The planar algebra associated to a Kac algebra

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Abstract

We obtain (two equivalent) presentations - in terms of generators and relations - of the planar algebra associated with the subfactor corresponding to (an outer action on a factor by) a finite-dimensional Kac algebra. One of the relations shows that the antipode of the Kac algebra agrees with the 'rotation on 2-boxes'.

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1 Introduction

For an arbitrary finite index inclusion $N \subset M$ of II_1 factors, the basic construction of V. Jones [J] gives a canonical construction of a tower of factors

$$N \subset M \subset M_1 \subset M_2 \subset \dots \ .$$

The double sequence of finite dimensional algebras

$$\begin{array}{ccccccc} N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset \dots \\ & & \cup & & \cup & \\ & & M' \cap M & \subset & M' \cap M_1 & \subset \dots \end{array},$$

known as the *standard invariant*, became an extremely important invariant in the study of subfactors (see for example [GHJ, JS, Po1, Po2]). In fact S. Popa determined that the standard invariant completely classifies certain kinds of subfactors [Po1]. In [Po2], Popa provided an abstract algebraic characterization of the standard invariant called a λ*-lattice*. Subsequently, Jones developed an equivalence between λ-lattices and certain structures that he called *planar algebras* [J1]. In the planar algebra framework, many of the seemingly complicated algebraic conditions associated with a λ-lattice can be simply described in terms of the geometry of the plane.

Given a Kac algebra H and a II_1 factor M, there is a well known construction of a subfactor $M^H \subset M$ where M^H is the algebra of fixed points of an outer action of H on M. In this paper, we give a presentation - in terms of generators and relations - of the planar algebra associated with $M^H \subset M$. Since the subfactor $M^H \subset M$ is known to 'remember H' - see [Sz] - this presentation contains a pictorial represention of the Kac algebra H which we hope might be useful to those who have occasion to do complicated calculations in H. The presentation given here is motivated by earlier work of one of the authors. In [L], a class of planar algebras, termed 'exchange relation planar algebras' was defined and it was shown that the planar algebra corresponding to a subfactor coming from a finite dimensional Kac algebra (i.e. a depth-two subfactor, see [O] or [Sz]) is an exchange relation planar algebra. In addition, a presentation of the planar algebra was given for the case when $H = \mathbb{C}G$, G a finite group.

We have attempted to render this paper 'accessible' and self-contained to a person who is familiar with little more than the basics of subfactor theory and the definition of Kac algebras. For this reason, we have chosen to include some material that appears elsewhere in the literature. Specifically, we include a presentation of Jones' planar algebras; we felt that it could not hurt to repeat the description for the sake of the reader who has not yet had the pleasure of making the acquaintance of Jones' planar algebras. In addition, we have always had trouble laying hands on references for basic definitions and facts concerning finite-dimensional Kac algebras and their actions on II_1 factors; so, in spite of the existing literature (cf. [D], [O], [Sa] and [Sz]) on the connections between subfactors and Kac algebras, we have worked out most of the details here in an effort to make the treatment essentially self-contained.

This paper is organised as follows:

We begin in Section 2 by setting up the notation and recalling some basic facts (from the operadic approach-see [J2]) of Jones' planar algebras. Section 3 is devoted to recapitulating various facts concerning presentations of planar algebras; here things are closer to Jones' initial - see [J1] - approach. We also recall some facts about 'exchange relation planar algebras.'

Section 4 is devoted to Kac algebras. After recalling various standard facts about these objects, we give a description of the first few stages of the tower of the basic construction that is associated to an outer action of a Kac algebra on a II_1 factor. A bonus of our presentation is the fact that 'actions of finite-dimensional Kac algebras on II_1 factors are automatically normal'. We work out a matricial description of the crossed-product of a II_1 factor by such an action (analogous to the one given in [JS] for the group-case), and which is used in the subsequent discussion.

In Section 5, we give two presentations of the planar algebra of the subfactor coming from a finite-dimensional Kac algebra H. The first one uses the entire algebra H as generators and has the advantage that various relations (as well as proofs) become much more transparent; while the second uses a certain kind of basis of H as generators and has the advantage of being a 'finite presentation' (but has the disadvantage of being 'unnatural'). It is this second finite presentation which specialises to the description in [L] for the case of the group algebra.)

Some concluding remarks form the content of the brief section 6; we give descriptions here, for instance - which constitute natural generalisations, to the Kac algebra case, of results (in [L]) for the group algebra case - of an orthonormal basis for $P_n^{M^H \subset M}$ as well as the partition function for $P^{M^H \subset M}$.

2 Planar algebras

This section is devoted to a survey of such facts about planar algebras as we will require. (See [J1] - where these objects first appear - and [J2] - where the operadic approach is discussed - and also [L] for a 'crash-course'.)

We define a set *Col*, whose members we shall loosely call 'colours', by

$$Col = \{0_+, 0_-, 1, 2, 3, \cdots\} \tag{2.1}$$

Some of the basic objects here are the so-called k-tangles (where $k \in Col$), towards whose definition we now head.

Consider a copy D_0 of the closed unit disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$, together with a collection $\{D_i : 1 \leq i \leq b\}$ of some number b (which may be zero) of pairwise disjoint (adequately compressed) copies of D in the interior of D_0. Suppose now that we have a pair (T, f), where:

(a) T is an oriented compact one-dimensional submanifold of $D_0 \setminus \cup_{i=1}^b Int(D_i)$ - where 'Int' denotes interior - with the following properties:

(i) $\partial(T) \subset \cup_{i=0}^b \partial(D_i)$ and all intersections of T with ∂D are transversal;

(ii) each connected component of the complement of T in $Int(D_0) \setminus \cup_{i=1}^b Int(D_i)$ comes equipped with an orientation which is consistent with the orientation of T;

(iii) $|T \cap \partial(D_i)| = 2k_i$ for some integers $k_i \geq 0$, for each $0 \leq i \leq b$; and

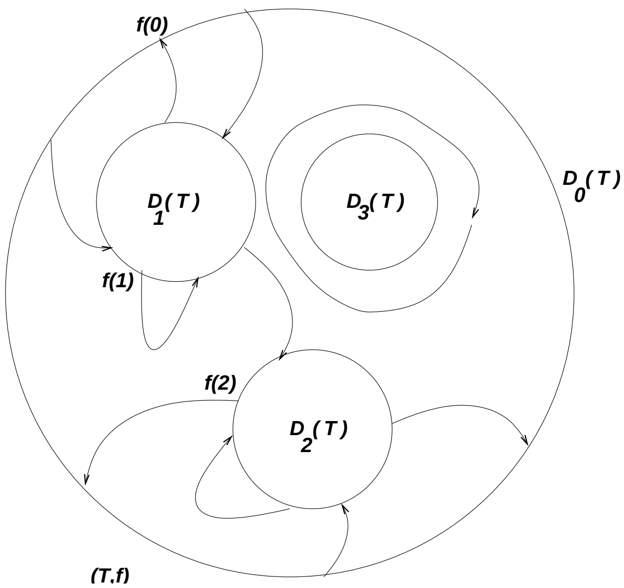
(b) f specifies some 'distinguished points' thus: whenever $0 \leq i \leq b$ is such that $k_i > 0$, we are given a 'distinguished point' $f(i) \in T \cap \partial(D_i)$; these distinguished points (which we will sometimes follow Jones and simply denote by $*$) are required to satisfy the following 'compatibility condition' with respect to the orientation of (a)(ii):

the component of T which contains $f(i)$ is required to be oriented (at $f(i)$) away from or towards ∂D_i according as $i > 0$ or $i = 0$.

The orientation requirement above ensures that there is a unique *chequerboard shading* of $Int(D_0) \setminus (\cup_{i=1}^{b} Int(D_i) \cup T)$ as follows: shade a component white or black according as it is equipped with the mathematically positive or negative orientation in (a)(ii) above. Thus, whenever one moves along any component of T in the direction specified by its orientation, the region immediately to one's right is shaded black. With the above notation, there are two possibilities, for each $0 \leq i \leq b$: (i) $k_i > 0$, in which case we shall say that D_i is of colour k_i; and (ii) $k_i = 0$; in this case, we shall say that D_i is of colour $0_\pm$ according as the region immediately adjacent to ∂D_i is shaded white or black in the 'chequerboard shading'.

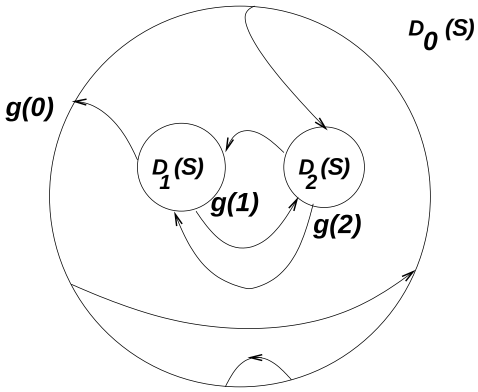
We shall consider two such pairs (T_i, f_i) to be *equivalent* if the T_i are isotopic via an isotopy which preserves the orientation and the 'distinguished points'. Finally, an equivalence class as above is called a k**-tangle** where k is the colour of the external disc D_0.

An example of a 3-tangle with 3 internal discs is illustrated here - in which $b = 3$ and the internal discs D_1, D_2 and D_3 have colours $3, 3$ and 0_- respectively:

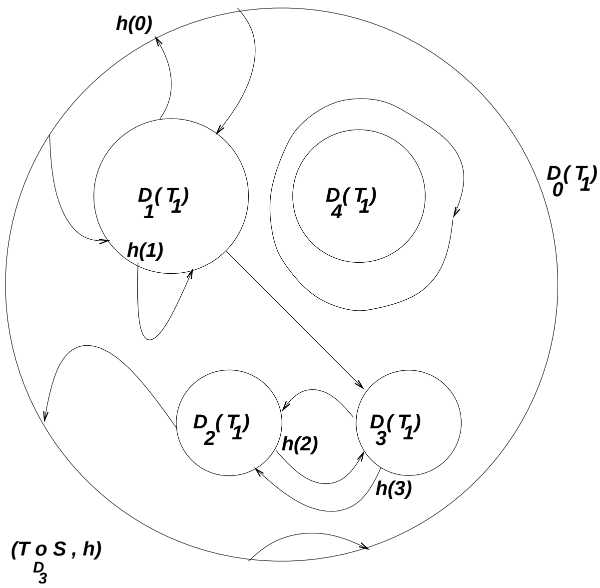


There is a natural way to 'compose' two tangles. For instance, suppose (T, f) is a k-tangle, with $b \geq 1$ internal discs; if one of these internal discs D_i has colour k_i, and if (S, g) is a k_i-tangle, then $T \circ_{D_i} S$ is the k-tangle obtained by 'glueing S into D_i' (taking care to attach $g(0)$ to $f(i)$ in case $k_i > 0$).

For example, if (T, f) is as above and if (S, g) is the 3-tangle given by



then a possible 'composite tangle' is the 3-tangle $(T_1 = T \circ_{D_2} S, h)$ given thus:



The collection of 'coloured tangles' with the 'composition' defined above is referred to as the *(coloured) planar operad*; and by a **planar algebra** is meant an 'algebra over this operad': in other words, a planar algebra P is a family $P = \{P_k : k \in Col\}$ of vector spaces with the following property: for every $k_0 = k_0(T)$-tangle (T, f) with $b = b(T)$ internal discs $D_1(T), \cdots, D_{b(T)}(T)$ of colours $k_1(T), \cdots, k_{b(T)}(T)$, there is associated a linear map

$$Z_T : \otimes_{i=1}^b P_{k_i} \rightarrow P_{k_0}$$

which is 'compatible with respect to composition of tangles' in the following obvious manner.

