Operator algebras - stage for non-commutativity
(Panorama Lectures Series)
II. K-theory for C^*-algebras

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By a **vector bundle of rank n** on a compact Hausdorff space X is meant an ordered pair (E, p) consisting of a topological space E and a continuous map $p : E \to X$, which satisfy some requirements which say loosely that:

- for each $x \in X$, the *fibre* $E_x = \pi^{-1}(x)$ over x has the structure of a vector space of dimension n
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Vector bundles

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We will, however, be concerned primarily with complex vector bundles here.

If (E, p) is a vector bundle on X, a *section* of E is a continuous function $s : X \to E$ such that $s(x) \in E_x \; \forall \; x \in X$. The set $\Gamma(E)$ of sections of E is naturally a vector space - with

$$(\alpha s + \beta t)(x) = \alpha s(x) + \beta t(x) \; ,$$

and with the linear combination on the right interpreted in the vector space E_x. In fact, $\Gamma(E)$ is naturally a *module* over $C(X)$ - with

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Serre-Swan theorem

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Theorem: (**Serre-Swan theorem:**)
If (E, p) is a vector bundle over a compact Hausdorff space X, then $\Gamma(X)$ is a *finitely generated projective module* over $C(X)$ (i.e., there exist finitely many elements $s_1, \cdots, s_n \in \Gamma(X)$ such that $\Gamma(E) = \sum_{i=1}^n C(X) \cdot s_i$).

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Further, **every** finitely generated projective module over $C(X)$ is of this form.

□

Notice next that if A is any unital C^*-algebra, so is $M_n(A)$ (in a natural way); the algebraic operations are the natural ones, while the norm may be obtained thus: if $A \hookrightarrow \mathcal{L}(\mathcal{H})$, then $M_n(A) \hookrightarrow M_n(\mathcal{L}(\mathcal{H})) \cong \mathcal{L}(\mathcal{H} \oplus \mathcal{H} \oplus \overset{n\ terms}{\cdots} \mathcal{H})$. We shall identify $M_n(A)$ with the 'northwest corner' of $M_{n+1}(A)$ via $x \sim \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$.

Projections and unitaries

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Write $\mathcal{P}_n(A) = \mathcal{P}(M_n(A))$, and $\mathcal{U}_n(A) = \mathcal{U}(M_n(A))$ where $\mathcal{P}(B))$ (resp., $\mathcal{U}(B)))$ denotes the set $\{p \in B : p = p^2 = p^*\}$, (resp., $\{u \in B : u^*u = uu^* = 1\}$) of **projections** (resp., **unitary elements**) in any C^*-algebra B.

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Regard $\mathcal{P}_n(A)$ (resp., $\mathcal{U}_n(A)$) as being included in $\mathcal{P}_{n+1}(A)$ (resp., $\mathcal{U}_{n+1}(A)$) via the identification

$$\mathcal{P}_n(A) \ni p \sim \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{P}_{n+1}(A) ,$$

(resp. $u \sim \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}$) and write $\mathcal{P}_\infty(A)$, $M_\infty(A)$ and $\mathcal{U}_\infty(A)$ for the indicated increasing union.

A finitely generated projective module over A is of the form

$$V_p = \{\xi \in M_{1\times n}(A) : \xi = \xi p\} ,$$

for some $p \in \mathcal{P}_n(A)$, and some positive integer n - where of course the A action on V_p is given by

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It is not hard to see that if $p, q \in \mathcal{P}_\infty(A)$, then a linear map $x : V_p \to V_q$ is *A-linear* if and only if there exists a matrix $X = ((x_{ij})) \in M_\infty(A)$ such that

$$x \cdot v = v \cdot X \quad \text{and} \quad X = pxq$$

where we think of elements of V_p and V_q as row vectors. (This assertion is an instance of the thesis 'what commutes with all left-multiplications must be a right-multiplication', many instances of which we will keep running into.) In particular, modules V_p and V_q are isomorphic iff there exists a $u \in M_\infty(A)$ such that $u^*u = p$ and $uu^* = q$; write $p \sim q$ when this happens.

