrix algebra.

Then $\tau$ is seen to be a positive ($\tau(X^*X) \geq 0 \ \forall X$) faithful (i.e., $X \neq 0 \Rightarrow \tau(X^*X) > 0$) and tracial ($\tau(XY) = \tau(YX)$) linear functional. Further $\tau$ ‘respects the inclusion of $M_k(M_n(\mathbb{C}))$ into $M_{k+1}(M_n(\mathbb{C}))$ in the sense that

$$\tau(X) = \tau\left( \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \right).$$
The fact that $\tau$ is a trace implies that the equation
\[\tilde{\tau}([p]) = \tau(p)\]
gives a well defined map $\tilde{\tau} : K_0(M_n(\mathbb{C})) \to \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.
The fact that $\tau$ is a trace implies that the equation

$$\tilde{\tau}([p]) = \tau(p)$$

gives a well defined map $\tilde{\tau} : K_0(M_n(\mathbb{C})) \to \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

The fact that $\tau$ is faithful implies that $\tilde{\tau}$ is an isomorphism of monoids; and since the Grothendieck group of $\mathbb{Z}_+$ is just $\mathbb{Z}$, it follows that $\tilde{\tau}$ gives rise to a unique isomorphism $\tau_# : K_0(M_n(\mathbb{C})) \to \mathbb{Z}$ such that $\tau_#([p_1]) = 1$, where $p_1 \in \mathcal{P}_1(M_n(\mathbb{C}))$ is a rank one projection.
The fact that $\tau$ is a trace implies that the equation

$$\tilde{\tau}([p]) = \tau(p)$$

gives a well defined map $\tilde{\tau} : K_0(M_n(\mathbb{C})) \to \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

The fact that $\tau$ is faithful implies that $\tilde{\tau}$ is an isomorphism of monoids; and since the Grothendieck group of $\mathbb{Z}_+$ is just $\mathbb{Z}$, it follows that $\tilde{\tau}$ gives rise to a unique isomorphism $\tau_# : K_0(M_n(\mathbb{C})) \to \mathbb{Z}$ such that $\tau_#([p_1]) = 1$, where $p_1 \in \mathcal{P}_1(M_n(\mathbb{C}))$ is a rank one projection.

The above argument can be made to work in much greater generality, thus:

Suppose $\tau_1$ is a positive, faithful, tracial linear functional on a general $C^*$-algebra. Then, the map defineby $\tau_n((x_{ij})) = \sum_{i=1}^n \tau_1(x_{ii})$ is seen to yield a faithful positive tracial functional $\tau_n$ on the $C^*$-algebra $M_n(A)$; and the $\tau_n$’s ‘patch up’ to yield a positive faithful tracial functional on $M_\infty(A)$ which ‘respects the inclusion of $M_n(A)$ into $M_{n+1}(A)$’ and to consequently define an isomorphism $\tau_#$ of $K_0(A)$ onto its image in $\mathbb{R}$. 
We wish to discuss one non-trivial example where some of these considerations help. Given a countable group $\Gamma$, let $\ell^2(\Gamma)$ denote a Hilbert space with a distinguished o.n. basis $\{\xi_t : t \in \Gamma\}$ indexed by $\Gamma$, and let $\lambda$ denote the so-called *left-regular* unitary representation of $\Gamma$ on $\ell^2(\Gamma)$ defined by

$$\lambda_s(\xi_t) = \xi_{st} \quad \forall \ s, t \in \Gamma$$

Define $C^*_\text{red}(\Gamma)$, the *reduced* $C^*$-algebra of $\Gamma$ to be the $C^*$-subalgebra of $L(\ell^2(\Gamma))$ generated by $\lambda(\Gamma)$. It is a fact that the equation

$$\tau_1(x) = \langle x \xi_1, \xi_1 \rangle$$

where $\xi_1$ denotes the basis vector indexed by the identity element $1$ in $\Gamma$ - defines a faithful positive tracial state on $C^*_\text{red}(\Gamma)$. 
We wish to discuss one non-trivial example where some of these considerations help. Given a countable group $\Gamma$, let $\ell^2(\Gamma)$ denote a Hilbert space with a distinguished o.n. basis $\{\xi_t : t \in \Gamma\}$ indexed by $\Gamma$, and let $\lambda$ denote the so-called left-regular unitary representation of $\Gamma$ on $\ell^2(\Gamma)$ defined by

$$\lambda_s(\xi_t) = \xi_{st} \ \forall \ s, t \in \Gamma$$

Define $C^*_\text{red}(\Gamma)$, the reduced $C^*$-algebra of $\Gamma$ to be the $C^*$-subalgebra of $\mathcal{L}(\ell^2(\Gamma))$ generated by $\lambda(\Gamma)$. 
We wish to discuss one non-trivial example where some of these considerations help. Given a countable group $\Gamma$, let $\ell^2(\Gamma)$ denote a Hilbert space with a distinguished o.n. basis $\{\xi_t : t \in \Gamma\}$ indexed by $\Gamma$, and let $\lambda$ denote the so-called left-regular unitary representation of $\Gamma$ on $\ell^2(\Gamma)$ defined by

$$\lambda_s(\xi_t) = \xi_{st} \forall s, t \in \Gamma$$

Define $C^*_{red}(\Gamma)$, the reduced $C^*$-algebra of $\Gamma$ to be the $C^*$-subalgebra of $\mathcal{L}(\ell^2(\Gamma))$ generated by $\lambda(\Gamma)$.