If $1 \leq i \leq b$ is fixed, and if (S, g) is a $k_i(T)$-tangle with $b(S)$ internal discs - call them $D_1(S), \cdots, D_{b(S)}(S)$ - with colours $k_1(S), \cdots, k_{b(S)}(S)$ (say) - then we know that the composite tangle $T_1 = T \circ_{D_i(T)} S$ is a k_0-tangle with the $(b(T) + b(S) - 1)$ internal discs given by

$$D_j(T_1) = \begin{cases} D_j(T) & \text{if } 1 \leq j < i \\ D_{j-i+1}(S) & \text{if } i \leq j \leq i + b(S) - 1 \\ D_{j-b(S)+1}(T) & \text{if } i + b(S) \leq j \leq b(T) + b(S) - 1 \end{cases} ;$$

it is required that the following diagram commutes:

$$
\begin{array}{lcr}
(\otimes_{j<i} P_{k_j(T)}) \otimes (\otimes_{j=1}^{b(S)} P_{k_j(S)}) \otimes (\otimes_{j>i} P_{k_j(T)}) & & \\
 & Z_{T \circ_{D_i(T)} S} \searrow & \\
id \otimes Z_S \otimes id \ \downarrow & & P_{k_0(T)} \\
 & Z_T \nearrow & \\
\otimes_{j=1}^{b(T)} P_{k_j(T)} & &
\end{array}
\tag{2.2}
$$

Strictly speaking, we need to exercise a little caution when 0 is involved. For instance, in order to make sense of the domain of Z_T, when the tangle T has no internal discs (i.e., $b(T) = 0$), we need to adopt the convention that the empty tensor product is the underlying field, which we shall always assume is $\mathbb{C}$. (So each P_k has a distinguished subset, *viz.,* $\{Z_T(1) : T$ a k-tangle without internal discs $\}$.)

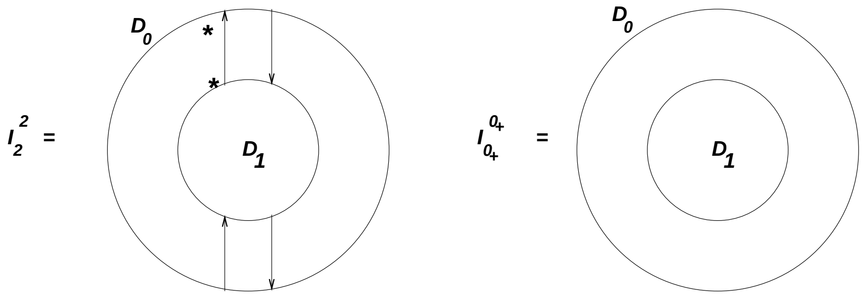
Next, our statement of the 'compatibility requirement (2.2)' needs to be slightly modified if the tangle S has no internal discs; thus, if $b(S) = 0$, the requirement (2.2) needs to be modified thus:

$$
\begin{array}{lcr}
\otimes_{j \neq i} P_{k_j(T)} & & \\
\cong \ \downarrow & Z_{T \circ_{D_i(T)} S} \searrow & \\
(\otimes_{j<i} P_{k_j(T)}) \otimes \mathbb{C} \otimes (\otimes_{j<i} P_{k_j(T)}) & & P_{k_0(T)} \\
id \otimes Z_S \otimes id \ \downarrow & Z_T \nearrow & \\
\otimes_{j=1}^{b} P_{k_j(T)} & &
\end{array}
\tag{2.3}
$$

Further, we need to make an additional assumption in order to 'rule out some degeneracies'. To see this, consider the k-tangles I_k^k[3], $k \in Col$, with one internal disc also of colour k - thus, in our notation, $b(I_k^k) = 1, k_0(I_k^k) = k_1(I_k^k) = k$ - defined

[3]In the sequel we shall consistently use the convention of writing T_n^m to denote a tangle T with $b(T) = 1, k_0(T) = m, k_1(T) = n$; so for instance we might have $T_k^m \circ S_n^k = (T_1)_n^m$ as with matrix multiplication

as in the figure below:



(The understanding is that $I_{0_+}^{0+}$ consists of the 'empty submanifold of $D_0 \setminus D_1$' and that the annular region $D_0 \setminus D_1$ is equipped with the 'mathematically positive orientation' and hence shaded white in the chequerboard shading. In the case of $I_{0_+}^{0+}$, the only difference is that the annular region is shaded black.)

It is easily seen that, for every k, and for every k-tangle T, we have $I_k^k \circ_{D_1} T = T$, and hence

$$Z_{I_k^k} \circ Z_T = Z_T \ . \tag{2.4}$$

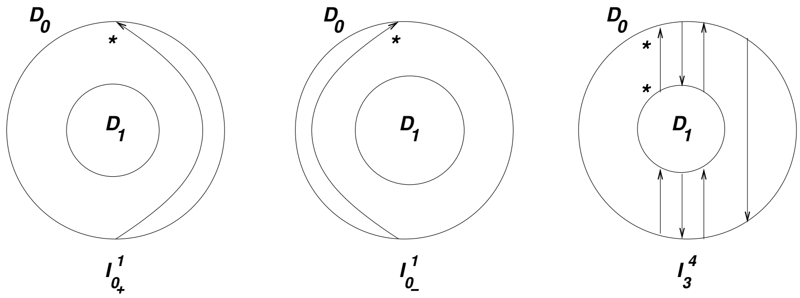
It follows that $Z_{I_k^k}$ is an idempotent endomorphism of P_k whose range contains the range of Z_T for every k-tangle T.

The *non-degeneracy condition* we wish to impose is that P_k is spanned by the ranges of the Z_T's, as T ranges over all k-tangles; in view of the above comments, this is equivalent to the following condition, which we shall henceforth assume is satisfied by all our planar algebras:

$$Z_{I_k^k} = id_{P_k} \ \forall \ k \in \ Col. \tag{2.5}$$

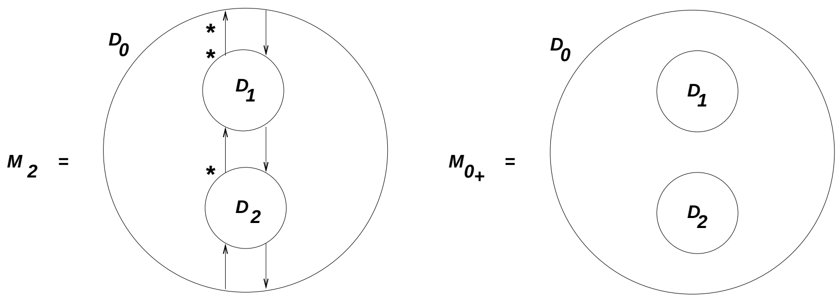
We shall need the following tangles:

The inclusion tangles: For every $k \in Col$, there is an associated $(k+1)$-tangle I_k^{k+1} with one internal disc of colour k - where of course $0_\pm + 1 = 1$; rather than giving the formal definition, we just illustrate $I_{0_+}^1, I_{0_-}^1$ and I_3^4 below - the idea being that an 'extra vertical line is stuck on to the far right (in all but one exceptional case)'.



It should be clear that $Z_{I_k^{k+1}} : P_k \to P_{k+1}$. It will turn out that these 'inclusion' tangles indeed induce injective maps in the case of 'good' planar algebras (the ones with a 'non-zero modulus').

The product tangles: For each $k \in Col$, these are k-tangles M_k with two internal discs, both of colour k, which equip P_k with a multiplication. We illustrate the cases $k = 2$ and $k = 0_+$ below:



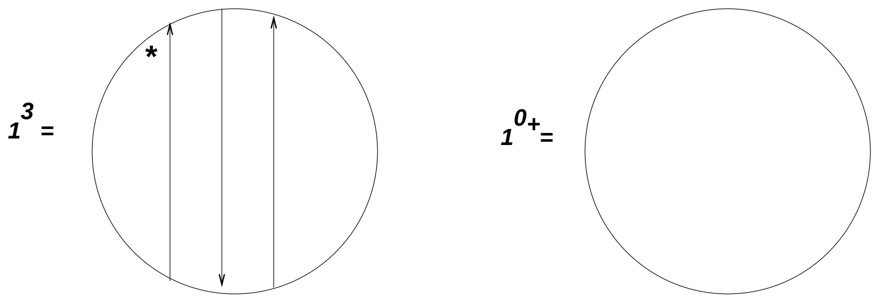
(As in the case of the 'identity annular tangles $I^{0_\pm}_{0_\pm}$, the tangles $M_{0_\pm}$ consist only of the empty submanifold (of $D_0 \setminus \cup_{i=1}^2 Int(D_i)$), the only distinction between $M_{0_\pm}$ being that the region $D_0 \setminus \cup_{i=1}^2 Int(D_i)$ is shaded white and black in M_{0_+} and M_{0_-}, respectively.)

It is easy to see that each P_k is an associative algebra, with respect to multiplication being defined by

$$x_1 x_2 = Z_{M_k}(x_1 \otimes x_2) \ .$$

It must be noted that this convention - of putting the first factor in the disc on top - is opposite to the one adopted in [BJ], for instance; and also that $P_{0_\pm}$ are even commutative.

We also wish to point out that the fact that the P_k's are unital algebras is a consequence of our 'non-degeneracy condition' and of the compatibility condition (2.3): in fact, consider the k-tangle 1^k, which has no internal discs, defined analogous to the case 1^3 illustrated below: (The tangle 1^{0+} again is the empty submanifold of D_0, with the interior of D_0 shaded white; and 1^{0-} is defined analogously except that 'white' is replaced by 'black'.)