Towards defining $K_0(A)$

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Proposition The set $\mathcal{K}_0(A) = \mathcal{P}_\infty(A)/\sim$ is an abelian monoid (=semigroup with identity) with respect to addition defined by

$$[p] + [q] = [p \oplus q] \ ,$$

the identity element being [0].

If S is an abelian semigroup, the set $\{a - b : a, b \in S\}$ of formal differences in S - with the convention that $a - b = c - d$ iff $a + d + f = c + b + f$ for some $f \in S$ - turns out to be an abelian group, called the **Grothendieck group** of S.

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Definition: If A is a unital C^*-algebra, then

(i) $K_0(A)$ is defined to be the Grothendieck group of $\mathcal{K}_0(A)$:

(ii) $K_1(A)$ is defined to be the quotient of the group $\mathcal{U}_\infty(A)$ by the normal subgroup $\mathcal{U}_\infty(A)^{(0)}$ (defined by the connected component of its identity element).

It turns out that $K_1(A)$ is also an abelian group, with the group law being given in two equivalent ways, thus: if $u \in \mathcal{U}_m(A), v \in \mathcal{U}_k(A)$, then

$$[uv] = [u][v] = [\begin{bmatrix} u & 0 \\ 0 & 1_k \end{bmatrix}][\begin{bmatrix} 1_m & 0 \\ 0 & v \end{bmatrix}] = [u \oplus v]$$

where we write 1_ℓ for the identity in $M_\ell(A)$.

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Some fundamental properties of the K-groups, which we shall briefly discuss below, are:

- Functoriality
- Normalisation
- Stability
- Homotopy invariance

Functoriality: $K_i, i = 0, 1$ define covariant functors from the category of C^*-algebras to abelian groups; i.e., if $\phi \in Hom(A, B)$ is a morphism of C^*-algebras, there exist group homomorphisms $K_i(\phi) = \phi_* : K_i(A) \rightarrow K_i(B)$ satisfying the usual functoriality requirements - of being well-behaved with respect to compositions and identity morphisms: i.e.,

$$K_i(\phi \circ \psi) = K_i(\phi) \circ K_i(\psi) \ , \ K_i(id_A) = id_{K_i(A)} \ .$$

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Homotopy invariance: If $\{\phi_t : t \in [0, 1]\}$ is a continously varying family of homomorphisms from A into B (or equivalently, if there exists a homomorphism $A \ni a \mapsto (t \mapsto \phi_t(a)) \in C([0, 1], B)$), then $(\phi_0)_* = (\phi_1)_*$.

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Use of homotopy of invariance

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Proof: Let $\{h_t : t \in [0,1]\}$ be a homotopy with $h_1 = id_X$ and $h_0(x) = x_0 \in X \ \forall x \in X$. Consider $\phi_t (= h_t^* =) : C(X) \rightarrow C(X)$ defined by $\phi_t(f) = f \circ h_t$. Then $\phi_1 = id_{C(X)}$ while $\phi_0(f)$ is the constant function identically equal to $f(x_0)$ So, if j denotes the inclusion map $j : \mathbb{C} \rightarrow C(X)$, and if we write $f(x_0) = ev_0(f)$, we have commutative diagrams of maps:

$$\begin{array}{ccc} C(X) & \stackrel{\phi_0}{\rightarrow} & C(X) \\ ev_0 \downarrow & \stackrel{j}{\nearrow} & \downarrow ev_0 \\ \mathbb{C} & \stackrel{\rightarrow}{id} & \mathbb{C} \end{array}$$

and

$$\begin{array}{ccc} K_i(C(X)) & \stackrel{(\phi_0)_*}{\rightarrow} & K_i(C(X)) \\ (ev_0)_* \downarrow & \stackrel{j_*}{\nearrow} & \downarrow (ev_0)_* \\ K_i(\mathbb{C}) & \stackrel{\rightarrow}{id} & K_i(\mathbb{C}) \end{array}$$

Since $\phi_0^* = \phi_i^* = id^*$, the second diagram shows that j^* is an isomorphism with inverse ev_0^*.

