Operator algebras - stage for non-commutativity
(Panorama Lectures Series)
I. C^*-algebras: the Gelfand Naimark Theorems

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The goal is to give a flavour of the world of non-commutativity, by touching on various aspects of **operator algebras** - the natural arena for such a discussion. *Operator algebras* were born at the hands of von Neumann in order to tackle several requirements of the mathematical foundations of (the then fledgeling) Quantum Mechanics.

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Operator algebras are the arena to tackle matrices involving infinitely many degrees of freedom.

Outline of talks

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- The Gelfand Naimark theorems (non-commutative general topology)
- K-theory of C^*-algebras (non-commutative algebraic topology)
- von Neumann algebras (non-commutative measure theory)
- II_1 factors and subfactors (the bridge to low-dimensional topology, conformal field theory, ...)
- The Jones polynomial invariant of knots

The slides of all tnese talks can be found on my home-page at

http://www.imsc.res.in/∼sunder/

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(non-commutative G-N th) A is a C^*-algebra if and only if A is isomorphic to a closed *-subalgebra of $\mathcal{L}(\mathcal{H})$ (the C^*-algebra of *bounded operators on Hilbert space*).

A **Banach algebra** is a triple $(A, \|\cdot\|, \cdot)$, where:

- $(A, \|\cdot\|)$ is a Banach space
- $(A, \cdot)$ is a ring
- The map $A \ni x \mapsto L_x \in \mathcal{L}(A)$ defined by $L_x(y) = xy$ is a linear map and a ring-homomorphism satisfying

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A is **unital** if it has a multiplicative identity 1; we always assume it satisfies $\|1\| = 1$. (This turns out to be little loss of generality.) We only consider unital algebras here.

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Lemma: $\|x\| < 1 \Rightarrow$

- $1 - x \in GL(A)$
- $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$
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Corollary: $GL(A)$ is open, and $x \mapsto x^{-1}$ is a continous self-map of $GL(A)$ of order two, hence a homeomorphism.

Proof: The lemma says 1 is an interior point of $GL(A)$ and $x \mapsto x^{-1}$ is continuous at 1; and if $x \in GL(A)$, then L_x is a homoemorphism of $GL(A)$ onto itself.

Define the **spectrum** of an element $x \in A$ by

$$sp(x) = \{\lambda \in \mathbb{C} : x - \lambda \notin GL(A)\}$$

and its **spectral radius** by

$$r(x) = \sup\{|\lambda| : \lambda \in sp(x)\}$$

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Theorem: (**spectral radius formula**)
The spectrum is always non-empty, and we have

$$r(x) = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}} .$$

Caution: We must exercise some caution and talk about $sp_A(x)$, since if D is a unital Banach subalgebra of A and if $x \in D$, it may be the case that $sp_D(x) \neq sp_A(x)$. For example, by the maximum modulus principle, it is seen that

$$D = \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is holomorphic}\}$$

imbeds isometrically as a Banach subalgebra of $A = C(\partial\mathbb{D})$, and

$$f \in D \Rightarrow sp_D(f) = f(\overline{\mathbb{D}}), sp_A(f) = f(\partial\mathbb{D}) \ .$$

But it turns out that there is no such pathology if our Banach algebras are C^*-algebras.

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1. $x \notin GL(A)$
2. $\exists J \in \mathcal{M}(A)$ such that $x \in J$.

Proof: For (1) $\Rightarrow$ (2), note that $I = Ax$ is a proper ideal; pick $J \in \mathcal{M}(A)$ such that $I \subset J$.

Note that maximal ideals are closed (since 1 is in the exterior of any proper ideal). This implies:

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Proposition: Write $\hat{A}$ for the collection of unital homomorphisms $\phi : A \to \mathbb{C}$. Then

(a)

$$J \in \mathcal{M}(A) \Leftrightarrow \exists \phi \in \hat{A} \text{ such that } J = ker\ \phi.$$

(b)

$$\phi \in \hat{A} \Rightarrow \phi(x) \in sp(x) \Rightarrow |\phi(x)| \leq r(x) \leq \|x\| \ ,$$

so $\hat{A} \subset ball(A^*)$.

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Proposition: The **Gelfand transform** of A, which is the map $\Gamma : A \to C(\hat{A})$ defined by

$$(\Gamma(x))(\phi) = \phi(x) \; \forall \phi \in \hat{A}$$

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Question: When is Γ an isometric isomorphism onto $C(\hat{A})$?

Answer: Precisely when A is a C^*-algebra!