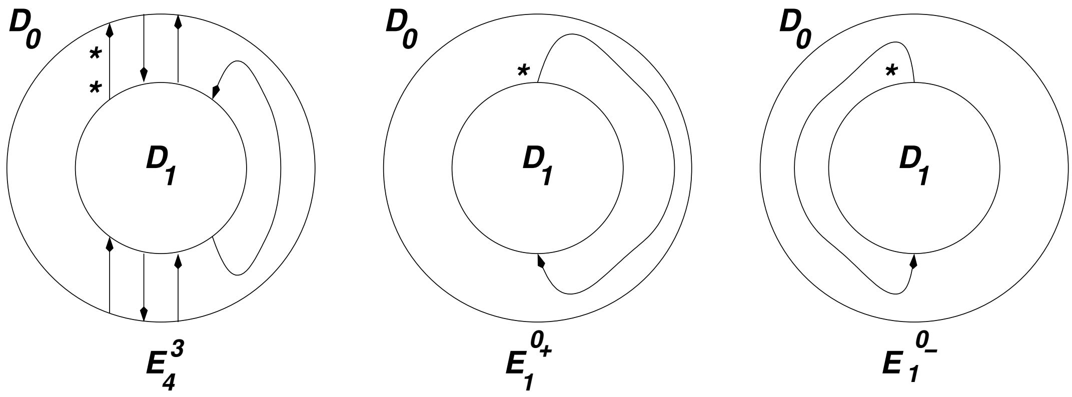


Notice that $M_k \circ_{D_2} 1^k = I_k^k$, and if we write $1_k = Z_{1^k}(1)$ (where the 1 on the right is the 1 in $\mathbb{C}$), then we may deduce from (2.3) that for arbitrary $x \in P_k$:

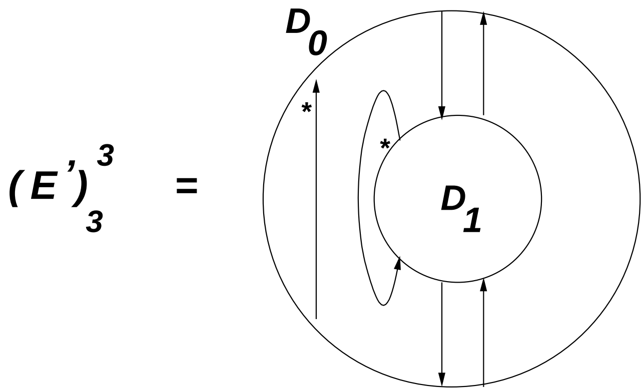
$$\begin{aligned} x \cdot 1_k &= Z_{M_k}(x \otimes 1_k) \\ &= Z_{M_k}(x \otimes Z_{1^k}(1)) \\ &= Z_{M_k \circ_{D_2} 1^k}(x) \\ &= Z_{I_k^k}(x) \\ &= x \; . \end{aligned}$$

A similar argument - with D_2 replaced by D_1 shows that 1_k is also a 'left-identity'. Hence P_k is a unital associative algebra with 1_k as the multiplicative identity. A similar argument also shows that the 'inclusion tangles' in fact induce homomorphisms of unital algebras - so that, in 'good cases', any planar algebra admits the structure of an associative unital algebra which is expressed as an increasing union of subalgebras.

The conditional expectation tangles: These are two families of tangles $\{E_{k+1}^k : k \in Col\}$, and $\{(E')_k^k : k \geq 1\}$, where (by our notational convention for 'annular tangles') (i) E_{k+1}^k is a k-tangle with one internal disc of colour $k+1$, which is defined by ‘capping off the last strand’; again, rather than giving a formal definition, we illustrate E^3_4, E^{0+}_1 and E^{0-}_1 below:



and (ii) $(E')^k_k$ is a k-tangle with one internal disc of colour k, which is defined by ‘capping to the left’; again, rather than giving a formal definition, we illustrate $(E')^3_3$ below:



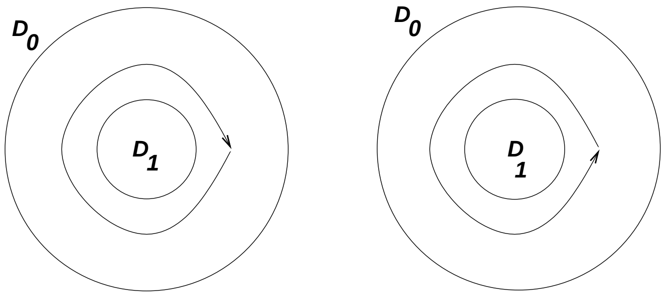
Clearly $Z_{E^k_{k+1}} : P_{k+1} \to P_k$ while $Z_{(E')^k_k} : P_k \to P_k$. (In fact, the range of $Z_{(E')^k_k}$ is contained in $Z_{I^k_{k-1}} \circ \cdots \circ Z_{I^2_1}(P_1)' \cap P_k$.)

The planar algebras that we will be encountering have various additional good features, which we now outline.

Connectedness: A planar algebra P is said to be *connected* if $dim\ P_{0_\pm} = 1$.

Since $P_{0_\pm}$ are unital $\mathbb{C}$-algebras, it follows that if P is connected, then there exist unique algebra isomorphisms $P_{0_\pm} \cong \mathbb{C}$; they will necessarily identify what we called $1_{0_\pm}$ (recall the 1_k above) with $1 \in \mathbb{C}$.

For the next definition, we need to introduce two more tangles. Consider the tangles $T_{\pm}$ of colours $0_{\pm}$, and without any internal discs, given by:



Modulus: A planar algebra P is said to have *modulus* δ if there exists a scalar δ such that $Z_{T_{\pm}}(1) = \delta \ 1_{\pm}$. We will primarily be interested in the case when the modulus is positive.

It must be noted that if P has modulus δ, then

$$Z_{E^k_{k+1}} \circ Z_{I^{k+1}_k} = \delta \ id_{P_k} \ \forall k \in \ Col \ ;$$

and in particular, if $\delta \neq 0$, then the 'inclusion tangles' do induce injective maps.

Finite-dimensionality: A planar algebra P is said to be finite-dimensional if $dim \ P_k \ < \infty \ \forall k \in \ Col$.

Suppose P is a connected planar algebra and that T is a 0-tangle (by which we shall mean a 0_+- or a 0_--tangle). If T has internal discs D_i of colour k_i, and if $x_i \in P_{k_i}$ for $1 \leq i \leq b$, then $Z_T(\otimes_{i=1}^b x_i) \ \in \mathbb{C}$, where we have made the canonical identifications $P_{0_{\pm}} = \mathbb{C}$. This assignment of scalars to 'labelled 0-tangles' is also referred to as the *partition function associated to the planar algebra.*

Sphericality: A planar algebra is said to be *spherical* if its partition function assigns the same value to any two 0-tangles which are isotopic as tangles on the 2-sphere (and not just the plane).

The last bit of terminology we will need is that of the 'adjoint of a tangle'. Suppose (T, f) is a k_0-tangle as defined earlier, with external disc D_0 and b internal discs D_i of colours k_i. We then define its adjoint to be the k_0-tangle (T^*, f^*) given thus:

(a) Let ϕ be any orientation reversing smooth map of D_0 onto a disc D_0^* and let T^* be defined by requiring that its external disc is D_0^*, its internal discs are $\{D_i^* = \phi(D_i) : 1 \leq i \leq b\}$, and its underlying one-submanifold of D_0^* is $\phi(T)$, and the orientation - of T^* as well as the components of its complement in $(D_0^* \setminus \cup_{i=1}^b D_i^*)$ - is opposite to the one inherited via ϕ. (In other words, a region $\phi(R)$ - in the complement of T^* in $(D_0^* \setminus \cup_{i=1}^b D_i^*)$ - has the same colour as R in the chequerboard shading.)

(b) If $k_i > 0$, define $\widetilde{f(i)}$ to be the 'first point' of $T \cap \partial(D_i)$ that is encountered as one proceeds anti-clockwise along $\partial(D_i)$ from $f(i)$; and define $f^*(i) = \phi(\widetilde{f(i)})$.

Finally, we shall say that P is a *subfactor planar algebra* if:

(i) P is connected, finite-dimensional, spherical, and has positive modulus;

(ii) each P_k is a C^*-algebra in such a way that, if (T, f) is a k_0-tangle as above, with external disc D_0 and b internal discs D_i of colours k_i, and if $x_i \in P_{k_i},\ 1 \leq i \leq b$, then

$$Z_T(x_1 \otimes \cdots \otimes x_b)^* \ = \ Z_{T^*}(x_1^* \otimes \cdots \otimes x_b^*) \ ;$$

and

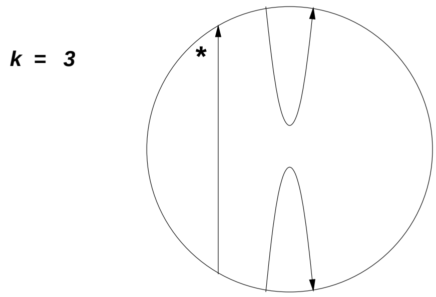
(iii) if we define the 'pictorial trace' on P by

$$tr_{k+1}(x)\ 1_+ \ = \ \delta^{-k-1} Z_{E_1^{0+}} Z_{E_2^1} \cdots Z_{E_{k+1}^k}(x) \qquad (2.6)$$

for $x \in P_{k+1}$, then tr_m is a faithful positive trace on P_m for all $m \geq 1$.

It should be obvious that if P is a subfactor planar algebra, the 'tr_m's are consistent and yield a 'global trace tr on P'.

Our primary interest in planar algebras stems from a beautiful result - Theorem 2.1 below - of Jones' (see [J1]). Before stating it, it will be convenient for us to introduce another family $\{\mathcal{E}^k : k \geq 2\}$, of tangles, where $\mathcal{E}^k$ is a k-tangle with no internal discs; we illustrate the case $k = 3$ below:



Theorem 2.1 *Let*

$$N \subset M (= M_0) \subset^{e_1} M_1 \subset \cdots \subset^{e_k} M_k \subset^{e_{k+1}} \cdots$$

be the tower of the basic construction associated to an extremal subfactor with $[M : N] = \delta^2 < \infty$. *Then there exists a unique subfactor planar algebra* $P = P^{N \subset M}$ *of modulus* δ *satisfying the following conditions:*

(0) $P_k^{N \subset M} = N' \cap M_{k-1}$ $\forall k \geq 1$ *- where this is regarded as an equality of *-algebras which is consistent with the inclusions on the two sides;*

(1) $Z_{\mathcal{E}^{k+1}}(1) = \delta \; e_k \; \forall \; k \geq 1$;

(2) $Z_{(E')_k^k}(x) = \delta \; E_{M' \cap M_{k-1}}(x) \; \forall \; x \in N' \cap M_{k-1}, \; \forall k \geq 1$;

(3) $Z_{E_{k+1}^k}(x) = \delta \; E_{N' \cap M_{k-1}}(x) \; \forall \; x \in N' \cap M_k$; *and this is required to hold for all* k *in* Col, *where for* $k = 0_\pm$, *the equation is interpreted as*

$$Z_{E_1^{0_\pm}}(x) \;=\; \delta \; tr_M(x) \; \forall \; x \in N' \cap M \; .$$

Conversely, any subfactor planar algebra P *with modulus* δ *arises from an extremal subfactor of index* δ^2 *in this fashion.*

Remark 2.2 (a) If P is any planar algebra with non-zero modulus, we have an induced tower

$$P_1 \subset \cdots \subset P_k \subset \cdots$$

of unital associative algebras. We shall - taking a cue from (1) of the above theorem - define

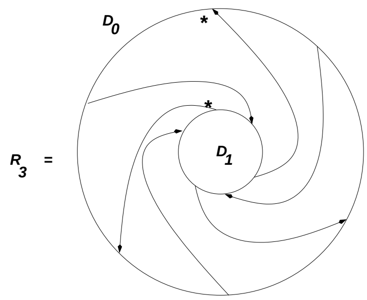
$$e_k = \delta^{-1} Z_{\mathcal{E}^{k+1}}(1) \; , \; \forall \; k \geq 1 \; .$$

It then follows that for all k, we have

$$\begin{aligned} e_k^2 &= e_k \\ e_k e_m &= e_m e_k \; if \; |k-m| > 1 \\ e_k e_{k\pm 1} e_k &= \delta^{-2} e_k \; . \end{aligned} \tag{2.7}$$

Of course if P is a C^*-planar algebras, the e_k's will be genuine (= self-adjoint) projections.