Before proceeding further, we need to discuss non-unital C^*-algebras. (This corresponds to studying vector bundles over locally compact non-compact spaces.) If A is any C^*-algebra - with or without identity - then $\tilde{A} = A \times \mathbb{C}$ becomes a unital C^*-algebra thus:

$$\begin{aligned} (x.\lambda) \cdot (y, \mu) &= (xy + \lambda y + \mu x, \lambda\mu) \\ \|(x, \lambda)\| &= \sup\{\|xa + \lambda a\| : a \in A, \|a\| = 1\} \end{aligned}$$

(Addition and involution are componentwise, and $(0, 1)$ is the identity.) Further $\epsilon : \tilde{A} \longrightarrow \mathbb{C}$ defined by $\epsilon(x, \lambda) = \lambda$ is a homomorphism of unital C^*-algebras, with $ker(\epsilon) = A$; thus A is an ideal of co-dimension 1 in $\tilde{A}$.

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Example: In case $A = C_0(X)$ is the algebra of continuous functions on a locally compact space X which 'vanish at infinity', the 'unitisation' $\tilde{A}$ can be identified with $C(\hat{X})$, where $\hat{X} = (X \cup \{\infty\})$ is the one-point compactification of X, and $\epsilon(f) = f(\infty)$.)

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For a possibly non-unital A, define

$$K_i(A) = ker\ K_i(\epsilon).$$

Six term exact sequence: If

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{\pi} B \longrightarrow 0$$

is a short exact sequence of C^*-algebras, then there exists an associated six term exact sequence of K-groups

$$\begin{array}{ccccc} K_0(J) & \stackrel{j_*}{\longrightarrow} & K_0(A) & \stackrel{\pi_*}{\longrightarrow} & K_0(B) \\ \partial_1 \uparrow & & & & \downarrow \partial_0 \\ K_1(B) & \stackrel{\pi_*}{\longleftarrow} & K_1(A) & \stackrel{j_*}{\longleftarrow} & K_1(J) \end{array}$$

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It is worth noting the special case when the short exact sequence splits - i.e., when there exists a *-homomorphism $s : B \longrightarrow A$ such that $\pi \circ s = id_B$; in this case, also π_* is surjective, whence both connecting maps must be the zero maps, so the six-term sequence above splits into two short exact sequences

$$0 \longrightarrow K_i(J) \xrightarrow{j_*} K_i(A) \xrightarrow{\pi_*} K_i(B) \longrightarrow 0$$

Example: Consider the short exact sequence

$$0 \to C_0((0,1]) \xrightarrow{j} C([0,1]) \xrightarrow{ev_0} \mathbb{C} \to 0$$

Since $K_i(ev_0) : K_i(C([0,1]) \cong K_i(\mathbb{C})$ it follows from the six term exact sequence that

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is seen to be

$$\begin{array}{ccccc} K_0(C_0((0,1))) & \stackrel{j_*}{\longrightarrow} & 0 & \stackrel{(ev_1)_*}{\longrightarrow} & K_0(\mathbb{C}) \\ \partial_1 \uparrow & & & & \downarrow \partial_0 \\ K_1(\mathbb{C}) & \stackrel{(ev_1)_*}{\longleftarrow} & 0 & \stackrel{j_*}{\longleftarrow} & K_1(C_0((0,1))) \end{array} ;$$

so $K_i(C_0(\mathbb{R})) \cong K_i(C_0((0,1))) = K_{i+1}(\mathbb{C})$ (mod 2).

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so $K_i(C_0(\mathbb{R})) \cong K_i(C_0((0,1))) = K_{i+1}(\mathbb{C}) \pmod 2$.