A **C^*-algebra** is a Banach algebra A equipped with an involution - i.e., a self-map $a \ni x \mapsto x^* \in A$ satisfying

- $(\alpha x + y)^* = \bar{\alpha} x^* + y^*$
- $(xy)^* = y^* x^*$
- $(x^*)^* = x$
- $\|x\|^2 = \|x^* x\|$. (C^*-**identity**)

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The commutative G-N theorem:

The Gelfand transform of a commutative Banach algebra A is an isometric surjection if and only if A has the structure of a commutative C^*-algebra. In this case, Γ is automatically an isomorphism of C^*-algebras.

Sketch of Proof: Suppose A is a C^*-algebra and $x = x^*$ is 'self-adjoint'. For $t \in \mathbb{R}$, define $u_t = e^{itx} = \sum_{n=0}^{\infty} \frac{(itx)^n}{n!}$ and note that $u_t^* = u_{-t} = u_t^{-1}$. So, by the C^*-identity,

$$\|u_t\|^2 = \|u_t^* u_t\| = 1.$$

Hence

$$\phi \in \hat{A} \Rightarrow 1 \geq |\phi(u_t)| = \left|e^{it\phi(x)}\right| .$$

Since $t \in \mathbb{R}$ is arbitrary, deduce that $\phi(x) \in \mathbb{R}$.
Also, for self-adjoint x, note that

$$\|x\| = \|x^* x\|^{\frac{1}{2}} = \|x^2\|^{\frac{1}{2}}$$

so

$$\|x\| = \|x^2\|^{\frac{1}{2}} = \cdots = \lim_{n \to \infty} \|x^{2^n}\|^{\frac{1}{2^n}} = r(x) = \|\Gamma(x)\|$$

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Proposition:
The following conditions on $x \in A$ are equivalent:

1. $C^*(x)$ is commutative
2. $x^*x = xx^*$ (such x's are called **normal**).
3. If x is normal, there exists a unique unital C^*-algebra isomorphism $\theta_x : C(sp(x)) \to C^*(x)$ such that $\theta_x(id_{sp(x)}) = x$.

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It is customary to write $\theta_x(f) = f(x)$ and call θ_x the *continuous functional calculus for x*.

Proposition: An element satisfies the algebraic condition in the second column of the table below if and only if it is normal and its spectrum is contained in the set listed in the third column.

Name	Alg. def.	$sp(x) \subset ?$
self-adjoint	$x = x^*$	$\mathbb{R}$
unitary	$x^*x = xx^* = 1$	$\mathbb{T}$
projection	$x^2 = x^* = x$	$\{0, 1\}$

Sub-classes of normal elements

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Study of general C^*-algebras is facilitated by applying the commutative theory to normal elements of these types. Normal elements can be dealt with as easily as functions. Here is a sample of such results:

- (*Cartesian decomposition*) Every element $z \in A$ admits a unique deomposition $z = x + iy$, with x, y self-adjoint; in fact, $x = \frac{z+z^*}{2}, y = \frac{z-z^*}{2i}$
- Every self-adjoint element $x \in A$ admits a unique decomposition $x = x^+ - x^-$, where $x^\pm$ are self-adjoint and satisfy $x^+x^- = 0$; in fact, $x^\pm = f^\pm(x)$, where $f^\pm \in C(\mathbb{R})$ are defined by $f^\pm(t) = \frac{t \pm |t|}{2}$

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Theorem: (a) The following conditions on an element $x \in A$ are equivalent:

1. $x = x^*$ and $sp(x) \subset [0, \infty)$
2. $\exists y = y^* \in A$ such that $x = y^2$
3. $\exists z \in A$ such that $x = z^* z$

Such x's are said to be 'positive'; the set A_+ of positive elements of A is a 'positive cone' (proved using (1) above).

(b) If $x \in A_+$, then the y of (2) above may be chosen to be positive, and such a 'positive square root of x' is unique, and in fact $y = x^{\frac{1}{2}}$.

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Proposition: The following conditions on a $\phi \in A^*$ are equivalent:

1. $\phi(A_+) \subset \mathbb{R}_+$
2. $\|\phi\| = \phi(1)$

Such ϕ's are said to be positive (linear functionals); the set A^*_+ of of positive elements of A^* is a 'positive cone'. A positive functional ϕ is said to be a **state** if it is normalised so that $\phi(1)(= \|\phi\|) = 1$.

Definition: A **representation** of a unital C^*-algebra A is a a morphism $\pi : A \to \mathcal{L}(\mathcal{H})$ of unital C^*-algebras.

En route to proving the non-commutative GN theorem, we will need the following fact about representations.

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Proposition:(repprop)

1. Every representation is contractive, i.e., $\|\pi(x)\| \leq \|x\| \ \forall x \in A$. (In fact, an injective representation is automatically norm-preserving.)
2. For each $x \in A$, there exists a representation π such that $\|\pi(x)\| = \|x\|$.