(b) We want to single out one specific class of tangles which play a very important role in the proof of the above theorems as well as in the general theory. These are the family of tangles $\{R_k : k \geq 2\}$; (since this tangle will occur frequently in the sequel, we shall, in the interest of convenience, drop our (otherwise) standing convention for annular tangles, and write R_k rather than R_k^k). This *rotation tangle* R_k is a k-tangle with one internal disc of colour k, and we shall just illustrate R_3:



3 Presentations of planar algebras

Jones' initial approach - in [J1] - to planar algebras was through a 'generators and relations approach' - which can be quickly shown to be equivalent to the 'operadic' approach which was later espoused by Jones (see [J2]) and was the one presented in §2 here. This is analogous to the two approaches one may adopt towards group theory. On the one hand, one could - analogous to our approach here - take the axiomatic definition; then construct the example of a 'free group on an arbitrary set L of generators'; then consider the quotient $< L : R >$, of the free group on the generating set L, by the smallest normal subgroup generated by a set R of relations; and finally prove that every group arises in this fashion. Or one could - analogous to Jones - simply define a group as something which is given as an $< L : R >$.

The analogue, for planar algebras, of the free group is the so-called *universal planar algebra on the label set* L which is defined thus: suppose $L = \coprod_{k \in Col} L_k$ is the disjoint union of an arbitrary collection L_k of 'label sets' - where some L_k may be empty. Define a *k_0-tangle labelled by* L to be a k_0-tangle (T, f) as above subject to one additional constraint - *viz.,* that T is allowed to have an internal disc of colour m only if $L_m \neq \emptyset$ - and equipped with the extra structure of a label from L_m associated to every internal disc of colour m. (Of course, two such tangles which are isotopic are considered to be the same.) Define $P_k(L)$ to be the vector-space with basis given by the collection of 'k-tangles labelled by L', and let $P(L) = \{P_k(L) : k \in Col\}$. It is not hard to see that $P(L)$ has a natural structure of a planar algebra.

A family $J = \{J_k : k \in Col\}$ is said to be a *planar ideal* of a planar algebra $P = \{P_k : k \in Col\}$ if (i) J_k is a vector subspace of P_k for each k, and (ii) if (T, f) is any k-tangle, with b internal discs D_i of colour k_i for $1 \leq i \leq b$, then it is demanded that $Z_T(\otimes_{i=1}^b x_i) \in J_k$ whenever $x_i \in J_{k_i}$ for any one i.

It is a simple matter to verify that if J is a planar ideal of a planar algebra as above, and if we define $P/J = \{P_k/J_k : k \in Col\}$, then P/J is naturally a planar algebra and that the natural quotient maps define a 'morphism of planar algebras' - where a morphism from the planar algebra P to a planar algebra Q is just a collection of linear maps $\pi_k : P_k \to Q_k$ which satisfy the obvious compatibility requirement that

$$\pi_{k_0}(Z_T^P(\otimes_{i=1}^b x_i)) = Z_T^Q(\otimes_{i=1}^b \pi_{k_i}(x_i))$$

whenever T is a k_0-tangle with b internal discs D_i of colour k_i, and $x_i \in P_{k_i}$ for $1 \leq i \leq b$.

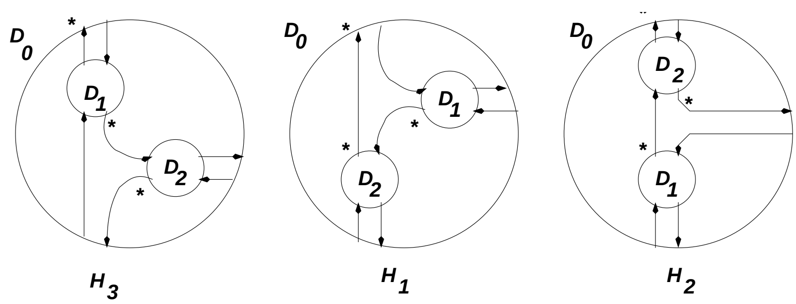
It is easy to see that these are the 'correct' definitions in the sense that planar ideals are precisely the kernels of morphisms of planar algebras, and images under epimorphisms are isomorphic to quotients by planar ideals. We are now ready for the analogue of what we called $< L : R >$ in the group context.

Definition 3.1 *Given an arbitrary 'label set'* $L = \coprod_{k \in Col} L_k$, *and an arbitrary 'subset'* $R = \{R_k : k \in Col\}$ *of the 'universal planar algebra* $P(L)$ *with generating set* L*', let* $J(R)$ *be the smallest planar ideal 'containing* R*', and define* $P(L;R)$ *to be the quotient* $P(L)/J(R)$.

An abstract planar algebra Q *will be said to be 'presented on the generating set* L *with the relations* R*' if there exists an isomorphism* $\Phi : P(L;R) \to Q$ *of planar algebras; and we shall say that* Q *is 'presented' by the map* Φ.

The planar algebras $P(L;R)$ will typically not possess many of the nice features of what we have called a subfactor planar algebra. However, a condition has been identified in [L], that is known to ensure some of these 'good' properties. We shall discuss this condition, which we will need. We pause with a digression concerning notation which might be slightly different from the notation of [L], in order to dispell possible confusion in the reader. We shall indicate the 'basis vector' $Z_T(g_1, \cdots, g_b)$ by drawing the tangle T and labelling the i-th internal disc with a g_i. Also, we shall indicate the 'distinguished points' $f(i)$ by simply marking a *; and rather than adopting Landau's convention of drawing labels which are 'upside down', we shall draw the labels 'upright' but the position of the * will indicate the difference.

Our description of the condition will be facilitated by the introduction of the following three 3-tangles, each having two internal discs of colour 2.

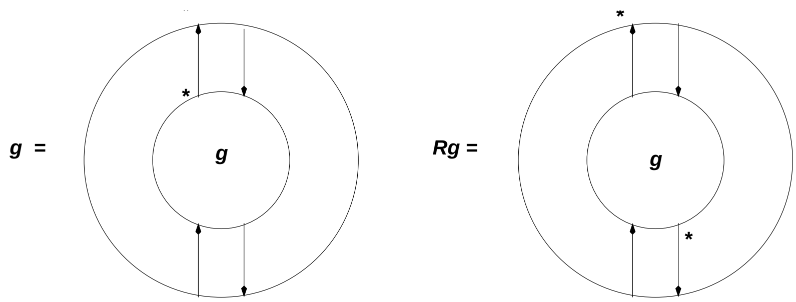


One way to remember our convention for these tangles is to remember that: (a) all the tangles have the internal discs straddling a pair of strings that 'go into the junction', and the distinguished point (*) of both internal discs 'face the junction'; (b) if the points of $\partial T \cap D_0$ are labelled $* = 1, 2, \cdots, 6$, the tangle called H_i does not have an internal disc on the strands labelled $2i-1$ and $2i$; and (c) for $1 \leq j \leq 2$ and $1 \leq i \leq 3$, the internal disc D_j of the tangle H_i straddles the strands labelled $2i+2j-1$ and $2i+2j$. We shall later use the following consequence of the circular symmetry of these definitions:

$$R_3 \circ H_i = H_{i+2} \ \forall i \ , \tag{3.8}$$

where all indices above are mod 3.

For a set L_2, we shall write $\tilde{L_2} = \{Z_{I_2^2}(g) : g \in L_2\} \coprod \{Z_{R_2}(g) : g \in L_2\}$. So, if $L_2 = \{g\}$ is a singleton, then $\tilde{L_2}$ consists of the two L_2-labelled tangles below:



We are finally ready for the condition.

Definition 3.2 *Assume that $L = L_2$ and that $L_k = \emptyset$ for $k \neq 2$. An* **exchange relation algebra** *is a planar algebra of the form $P(L; R)$, where it is assumed that the set $R \subset P(L)$ of relations satisfies the following conditions:*

(0) there exists a positive number δ such that

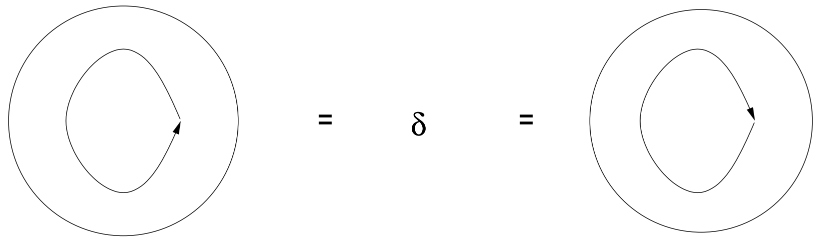
$$Z_{T_+}(1) - \delta 1_{0_+} \ , Z_{T_-}(1) - \delta 1_{0_+} \ \in R \ ;$$

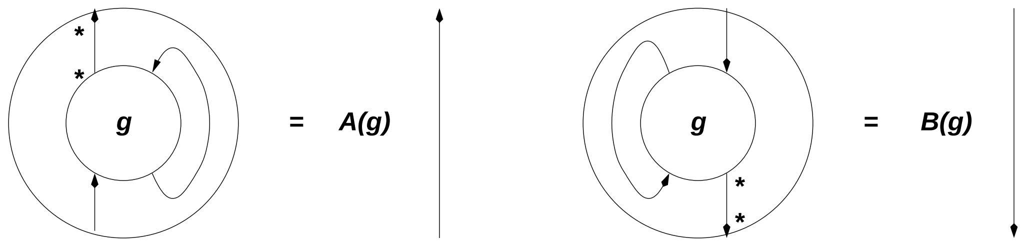
(1) for every $g \in \tilde{L_2}$, there exist scalars $A(g), B(g)$ such that
(i) $Z_{E_2^1}(g) - A(g)1_1 \in R$;
(ii) $Z_{(E')_2^2}(g) - B(g)1_1 \in R$;
and

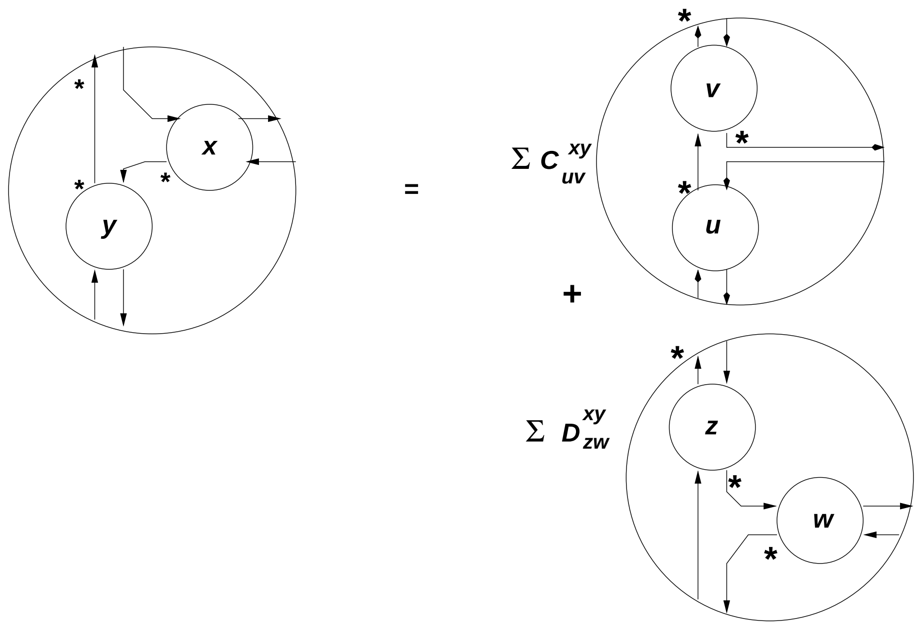
(2) For all $x, y, u, v, z, w \in \tilde{L_2}$, there exists scalars C^{xy}_{uv}, D^{xy}_{zw} such that

$$Z_{H_1}(x, y) - \left(\sum_{u,v \in \tilde{L_2}} C^{xy}_{uv} Z_{H_2}(u, v) + \sum_{z,w \in \tilde{L_2}} D^{xy}_{zw} Z_{H_3}(z, w) \right) \ \in \ R \ .$$