Similar reasoning, applied to $C_0(\mathbb{R}; A)$, essentially yields the **Bott periodicity theorem**:

$$K_i(C_0(\mathbb{R}; A)) = K_{i+1}(A) \mod 2.$$

Applied inductively to $A = C_0(\mathbb{R}^n)$, we conclude that

$$K_i(C_0(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z} & \text{if } (n-i) \text{ is even} \\ 0 & \text{otherwise} \end{cases} .$$

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The short exact sequence

$$0 \to C_0(\mathbb{R}^n) \stackrel{j}{\to} C(S^n) \stackrel{ev_\infty}{\to} \mathbb{C} \to 0$$

is split by the inclusion morphism $\eta : \mathbb{C} \to C(S^n)$, so that we have a short exact sequence

$$0 \to K_i(C_0(\mathbb{R}^n))) \stackrel{j_*}{\to} K_i(C(S^n)) \stackrel{\pi_*}{\to} K_i(\mathbb{C}) \to 0$$

which also splits and we may deduce that

$$K_i(C(S^n)) \cong K_i(C_0(\mathbb{R}^n)) \oplus K_i(\mathbb{C}) .$$

The simplest non-abelian C^*-algebras are the $M_n(\mathbb{C})$'s, and we may conclude from the 'stability' of K-groups that

$$K_i(M_n(\mathbb{C})) \cong K_i(\mathbb{C}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n = 1 \end{cases} .$$

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We shall give another proof that $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$. Consider the map $\tau : M_\infty(M_n(\mathbb{C})) \to \mathbb{C}$ by

$$\tau((x_{ij})) = \sum_i Tr(x_{ii}) \; ,$$

where Tr denotes the usual trace ($=$ sum of diagonal entries) on the matrix algebra.

The virtue of traces

The simplest non-abelian C^*-algebras are the $M_n(\mathbb{C})$'s, and we may conclude from the 'stability' of K-groups that

$$K_i(M_n(\mathbb{C})) \cong K_i(\mathbb{C}) \cong \begin{cases} \mathbb{Z} & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n = 1 \end{cases} .$$

We shall give another proof that $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$. Consider the map $\tau : M_\infty(M_n(\mathbb{C})) \to \mathbb{C}$ by

$$\tau((x_{ij})) = \sum_i Tr(x_{ii}) \ ,$$

where Tr denotes the usual trace (= sum of diagonal entries) on the matrix algebra.

Then τ is seen to be a positive ($\tau(X^*X) \geq 0 \ \forall X$) faithful (i.e., $X \neq 0 \Rightarrow \tau(X^*X) > 0$) and tracial ($\tau(XY) = \tau(YX)$) linear functional. Further τ 'respects the inclusion of $M_k(M_n(\mathbb{C}))$ into $M_{k+1}(M_n(\mathbb{C}))$ in the sense that

$$\tau(X) = \tau(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} .$$

The fact that τ is a trace implies that the equation

$$\tilde{\tau}([p]) = \tau(p)$$

gives a well defined map $\tilde{\tau} : \mathcal{K}_0(M_n(\mathbb{C})) \to \mathbb{Z}_+ = \{0, 1, 2, ...\}$.

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The fact that τ is faithful implies that $\tilde{\tau}$ is an isomorphism of monoids; and since the Grothendieck group of $\mathbb{Z}_+$ is just $\mathbb{Z}$, it follows that $\tilde{\tau}$ gives rise to a unique isomorphism $\tau_\# : K_0(M_n(\mathbb{C})) \to \mathbb{Z}$ such that $\tau_\#([p_1]) = 1$, where $p_1 \in \mathcal{P}_1(M_n(\mathbb{C}))$ is a rank one projection.