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To prove (1) above, note first that if x is invertible, so should be $\pi(x)$; in other words, $sp(\pi(x)) \subset sp(x)$. The assertion (1) is a consequence of (i) the fact that $\|x\| = r(x)$ for self-adjoint x, and (ii) the C^*-identity.

As for (2), a clue to constructing representations is given by the following observation, regarding the commutative case $A = C(X)$:

1. States on A are identifiable with probability measures μ defined on the σ-algebra $\mathcal{B}_X$ of Borel sets in X, via $\phi_\mu(f) = \int f\ d\mu$.
2. $C(X)$ is 'dense' in $L^2(\mu)$; more accurately, $L^2(\mu)$ is the completion of $C(X)$ with respect to the semi-norm $\|f\|_\mu = \left(\int |f|^2 d\mu\right)^{\frac{1}{2}}$; and
3. the equation $\pi_\mu(f)g = fg$ defines a (cyclic) representation π_μ of $C(X)$.

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Cauchy-Schwarz inequality

$$|\phi(y^*x)|^2 \leq \phi(x^*x)\phi(y^*y) \ \ \forall \phi \in A_+^*, x, y \in A \ .$$

Proof: The equation

$$\langle x, y\rangle_\phi = \phi(y^*x)$$

defines a semi-inner product on A ,i.e., satisfies all requirements of an inner product except possibly positive - definiteness, and the CS-inequality is valid in any semi-inner-product space.

Theorem: T.F.A.E.:

1. ϕ is a state on A;
2. there exists a triple $(\mathcal{H}, \pi, \Omega)$ (essentially unique) consisting of and a unit vector $\Omega \in \mathcal{H}$ such that
   - a Hilbert space $\mathcal{H}$,
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   such that $\phi(x) = \langle \pi(x)\Omega, \Omega \rangle \ \forall x \in A$.

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   such that $\phi(x) = \langle \pi(x)\Omega, \Omega \rangle \ \forall x \in A$.

Sketch of proof: The fact that ϕ satisfies Cauchy-Schwarz inequality implies that the **radical of ϕ** defined by

$$\mathcal{N}_\phi = \{x \in A : \|x\|_\phi^2 = \langle x, x \rangle_\phi = 0\}$$

is a left-ideal in A (i.e., a subspace closed under left multiplication by elements of A). Then $A/\mathcal{N}_\phi$ is a genuine inner product space, whose completion is the desired $\mathcal{H}_\phi$, while the equation

$$\pi_0(x)(y + \mathcal{N}_\phi) = xy + \mathcal{N}_\phi$$

happens to define a bounded operator $\pi_0(x)$ on $A/\mathcal{N}_\phi$ which has a unique continuous extension $\pi_\phi(x)$ to $\mathcal{H}_\phi$.

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Proof: Let $A_0 = C^*(\{x\})$. Then pick $\phi_0 \in \widehat{A_0} \subset$ *ball*(A^*_0) such that $|\phi_0(x)| = \|x\|$. Use Hahn-Banach theorem to find a $\phi \in A^*$ such that $\phi|_{A_0} = \phi_0$ and $\|\phi\| = \|\phi_0\| (= \phi_0(1) = \phi(1))$. It follows that this $\phi \in A^*_+$ does the job.

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Proof of Prop (repprop) (2): Fix $x \in A$. Apply the previous Lemma to find a $\phi \in A_+^*$ such that $|\phi(x^*x)| = \|x\|^2$, and let $(\mathcal{H}_x, \pi_x, \Omega_x)$ be 'the GNS triple' associated to ϕ. First observe that

$$\|\Omega_x\|^2 = \langle \pi_x(1)\Omega_x, \Omega_x \rangle = \phi(1) = 1$$

and then deduce that

$$\|\pi_x(x)\|^2 = \|\pi_x(x)\pi_x(x)^*\| = \|\pi_x(x^*x)\| \geq \langle \pi_x(x^*x)\Omega_x, \Omega_x \rangle = \|x\|^2.$$

The reverse inequality follows from Proposition (repprop) (1).

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Next, it follows from Proposition (repprop) (1) that the equation

$$\pi(y) = \oplus_{x \in D} \pi_x(y)$$

meaningfully defines a representation π of A on $\mathcal{H} = \oplus_{x \in D} \mathcal{H}_x$ which is contractive on the one hand (by Proposition (repprop) (1)), and which is isometric on a dense set on the other, by ($\dagger$). Consequently π is an isometric isomorphism.

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Several details, which might have been omitted here, can be found in my book

Functional Analysis: Spectral Theory, TRIM Series No. 13, Hindustan Book Agency, Delhi, 1997; international edition: Birkhaüser Advanced Texts, Basel, 1997.