In other words, the following ‘identities’ are assumed to hold in $P(L;R)$ if R is an ‘exchange relation’:







We will need the following facts from [L], which we state as a proposition for convenience of reference:

Proposition 3.3 *Suppose $P(L;R)$ is an exchange relation planar algebra. Then,* *(i) dim* $P_n(L;R) < \infty \ \forall n$*;*

(ii) $P(L;R)$ *is connected and has positive modulus* $(= \delta)$*; and further,*

(iii) $P_2(L;R)$ *is linearly spanned by* $\{1, e_1\} \cup \tilde{L_2}$ *- where, of course,* $e_1 = Z_{\mathcal{E}_1}(1)$*.*

We conclude this section with some remarks on how to get a *-algebra structure on $P(L;R)$. As a first step, let us try to impose a *-structure on the universal planar algebra $P(L)$ thus: assume that each L_k has an involution, denoted $L_k \ni g \mapsto g^* \in L_k$; note that the typical basis-vector of $P_k(L)$ may be envisaged as $Z_T(g_1, \cdots, g_b)$, where T is a k-tangle with b internal discs $D_1, \cdots, D_b$, and $g_i \in L_{k_i}$ (where of course k_i is the colour of D_i); define $Z_T(g_1, \cdots, g_b)^* = Z_{T^*}(g_1^*, \cdots, g_b^*)$; and extend the adjoint conjugate-linearly to all of $P(L)$. (If we start with a k-tangle T with no internal discs, the corresponding basis vector should be thought of as $Z_T(1)$, whose adjoint is then given by $Z_{T^*}(1)$.) It is an easy matter to see that this gives each $P_k(L)$ the structure of a *-algebra; and further, that if, $\cup_k R_k$ is a subset of $P(L)$ which is closed under the involution, then each $P_k(L;R)$ also has a natural *-algebra structure.

4 Kac algebras

We recall some standard facts concerning finite-dimensional Kac algebras. (The reader may consult [vD], for instance, for proofs and details.) We employ standard notation; a finite-dimensional[4] Hopf algebra will be a tuple $(H, \mu, \eta, \Delta, \epsilon, S)$. We shall consistently reserve the symbol n thus:

$$dim \ H = n \ .$$

A complex (finite-dimensional) Hopf algebra is said to be a Kac algebra if it is a C^*-algebra in such a way that the comultiplication Δ is a *-homomorphism. It is true that if H is a Kac algebra, then

[4]We shall not consider any other kind; thus we shall tacitly assume that all our Hopf algebras are finite-dimensional, and over the field $\mathbb{C}$.

(a) S and $*$ are commuting involutory product-reversing maps of H (the one being linear and the other conjugate-linear); and (b) the dual Hopf algebra $(H^*, \Delta^*, \epsilon^*, \mu^*, \eta^*, S^*)$ is also a Kac algebra with respect to adjunction given by $\psi^*(a) \ = \ \overline{\psi(Sa^*)}$.

Define $h \in H$ (resp., $\phi \in H^*$) to be the unique central minimal projection satisfying $hx = \epsilon(x)h, \ \forall x \in H$ (resp., $\psi\phi = \psi(1_H)\phi, \ \forall \psi \in H^*$) - where we naturally write $1_H = \eta(1_{\mathbb{C}})$. It is a fact that h (resp., ϕ) defines a faithful tracial state on H^* (resp., H), and that

$$\phi(h) = 1/n \ . \tag{4.9}$$

Further, we shall regard H (resp., H^*) as being equipped with the inner product derived from the trace ϕ (resp., h) thus:

$$\langle a, b\rangle_H = \phi(b^*a) \ ; \langle \psi, \rho\rangle_{H^*} = h(\rho^*\psi) \ .$$

Once and for all, we shall choose a 'system of matrix units' for the 'multi-matrix algebra' H^*, call it $\{e^\gamma_{kl} : 1 \leq k, l \leq d_\gamma, \gamma \in \hat{H}^*\}$ and denote the associated dual basis for H by $\{\gamma_{kl} : 1 \leq k, l \leq d_\gamma, \gamma \in \hat{H}^*\}$. In terms of these bases, the 'integral' h is known to decompose as:

$$h \ = \ 1/n \sum_{\gamma \in \hat{H}^*} d_\gamma \sum_{k=1}^{d_\gamma} \gamma_{kk} \ . \tag{4.10}$$

Further, these bases are known to constitute orthogonal bases for the underlying Hilbert spaces; thus, if we define

$$\widetilde{e^\gamma_{kl}} \ = \ \sqrt{\frac{n}{d_\gamma}} \, e^\gamma_{kl} \tag{4.11}$$

$$\widetilde{\gamma_{kl}} \ = \ \sqrt{d_\gamma} \, \gamma_{kl} \ , \tag{4.12}$$

then $\{\widetilde{e^\gamma_{kl}} : 1 \leq k, l \leq d_\gamma, \gamma \in \hat{H}^*\}$ is an orthonormal basis for H^*, and $\{\widetilde{\gamma_{kl}} : 1 \leq k, l \leq d_\gamma, \gamma \in \hat{H}^*\}$ is an orthonormal basis for H.

We shall write $triv$ for the trivial representation of H^* - given by evaluation at 1_H; since $d_{triv} = 1$, we shall simply write $triv$ rather than $triv_{11}$ (which is the same as $\widetilde{triv_{11}}$). Similarly $e^{triv} = e^{triv}_{11} = \sqrt{\frac{1}{n}}\widetilde{e^{triv}_{11}}$. What needs to be noted is that

$$triv = 1_H \ , \ e^{triv} = \phi \ . \tag{4.13}$$

We list a few simple facts as a lemma, for convenience of reference.

LEMMA 4.1 *With the above notation, we have:*

(i) $\Delta(\gamma_{kl}) = \sum_m \gamma_{km} \otimes \gamma_{ml}$, *and* $\epsilon(\gamma_{kl}) = \delta_{kl}$;

(ii) $\sum_m \gamma_{km} S\gamma_{ml} = \delta_{kl} 1_H$;

(iii) $S\gamma_{kl}^* = \gamma_{lk}$;

(iv) $\Delta(h) = \Delta^{op}(h)$, *where* $\Delta^{op} = \tau \circ \Delta$ *and* τ *denotes the 'flip' on* $H \otimes H$; *and*

(v) $\Delta(h)(1 \otimes a) = \Delta(h)(Sa \otimes 1)$, $\forall a \in H$.

The assertions (i), (iii) and (iv) are immediate consequences of the definitions, while (ii) follows from (i) and the defining property of S, and the proof of (v) may be found in [vD]. □

We should mention that we will, when necessary, use Sweedler's notation - according to which, for example, $\Delta(h) = \sum h_1 \otimes h_2$, while assertion (iv) of the above Lemma says that this is also equal to $\sum h_2 \otimes h_1$.

For the sake of completeness, and for establishing notation, we shall go through the construction of 'the crossed product of a II_1 factor M by an outer action of a Kac algebra H'.

DEFINITION 4.2 *By an action of a Kac algebra* H *on a* II_1 *factor*[5] M *will be meant a linear map* $\alpha : H \to End_{\mathbb{C}}(M)$ *(where we shall write* α_a *rather than* $\alpha(a)$ *for* $a \in H$ *and* $End_{\mathbb{C}}(M)$ *denotes the set of linear self-maps of* M*) satisfying the following conditions, for all* $a, b \in H, x, y \in M$:

(i) $\alpha_1 = id_M$;

(ii) $\alpha_{ab} = \alpha_a \circ \alpha_b$;

(iii) $\alpha_a(1_M) = \epsilon(a) 1_M$;

(iv) $\alpha_a(xy) = \sum \alpha_{a_1}(x) \alpha_{a_2}(y)$; *and*

(v) $\alpha_a(x)^* = \alpha_{Sa^*}(x^*)$.

We shall be working in the Hilbert space $L^2(M) \otimes H$ which we shall also want to think of as the direct sum of n copies of $L^2(M)$ - where of course $L^2(M) = L^2(M, tr_M)$ - these copies

[5] We restrict to II_1 factors rather than general von Neumann algebras, and we shall soon see that various maps are automatically normal.

being indexed by $\{(\rho kl) : 1 \leq k, l \leq d_\rho, \rho \in \hat{H}^*\}$. More precisely, we identify the element $\xi \otimes a$ with the vector which has $\phi(\widetilde{\rho_{kl}}^* a)\xi$ in the '(ρkl)-th coordinate'.

The above identification shows that we have a natural bijection between $End_{\mathbb{C}}(L^2(M) \otimes H)$ (the space of linear self-maps of $L^2(M) \otimes H$) and $M_n(End_{\mathbb{C}}(L^2(M)))$ given by

$$T \leftrightarrow ((t^{\rho kl}_{\sigma mn})) \;\Leftrightarrow\; T(\xi \otimes \widetilde{\sigma_{mn}}) = \sum_{\rho kl} t^{\rho kl}_{\sigma mn} \xi \otimes \widetilde{\rho_{kl}} \;. \qquad (4.14)$$

It should be clear that in equation (4.14), the map T is a bounded operator if and only if each $t^{\rho kl}_{\sigma mn}$ is. Notice that an element $x \in M$ can be - and is - identified with the (bounded) operator of 'left-multiplication by x' on $L^2(M)$. It follows then that $M_n(M)$ is a II_1-subfactor of $\mathcal{L}(L^2(M) \otimes H)$.

Consider the maps $\pi : M \to M_n(M)$ and $\lambda : H \to M_n(M)$ defined by

$$\pi(x)^{\rho kl}_{\sigma mn} \;=\; \delta_{(\rho,l),(\sigma,n)} \alpha_{S\rho_{mk}}(x) \qquad (4.15)$$

and

$$\lambda(a)^{\rho kl}_{\sigma mn} \;=\; \phi(\widetilde{\rho_{kl}}^* a \widetilde{\sigma_{mn}}) 1_M \;. \qquad (4.16)$$

It follows from the preceding paragraph that in fact $\pi(x), \lambda(a)$ can be thought of as bounded operators on $L^2(M) \otimes H$, these being the maps given by

$$\pi(x)(\xi \otimes b) \;=\; \sum \alpha_{S(b_1)}(x)\xi \otimes b_2 \qquad (4.17)$$

and

$$\lambda(a)(\xi \otimes b) \;=\; \xi \otimes ab \;. \qquad (4.18)$$

PROPOSITION 4.3 *With the above notation, we have:*
(a) π and λ are morphisms of C^*-algebras;
(b) $tr_{M_n(M)} \circ \lambda \;=\; \phi$; and
(c) $\lambda(a)\pi(x) = \sum \pi(\alpha_{a_1}(x))\lambda(a_2)$.