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The above argument can be made to work in much greater generality, thus:

Suppose τ_1 is a positive, faithful, tracial linear functional on a general C^*-algebra. Then, the map defineby $\tau_n((x_{ij})) = \sum_{i=1}^n \tau_1(x_{ii})$ is seen to yield a faithful positive tracial functional τ_n on the C^*-algebra $M_n(A)$; and the τ_n's 'patch up' to yield a positive faithful tracial functional on $M_\infty(A)$ which 'respects the inclusion of $M_n(A)$ into $M_{n+1}(A)$' and to consequently define an isomorphism $\tau_\#$ of $K_0(A)$ onto its image in $\mathbb{R}$.

We wish to discuss one non-trivial example where some of these considerations help. Given a countable group Γ, let $\ell^2(\Gamma)$ denote a Hilbert space with a distinguished o.n. basis $\{\xi_t : t \in \Gamma\}$ indexed by Γ, and let λ denote the so-called *left-regular* unitary representation of Γ on $\ell^2(\Gamma)$ defined by

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It is a fact that the equation

$$\tau_1(x) = \langle x\xi_1, \xi_1 \rangle$$

- where ξ_1 denotes the basis vector indexed by the identity element 1 in Γ - defines a faithful positive tracial state on $C^*_{red}(\Gamma)$.

K theory distinguishes the $C^*_{red}(\mathbb{F}_n)$s

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Theorem:(Pimsner-Voiculescu)
Let $\mathbb{F}_n$ be the free group on n generators $\{u_1, \cdots, u_n\}$, and $A_n = C^*_{red}(\mathbb{F}_n), n \geq 2$. Then,

(a) $K_0(A_n) \cong \mathbb{Z}$ is generated by $[1_{A_n}]$ where $1_{A_n} \in \mathcal{P}_1(A_n) \subset \mathcal{P}_\infty(A_n)$; and

(b) $K_1(A_n) \cong \mathbb{Z}^n$ is generated by $\{[u_1], \cdots, [u_n]\}$ where $u_j \subset \mathcal{U}_1(A_n) \subset \mathcal{U}_\infty(A_n)$. □

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Corollary: (i) A_n has no non-trivial idempotents; and

(ii) $A_n \cong A_m \Rightarrow m = n$.

Proof: (i) Assertion (a) of the theorem implies that every $p \in \mathcal{P}_\infty(A)$ is equivalent to the identity of some $M_k(A_n)$. If τ be the faithful trace on A_n defined earlier, note that $\tau(1) = 1$ (since ξ_1 is a unit vector), so

$$p \in \mathcal{P}_1(A_n), p, 1-p \neq 0 \Rightarrow 0 < \tau(p) < 1 \ ;$$

this completes the proof.

(ii) follows immediately from (b) of the theorem.

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Theorem: **(Kasparov)**

Let Σ_g denote a compact surface of genus g, and $B_g = C^*_{red}(\pi_1(\Sigma_g))$. Then

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Let Σ_g denote a compact surface of genus g, and $B_g = C^*_{red}(\pi_1(\Sigma_g))$. Then

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(ii) $B_g \cong B_k \Rightarrow g = k$. □

We conclude with a brief mention of Kasparov's homotopy invariant bifunctor $KK(\cdot, \cdot)$ which:

1. assigns abelian groups to a pair of C^*-algebras
2. is covariant in the second variable and contravariant in the first variable.
3. $K_0(B) = KK(\mathbb{C}, B)\ \forall B$
4. $K_1(B) = KK(\mathbb{C}, C_0(\mathbb{R}, B))\ \forall B$

This ***KK*-theory** has led to a much better understanding of K theory and led to the computation of the K-groups of many algebras.

A few references

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(A classic text on topological K theory - of vector bundles on spaces.)

2. *K theory for operator algebras*, B. Blackadar.
(Probably the first book on the subject.)

3. Georges Skandalis, *Kasparov's bivariant theory and applications*, Exposit. Mat. 9 (1991) 193–250.

4. *Elements of KK theory*, K.K. Jensen and K. Thomsen
(A description with complete details of Kasparov's bivariate theory.).