It is a fact that the equation

$$\tau_1(x) = \langle x\xi_1, \xi_1 \rangle$$

- where $\xi_1$ denotes the basis vector indexed by the identity element $1$ in $\Gamma$ - defines a faithful positive tracial state on $C^*_{red}(\Gamma)$. 

Reduced $C^*$-algebra of a discrete group
We have the following beautiful result on the $K$-theory of some of these algebras.
We have the following beautiful result on the $K$-theory of some of these algebras.

**Theorem:** *(Pimsner-Voiculescu)*

Let $\mathbb{F}_n$ be the free group on $n$ generators \{u$_1$, $\cdots$, u$_n$\}, and $A_n = C^*_{{\text{red}}} (\mathbb{F}_n), n \geq 2$. Then,

(a) $K_0(A_n) \cong \mathbb{Z}$ is generated by \([1_{A_n}]\) where \(1_{A_n} \in \mathcal{P}_1(A_n) \subset \mathcal{P}_\infty(A_n)\); and

(b) $K_1(A_n) \cong \mathbb{Z}^n$ is generated by \{\([u_1], \cdots, [u_n]\)\} where 
\(u_j \subset \mathcal{U}_1(A_n) \subset \mathcal{U}_\infty(A_n)\).

\(\square\)
We have the following beautiful result on the $K$-theory of some of these algebras.

**Theorem:** *(Pimsner-Voiculescu)*

Let $F_n$ be the free group on $n$ generators $\{u_1, \cdots, u_n\}$, and $A_n = C^*_{\text{red}}(F_n), n \geq 2$. Then,

(a) $K_0(A_n) \cong \mathbb{Z}$ is generated by $[1_{A_n}]$ where $1_{A_n} \in \mathcal{P}_1(A_n) \subset \mathcal{P}_\infty(A_n)$; and

(b) $K_1(A_n) \cong \mathbb{Z}^n$ is generated by $\{[u_1], \cdots, [u_n]\}$ where $u_j \subset \mathcal{U}_1(A_n) \subset \mathcal{U}_\infty(A_n)$.

**Corollary:** (i) $A_n$ has no non-trivial idempotents; and

(ii) $A_n \cong A_m \Rightarrow m = n$.

**Proof:** (i) Assertion (a) of the theorem implies that every $p \in \mathcal{P}_\infty(A)$ is equivalent to the identity of some $M_k(A_n)$. If $\tau$ be the faithful trace on $A_n$ defined earlier, note that $\tau(1) = 1$ (since $\xi_1$ is a unit vector), so

$$p \in \mathcal{P}_1(A_n), p, 1 - p \neq 0 \Rightarrow 0 < \tau(p) < 1;$$

this completes the proof.

(ii) follows immediately from (b) of the theorem.
Another very pretty result along these lines is:

**Theorem:** (Kasparov)

Let $\Sigma_g$ denote a compact surface of genus $g$, and $B_g = C^*_{\text{red}}(\pi_1(\Sigma_g))$. Then

1. $B_g$ has no non-trivial idempotents; and
2. $B_g \sim B_k \Rightarrow g = k$.

We conclude with a brief mention of Kasparov’s homotopy invariant bifunctor $KK(\cdot, \cdot)$ which:

1. assigns abelian groups to a pair of $C^*$-algebras
2. is covariant in the second variable and contravariant in the first variable.

$K_0(B) = KK(C, B)$ $\forall B$

$K_1(B) = KK(C, C_0(R, B))$ $\forall B$

This $KK$-theory has led to a much better understanding of $K$-theory and led to the computation of the $K$-groups of many algebras.
Another very pretty result along these lines is:

**Theorem:** *(Kasparov)*

Let $\Sigma_g$ denote a compact surface of genus $g$, and $B_g = C^*_\text{red}(\pi_1(\Sigma_g))$. Then

(i) $B_g$ has no non-trivial idempotents; and

(ii) $B_g \cong B_k \Rightarrow g = k$. 

\[ \square \]
Another very pretty result along these lines is:

**Theorem:** *(Kasparov)*

Let $\Sigma_g$ denote a compact surface of genus $g$, and $B_g = C^*_\text{red}(\pi_1(\Sigma_g))$. Then

(i) $B_g$ has no non-trivial idempotents; and

(ii) $B_g \cong B_k \Rightarrow g = k$.  

We conclude with a brief mention of Kasparov's homotopy invariant bifunctor $KK(\cdot, \cdot)$ which:

1. assigns abelian groups to a pair of $C^*$-algebras
2. is covariant in the second variable and contravariant in the first variable.
3. $K_0(B) = KK(\mathbb{C}, B) \forall B$
4. $K_1(B) = KK(\mathbb{C}, C_0(\mathbb{R}, B)) \forall B$

This $KK$-theory has led to a much better understanding of $K$ theory and led to the computation of the $K$-groups of many algebras.
A few references

1. *K theory*, M. Atiyah. (A classic text on topological $K$ theory - of vector bundles on spaces.)