Proof: (a) The assertion regarding λ is obvious, since λ is just an ampliation of the left-regular representation of H. As for π, it is clear that it is a linear map, while if $x, y \in M$,

$$\begin{aligned}
\pi(xy)(\xi \otimes b) &= \sum \alpha_{S(b_1)}(xy)\xi \otimes b_2 \\
&= \sum \alpha_{S(b_2)}(x)\alpha_{S(b_1)}(y)\xi \otimes b_3 \\
&= \pi(x)(\sum \alpha_{S(b_1)}(y)\xi \otimes b_2) \\
&= \pi(x)\pi(y)(\xi \otimes b) \ ;
\end{aligned}$$

and also,

$$\begin{aligned}
\pi(x^*)^{\rho kl}_{\sigma mn} &= \delta_{(\rho,l),(\sigma,n)}\alpha_{S\rho_{mk}}(x^*) \\
&= \delta_{(\rho,l),(\sigma,n)}\alpha_{\rho^*_{mk}}(x)^* \\
&= \delta_{(\rho,l),(\sigma,n)}\alpha_{S\rho_{km}}(x)^* \\
&= (\pi(x)^{\sigma mn}_{\rho kl})^* \ .
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad tr_{M_n(M)}(\lambda(a)) &= 1/n \sum_{\rho kl} tr_M(\lambda(a)^{\rho kl}_{\rho kl}) \\
&= 1/n \sum_{\rho kl} \phi(\widetilde{\rho_{kl}}^* a \widetilde{\rho_{kl}}) \\
&= 1/n \sum_{\rho k} \phi(a \sum_l \widetilde{\rho_{kl}} S\widetilde{\rho_{lk}}) \\
&= 1/n \sum_{\rho k} \phi(d_\rho a) \text{ by Lemma 4.1(ii)} \\
&= 1/n \sum_{\rho} d_\rho^2 \phi(a) \\
&= \phi(a) \ .
\end{aligned}$$

$$\begin{aligned}
\text{(c)} \quad \sum \pi(\alpha_{a_1}(x))\lambda(a_2)(\xi \otimes b) &= \sum \pi(\alpha_{a_1}(x))(\xi \otimes a_2 b) \\
&= \sum \alpha_{S(b_1)S(a_2)a_1}(x)\xi \otimes a_3 b_2 \\
&= \sum \alpha_{\epsilon(a_1)S(b_1)}(x)\xi \otimes a_2 b_2 \\
&= \sum \alpha_{S(b_1)}(x)\xi \otimes \epsilon(a_1)a_2 b_2 \\
&= \sum \alpha_{S(b_1)}(x)\xi \otimes ab_2 \\
&= \lambda(a)\pi(x)(\xi \otimes b) \ .
\end{aligned}$$

$\square$

Lemma 4.4 *(a)* $tr_M \circ \alpha_h = tr_M$.

(b) There exists a unique morphism of C^**-algebras* $H \ni a \mapsto L_a \in \mathcal{L}(L^2(M))$ *such that* $L_a(x\Omega) = \alpha_a(x)\Omega \ \forall x \in M$ *(where of course* Ω *denotes the 'vacuum vector' in* $L^2(M)$*).*

Proof: (a) By the uniqueness of the trace on a finite factor, it suffices to verify that $tr_M \circ \alpha_h$ is a normalised trace on M; for this, note that for arbitrary $x, y \in M$,

$$\begin{aligned}
tr_M \circ \alpha_h(xy) &= \sum tr_M(\alpha_{h_1}(x)\alpha_{h_2}(y)) \\
&= tr_M(\sum \alpha_{h_2}(x)\alpha_{h_1}(y)) \text{ by Lemma 4.1(iv)} \\
&= tr_M(\sum \alpha_{h_1}(y)\alpha_{h_2}(x)) \\
&= tr_M \circ \alpha_h(yx) \ ;
\end{aligned}$$

and $tr_M \circ \alpha_h$ is normalised since $\alpha_h(1_M) = 1_M$. (*Reason:* the definition of h shows that $\epsilon(h) = 1$, and Definition 4.2(iii) now does the trick.)

(b) Begin by observing that (a) implies that for $x \in M$,

$$\begin{aligned}
tr(x^*x) &= tr(\alpha_h(x^*x)) \\
&= tr(\sum \alpha_{h_1}(x^*)\alpha_{h_2}(x)) \\
&= tr(\sum_{\rho kl} \frac{d_\rho}{n}\alpha_{\rho_{kl}}(x^*)\alpha_{\rho_{lk}}(x)) \\
&= \frac{1}{n} tr(\sum_{\rho kl} \alpha_{\widetilde{\rho_{kl}}}(x^*)\alpha_{\widetilde{\rho_{lk}}}(x)) \\
&= \frac{1}{n} tr(\sum_{\rho kl} \alpha_{\widetilde{\rho_{lk}}}(x)^*\alpha_{\widetilde{\rho_{lk}}}(x)) \ ,
\end{aligned}$$

and conclude that the operator $x\Omega \mapsto \alpha_{\widetilde{\rho_{lk}}}(x)\Omega$ extends uniquely to a bounded operator on $L^2(M)$. Since this is true for every k, l, we only need, in order to complete the proof of the lemma, to verify that $L_{a^*} = L_a^* \ \forall \ a \in H$; and this is because:

$$\begin{aligned}
\langle L_a(x\Omega), y\Omega\rangle_{L^2(M)} &= tr(y^*\alpha_a(x)) \\
&= tr(\sum \alpha_{h_1}(y^*)\alpha_{h_2 a}(x)) \text{ by Lemma 4.4(a)}
\end{aligned}$$

$$
\begin{aligned}
&= tr(\sum \alpha_{h_1 Sa}(y^*)\alpha_{h_2}(x)) \ \text{ by Lemma 4.1(v)} \\
&= tr(\alpha_{Sa}(y^*)x) \ \text{ by Lemma 4.4(a)} \\
&= tr(\alpha_{a^*}(y)^*x) \\
&= \langle x\Omega, L_{a^*}(y\Omega)\rangle_{L^2(M)} \ .
\end{aligned}
$$

□

Before proceeding further, we digress with a lemma and its corollary, which we will need (more than once). (This lemma is actually only about co-semisimple Hopf algebras, meaning the *-structure is irrelevant.) We omit the elementary proof which relies on Lemma 4.1 (ii) to check that the asserted inverse is a left-inverse, and then appeals to the underlying finite dimensionality.

LEMMA 4.5 *The element $C \in M_n(H)$ defined by*

$$C^{\rho kl}_{\mu rs} \ = \ \delta_{(\rho,l)(\mu,s)} S\rho_{rk}$$

is invertible and

$$(C^{-1})^{\mu rs}_{\rho kl} \ = \ \delta_{(\rho,l)(\mu,s)}\rho_{kr} \ .$$

COROLLARY 4.6 *Consider the matrices α_C and L_C defined by*

$$(\alpha_C)^{\rho kl}_{\mu rs} \ = \ \alpha_{C^{\rho kl}_{\mu rs}}$$

and

$$(L_C)^{\rho kl}_{\mu rs} \ = \ L_{C^{\rho kl}_{\mu rs}} \ .$$

Then α_C and L_C define invertible elements of $M_n(End\ M)$ and $\mathcal{L}(L^2(M)\otimes H)$; and their inverses are given by the obvious matrices $\alpha_{C^{-1}}$ and $L_{C^{-1}}$ respectively.

We shall also need the right regular representation of H; thus, we define $\rho : H \to M_n(\mathbb{C})$ by

$$\rho(a)^{\rho kl}_{\sigma mn} \ = \ \phi(\widetilde{\rho_{kl}}^* \widetilde{\sigma_{mn}} Sa) \ ; \tag{4.19}$$

as before, we may regard ρ as the C^*-morphism from H to $\mathcal{L}(H)$ given by

$$\rho(a)(b) \ = \ bSa \ . \tag{4.20}$$

With the foregoing notation, consider the following sets:

$$\begin{aligned}\mathcal{A}_1 &= (\pi(M)\cup\lambda(H))''\\ \mathcal{A}_2 &= \text{ the algebra generated by } (\pi(M)\cup\lambda(H))\\ \mathcal{A}_3 &= (L\otimes\rho)(\Delta^{op}(H))'\cap M_n(M)\ .\end{aligned}$$

We shall prove in Proposition 4.8 that all the $\mathcal{A}_i$'s are the same. We begin with a lemma.

Lemma 4.7 *(a) Every element of $\mathcal{A}_2$ is uniquely expressible in the form*

$$\sum_{\mu rs}\pi(y_{\mu rs})\lambda(\widetilde{\mu_{rs}}),\ \ y_{\mu rs}\in M\ .$$

(b) Every element of $\mathcal{A}_3$ is uniquely determined by its 'triv'-column.

Proof: (a) Begin by observing that since $\{\widetilde{\gamma_{kl}}\}$ forms a basis for the algebra H, the commutation relation in Proposition 4.3(c) guarantees that the set

$$\{\sum_{\mu rs}\pi(y_{\mu rs})\lambda(\widetilde{\mu_{rs}}) : y_{\mu rs}\in M\} \tag{4.21}$$

is an algebra and is consequently equal to $\mathcal{A}_2$.

Observe now that

$$\begin{aligned}\left(\sum_{\mu rs}\pi(y_{\mu rs})\lambda(\widetilde{\mu_{rs}})\right)^{\rho kl}_{triv} &= \sum_{\beta ij,\mu rs}\pi(y_{\mu rs})^{\rho kl}_{\beta ij}\lambda(\widetilde{\mu_{rs}})^{\beta ij}_{triv}\\ &= \sum_{\beta ij,\mu rs}\delta_{(\rho,l)(\beta,j)}\alpha_{S\rho_{ik}}(y_{\mu rs})\phi(\widetilde{\beta_{ij}}^{*}\widetilde{\mu_{rs}})\\ &= \sum_{\beta ij,\mu rs}\delta_{(\rho,l)(\beta,j)}\alpha_{S\rho_{ik}}(y_{\mu rs})\delta_{(\beta ij),(\mu rs)}\\ &= \sum_{\mu rs}\delta_{(\rho,l)(\mu,s)}\alpha_{S\rho_{rk}}(y_{\mu rs})\\ &= \sum_{\mu rs}(\alpha_C)^{\rho kl}_{\mu rs}(y_{\mu rs})\ .\end{aligned}$$

In view of Corollary 4.6, it follows that each $y_{\rho kl}$ is determined by $\sum_{\mu rs}\pi(y_{\mu rs})\lambda(\widetilde{\mu_{rs}})$, and (a) is proved.

(b) We need to show that if $A \in \mathcal{A}_3$ has its *triv*-column identically 0, then A must be identically 0; for which, it will suffice to show that $AL_C = 0$.

For this, we shall first show that

$$(L_C)^{\sigma mn}_{\mu rs} \;=\; ((L \otimes \rho)\Delta^{op}(S\widetilde{\mu_{rs}}))^{\sigma mn}_{triv} \;. \tag{4.22}$$

Indeed, note that, for any $a \in H$,

$$\begin{aligned}((L \otimes \rho)\Delta^{op}(a))\,(\xi \otimes triv) &= \sum L_{a_2}(\xi) \otimes S(a_1) \\ &= \sum \sum_{\sigma mn} \langle S(a_1), \widetilde{\sigma_{mn}} \rangle_H \; L_{a_2}(\xi) \otimes \widetilde{\sigma_{mn}} \;;\end{aligned}$$

and hence

$$((L \otimes \rho)\Delta^{op}(a))^{\sigma mn}_{triv} \;=\; \sum \langle S(a_1), \widetilde{\sigma_{mn}} \rangle_H \; L_{a_2} \;;$$

but notice that for $a = S\widetilde{\mu_{rs}}$,

$$\Delta a = (S \otimes S)\Delta^{op}(\widetilde{\mu_{rs}}) = \sum_u S\widetilde{\mu_{us}} \otimes S\mu_{ru} \;;$$

and hence

$$\begin{aligned}((L \otimes \rho)\Delta^{op}(S\widetilde{\mu_{rs}}))^{\sigma mn}_{triv} &= \sum_u \langle \widetilde{\mu_{us}}, \widetilde{\sigma_{mn}} \rangle_H \; L_{S\mu_{ru}} \\ &= \delta_{(\mu,s),(\sigma,n)} L_{S\mu_{rm}} \\ &= (L_C)^{\sigma mn}_{\mu rs} \;,\end{aligned}$$

so that equation 4.22 indeed holds.

Finally, conclude that for arbitrary ρ, k, l, μ, r, s,

$$\begin{aligned}(AL_C)^{\rho kl}_{\mu rs} &= \sum_{\sigma mn} (A)^{\rho kl}_{\sigma mn} \, ((L \otimes \rho)\Delta^{op}(S\widetilde{\mu_{rs}}))^{\sigma mn}_{triv} \\ &= \sum_{\sigma mn} ((L \otimes \rho)\Delta^{op}(S\widetilde{\mu_{rs}}))^{\rho kl}_{\sigma mn} \, A^{\sigma mn}_{triv} \\ &= 0 \;,\end{aligned}$$

and the proof is complete. $\square$

PROPOSITION 4.8

$$\mathcal{A}_1 \;=\; \mathcal{A}_2 \;=\; \mathcal{A}_3 \;.$$

Proof: We only need to establish that $\mathcal{A}_2 \supset \mathcal{A}_3 \supset \mathcal{A}_1$.

For the second inclusion, since $M_n(M)$ is a von Neumann algebra containing $(\pi(M) \cup \lambda(H))$, it suffices to show that $(L \otimes \rho)(\Delta^{op}(H)) \subset (\pi(M) \cup \lambda(H))'$. Notice that as the left- and right-regular representations have commuting ranges, it is clear that $(L \otimes \rho)(\Delta^{op}(H)) \subset \lambda(H)'$. So it suffices to verify that

$$a \in H,\ x \in M \ \Rightarrow \ \sum (L_{a_2} \otimes \rho(a_1)) \text{ commutes with } \pi(x)\ .$$

Compute as follows:

$$\begin{aligned} \sum \pi(x)(L_{a_2} \otimes \rho(a_1))(y\Omega \otimes b) &= \sum \pi(x)(\alpha_{a_2}(y)\Omega \otimes bS(a_1)) \\ &= \sum \alpha_{a_2 S(b_1)}(x)\alpha_{a_3}(y)\Omega \otimes b_2 S(a_1) \\ &= \sum \alpha_{a_2}(\alpha_{S(b_1)}(x)y)\Omega \otimes b_2 S(a_1) \\ &= \sum ((L_{a_2} \otimes \rho(a_1))\pi(x))(y\Omega \otimes b)\ . \end{aligned}$$

For the first inclusion, suppose now that $A = ((A^{\rho kl}_{\sigma mn})) \in \mathcal{A}_3$. Define

$$y_{\mu rs} \ = \ \sum_{\rho kl} \left((\alpha_{C^{-1}})^{\mu rs}_{\rho kl}\right) \left(A^{\rho kl}_{triv}\right)\ ,$$

(in the notation of Corollary 4.6).

Deduce then from Lemma 4.7(a) that the matrix $A_1 = A - \left(\sum_{\mu rs} \pi(y_{\mu rs})\lambda(\widetilde{\mu_{rs}})\right)$ has *triv* column identically 0, and belongs to $\mathcal{A}_3$ since $\mathcal{A}_1 \subset \mathcal{A}_3$. Since $A_1 \in \mathcal{A}_3$, we may conclude from Lemma 4.7(b) that $A_1 = 0$, and the proof is complete. □

DEFINITION 4.9 *(a) The* **crossed product** *of M by H with respect to the action α - denoted by $M \rtimes_\alpha H$ - is defined to be the set given by any of the $\mathcal{A}_i$'s of the previous proposition.*

(b) The **fixed-point subalgebra** *- denoted M^H - is defined by*

$$M^H = \{x \in M : \alpha_a(x) = \epsilon(a)x\ ,\ \forall a \in H\}\ . \qquad (4.23)$$

(c) The action α is called **outer** *if $(M^H)' \cap M = \mathbb{C}1_M$.*

PROPOSITION 4.10 *(a) $M \rtimes_\alpha H$ is a von Neumann algebra.*

(b) Every element of $M \rtimes_\alpha H$ is uniquely expressible in the form

$$\sum_{\mu rs} \pi(y_{\mu rs})\lambda(\widetilde{\mu_{rs}}) \ . \tag{4.24}$$

(c) For $x \in M$, the following conditions are equivalent:
(i) $x \in M^H$;
(ii) $\alpha_h(x) = x$;
(iii) $x \in L(H)'$.
(In particular, $M^H = M \cap L(H)'$ is a von Neumann algebra.)
(d) $\pi(M) = (M \rtimes_\alpha H) \cap \{e_2\}'$, where $(e_2)^{\rho kl}_{\sigma mn} = \delta_{\rho,triv}\delta_{\sigma,triv}1_M$.
(In particular, $\pi(M)$ is a von Neumann subalgebra of $M \rtimes_\alpha H$.)
(e) π and hence each $\alpha_a, a \in H$ is a normal map on M.
(f) If α is an outer action, then also $\pi(M)' \cap (M \rtimes H) = \mathbb{C}$.

Proof: (a) The set $\mathcal{A}_1$ of Proposition 4.8 is a von Neumann algebra.
(b) This follows from Proposition 4.8 and Lemma 4.7.
(c) If $x \in M^H, a \in H,$ and $y \in M$, then

$$\begin{aligned} L_a x(y\Omega) &= \alpha_a(xy)\Omega \\ &= \sum \alpha_{a_1}(x)\alpha_{a_2}(y)\Omega \\ &= \sum \epsilon(a_1)x\alpha_{a_2}(y)\Omega \\ &= x\alpha_a(y)\Omega \\ &= xL_a(y\Omega) \ , \end{aligned}$$

and hence $(i) \Rightarrow (iii)$.
For $(iii) \Rightarrow (ii)$,

$$\begin{aligned} xL_h = L_h x &\Rightarrow xL_h(\Omega) = L_h x(\Omega) \\ &\Rightarrow x\alpha_h(1)\Omega = \alpha_h(x)\Omega \\ &\Rightarrow x = \alpha_h(x) \ . \end{aligned}$$

For $(ii) \Rightarrow (i)$, notice that for any $a \in H$, $a = ah + a(1-h) = \epsilon(a)h + a(1-h)$; so if $\alpha_h(x) = x$, we see that $\alpha_a(x) = \epsilon(a)x + \alpha_a(\alpha_{1-h}(x)) = \epsilon(a)x$.

(d) Note first that if $y \in M$, then

$$\begin{aligned}
(\pi(y)e_2)^{\rho kl}_{\sigma mn} &= \sum_{\mu rs} (\pi(y))^{\rho kl}_{\mu rs} \, (e_2)^{\mu rs}_{\sigma mn} \\
&= (\pi(y))^{\rho kl}_{triv} \, \delta_{triv,\sigma} \\
&= \delta_{\rho,triv} \delta_{triv,\sigma} y \\
&= \cdots \\
&= (e_2 \pi(y))^{\rho kl}_{\sigma mn} \ ,
\end{aligned}$$

thereby showing that $\pi(M) \subset (M \rtimes_\alpha H) \cap \{e_2\}'$.

Conversely if $A \in (M \rtimes_\alpha H) \cap \{e_2\}'$, and if we define $y = A^{triv}_{triv}$, it is seen that A and $\pi(y)$ have the same $triv$-column, and Lemma 4.7(b) clinches matters.

(e) The map π is injective since M is a factor; and it defines an isomorphism of one von Neumann algebra onto another, and is consequently necessarily normal. The assertion regarding the α_a's is a consequence of the identity

$$\alpha_{S\rho_{k'k}}(y) = (\pi(y))^{\rho kl}_{\rho k' l}$$

and the fact that $\{S\rho_{k'k} : \rho k k'\}$ spans H.

(f) This is a simple (matrix-) computation. $\square$

We assume henceforth that we have a fixed outer action α of H on the II_1 factor M. Our objective is to establish the following facts concerning the initial stages of the tower of the basic construction associated to the subfactor $N = M^H \subset M$.

Theorem 4.11 *The 'tower'*

$$\pi(M^H) \subset \pi(M) \subset^{e_1} M \rtimes H \subset^{e_2} M_n(M) \tag{4.25}$$

is isomorphic to the tower $N \subset M \subset M_1 \subset M_2$, with the 'Jones projections' e_1 and e_2 being given by $e_1 = \lambda(h)$ and e_2 being as in Proposition 4.10(d).

We pave the way with a few intermediate propositions.

Proposition 4.12 *(a) The restriction - call it tr - to $M \rtimes H$, of $tr_{M_n(M)}$ satisfies*

$$tr(\lambda(a)\pi(y)) \ = \ \phi(a) \ tr_M(y) \ .$$

(b) $\{\lambda(\widetilde{\rho}_{kl}) : \rho kl\}$ *is an* orthonormal Pimsner-Popa basis *for* $M \rtimes H$ *over* $\pi(M)$*, meaning that*

$$E_{\pi(M)}(\lambda(\widetilde{\rho}_{kl})^* \lambda(\widetilde{\sigma}_{mn})) \;=\; \delta_{(\rho kl),(\sigma mn)}\; \pi(1_M). \tag{4.26}$$

In particular, for any $A \in M \rtimes H$*, we have*

$$A = \sum_{\mu rs} \lambda(\widetilde{\mu}_{rs})\; E_{\pi(M)}(\lambda(\widetilde{\mu}_{rs})^*\; A).$$

(c) $[M \rtimes H : \pi(M)] = n.$

Assertion (a) is an easy computation and implies (b), which implies (c).

PROPOSITION 4.13 *(a) There exist normal representations* $\pi : M \to M \rtimes_\alpha H$ *and* $\lambda : H \to M \rtimes_\alpha H$ *satisfying the commutation relation in Proposition 4.3(c).*

(b) If P *is a von Neumann algebra and if there exist normal representations* $\pi' : M \to P$ *and* $\lambda' : H \to P$ *satisfying the commutation relation in Proposition 4.3(c) (suitably primed), then there exists a unique normal representation* $\chi : M \rtimes_\alpha H \to P$ *such that* $\pi' = \chi \circ \pi$ *and* $\lambda' = \chi \circ \lambda$.

Proof: (a) has already been noted and requires no proof. For (b), simply define

$$\chi(A) = \sum_{\mu rs} \lambda'(\widetilde{\mu}_{rs}) \;\; \pi'(\pi^{-1}(E_{\pi(M)}(\lambda(\widetilde{\mu}_{rs})^*)A)),$$

and verify that this χ works. □

Proof of Theorem 4.11: Let $N = M^H \subset M \subset^{e_N} M_1$ be the basic construction. Define $\pi' : M \to M_1$ and $\lambda' : H \to M_1$ by $\pi'(x) = x$ and $\lambda'(a) = L_a$. Notice, by Proposition 4.10(c), that $M_1 = J_M((M^H)')J_M = J_M(M' \cup L(H)'')''J_M = (M \cup J_M L(H) J_M)'' = (M \cup L(H))''$ (since $J_M L_a J_M x\Omega = \alpha_a(x^*)^*\Omega = \alpha_{Sa^*}(x)\Omega = L_{Sa^*}x\Omega$ so that $J_M L_a J_M = L_{Sa^*}$). Hence the representations π', λ' do indeed land in M_1. The fact that these representations satisfy the commutation relation in Proposition 4.3(c), is a consequence of the definition of an action.

Hence there exists a unique morphism $\chi : M \rtimes H \to M_1$ (of von Neumann algebras) such that $\chi(\pi(x)) = x$ and $\chi(\lambda(a)) = L_a$. This χ is 1-1 since $M \rtimes H$ is a factor (which follows from Proposition 4.10(f)), and it is onto since - see the last paragraph - $M_1 = (M \cup L(H))''$; thus χ is an isomorphism. But $\chi(\lambda(h)) = L_h$, whereas Proposition 4.10(c)(ii) says that $e_N = L_h$. This completes the verification that $\pi(M^H) \subset \pi(M) \subset^{\lambda(h)} (M \rtimes H)$ is a basic construction.

To prove that $\pi(M) \subset (M \rtimes H) \subset^{e_2} M_n(M)$ is a basic construction, we need to verify three facts:

(i) $e_2 A e_2 = E_{\pi(M)}(A) e_2$, $\forall A \in M \rtimes H$; both sides describe the matrix whose only possible non-zero entry is in the $(triv, triv)$ spot, with that entry being A^{triv}_{triv};

(ii) $tr_{M_n(M)}(e_2) = 1/n$; and

(iii) $M_n(M) = Alg((M \rtimes H) \cup \{e_2\})$.

For (i), if $A = \sum_{\mu rs} \pi(y_{\mu rs}) \lambda(\widetilde{\mu_{rs}})$, then notice from Proposition 4.12(b) that $E_{\pi(M)}(A) = \pi(y_{triv}) = \pi(A^{triv}_{triv})$. Now calculate to get

$$\begin{aligned}
\left(E_{\pi(M)}(A) e_2\right)^{\rho kl}_{\sigma mn} &= (E_{\pi(M)}(A))^{\rho kl}_{triv}\ \delta_{triv,\sigma} 1_M \\
&= \delta_{triv,\rho} \delta_{triv,\sigma} y_{triv} \\
&= (e_2 A e_2)^{\rho kl}_{\sigma mn} .
\end{aligned}$$

The statement (ii) is obvious and requires no proof. As for (iii), observe that $M_n(M)$ is generated, as an algebra, by the following three kinds of matrices (all of which belong to $Alg((M \rtimes H) \cup \{e_2\})$): (a) $e_2 \pi(y) e_2, y \in M$, which has a unique non-zero entry at the $(triv, triv)$-spot, (which entry can be any element of M); (b) $e_2 \lambda(\widetilde{\mu_{rs}}^*)$, which has a unique non-zero entry at the $(triv, \mu rs)$-spot, that entry being 1_M; and (c) $\lambda(\widetilde{\mu_{rs}}) e_2$, which has a unique non-zero entry at the $(\mu rs, triv)$-spot, that entry being 1_M. □

LEMMA 4.14 *The equations*

$$\Gamma(e^{\mu}_{sr})^{\rho kl}_{\sigma mn} = \delta_{\mu\rho} \delta_{\mu\sigma} \delta_{nr} \delta_{mk} \delta_{sl} , \tag{4.27}$$

and

$$\Theta(T)^{\rho kl}_{\sigma mn} = \langle T \widetilde{\sigma_{mn}}, \widetilde{\rho_{kl}} \rangle_H \tag{4.28}$$

define faithful representations (of C^*-algebras) $\Gamma : H^* \to M_n(M)$ *and* $\Theta : End_{\mathbb{C}}(H) \to M_n(\mathbb{C}) \subset M_n(M)$ *. Further,* $tr_{M_n(M)} \circ \Gamma = h$*.*

We omit the proof, which is a routine verification. We conclude this section by explicitly identifying some relative commutants in the tower 4.25.

PROPOSITION 4.15 *(a)* $\pi(M^H)' \cap (M \rtimes H) = \lambda(H)$*;*
(b) $\pi(M)' \cap M_n(M) = \Gamma(H^*)$*; and*
(c) $\pi(M^H)' \cap M_n(M) = \Theta(End_{\mathbb{C}}(H)) = M_n(\mathbb{C})$*.*

Proof: (a) In the equation $\lambda(a)\pi(x) = \sum \pi(\alpha_{a_1}(x))\lambda(a_2)$ valid for all $a \in H, x \in M$, if we specialise to the case when $x \in M^H$, then we find that $\lambda(a)\pi(x) = \sum \epsilon(a_1)\pi(x)\lambda(a_2) = \pi(x)\lambda(a)$; i.e., $\pi(M^H)' \cap (M \rtimes H) \supset \lambda(H)$. Conversely, it follows from Proposition 4.10(b) that

$$\sum_{\mu rs} \pi(y_{\mu rs})\lambda(\widetilde{\mu_{rs}}) \in \pi(M^H)' \ \Leftrightarrow \ \pi(y_{\mu rs}) \in \pi(M^H)' \ , \ \forall \ \mu, r, s$$

but the outerness of the action ensures that $\pi(M^H)' \cap \pi(M) = \mathbb{C}$, and the proof of (a) is complete.

(b) A simple matrix-computation and the definitions show that $\pi(M)' \cap M_n(M) \supset \Gamma(H^*)$. However, the 'Fourier transform for subfactors of finite index' (together with Theorem 4.11) shows that $(n \leq)$ $dim\ (\pi(M)' \cap M_n(M)) = dim\ (\pi(M^H)' \cap (M \rtimes H)) = n$, and the proof is complete.

(c) Note that $\pi(M^H)$ consists precisely of those diagonal matrices in $M_n(M)$ all of whose entries are equal and belong to M^H; it follows that $\pi(M^H)' \cap M_n(M) = M_n((M^H)' \cap M) = M_n(\mathbb{C}) = \Theta(End_{\mathbb{C}}(H))$. $\square$

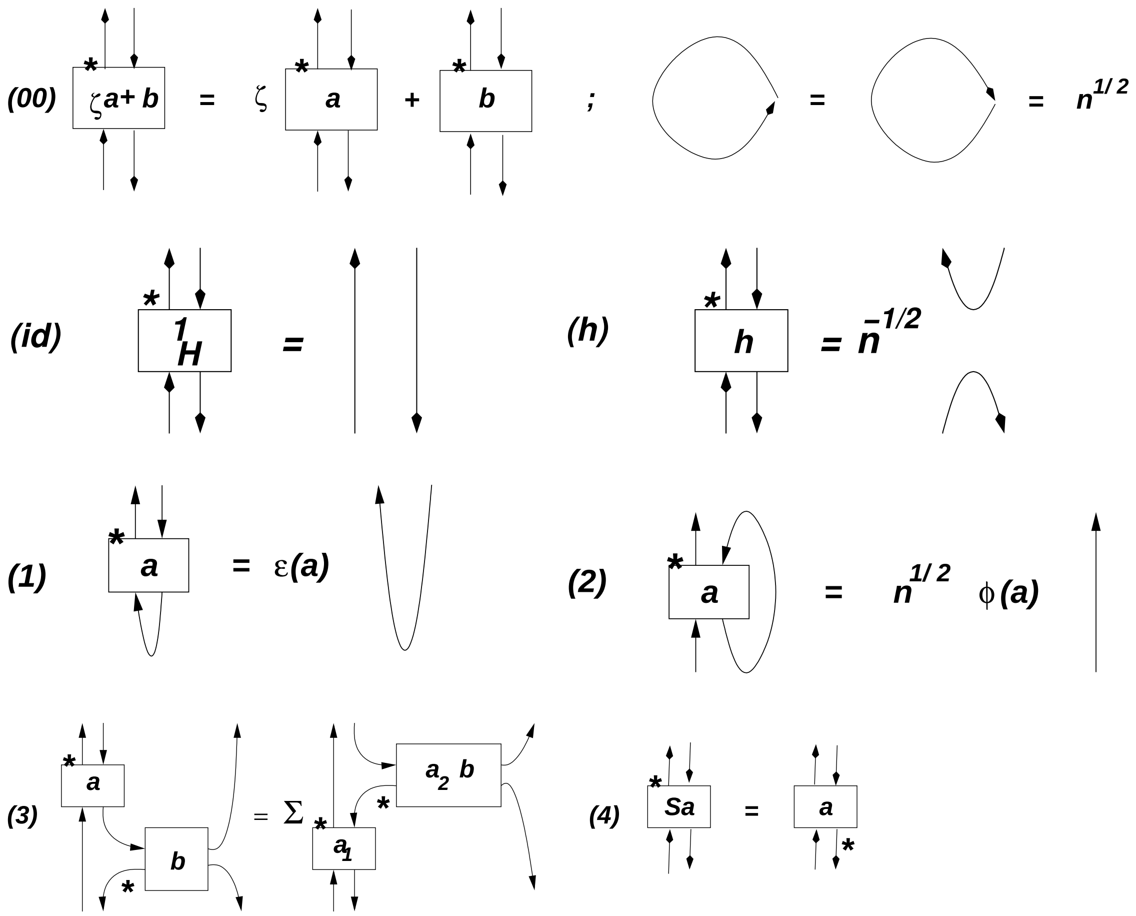
5 The main result

The purpose of this section is to obtain a presentation of the planar algebra $P^{M^H \subset M}$, where M^H is the fixed-point algebra for an outer action of a Kac algebra H on a II_1 factor as in §4. Our goal is to prove the following theorem:[6]

THEOREM 5.1 *Let H be an n-dimensional Kac algebra acting outerly on the hyperfinite II_1 factor M as in §4. Then there is a presentation $\Phi : P(L; R) \to P^{M^H \subset M}$ - which is a $*$-isomorphism - where:*

(a) $L = L_2 = H$ acquires the involution from the Kac algebra structure of H; and

(b) R is given by the following set of relations (where (i) we write the relations as identities - so the statement $a = b$ is interpreted as $a - b \in R$; and (ii) $\zeta \in \mathbb{C},\ a, b \in H$):



[6] We adopt the convention, throughout this section, that L-labelled k-tangles will be drawn without their external disc, and the distuingished * on the external disc (in case $k \neq 0_\pm$) will be taken as the 'top-left point'.

□

REMARK 5.2 There are two advantages with stating the result in the above form:

(i) The adjoint appears nowhere in the relations; so although the above presentation is an isomorphism of *-algebras (with the * on the label set H being defined as in the Kac algebra), this theorem leads us to a natural planar algebra associated with any n-dimensional complex semisimple Hopf algebra.

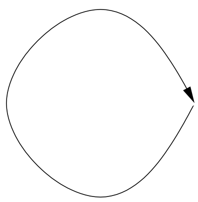
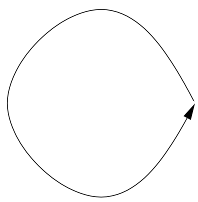
(ii) The description is basis-free and naturally highlights the role of each of the Hopf algebra ingredients; for instance, relation (4) might be thought of yet another vindication of the use of the word 'antipode' in Hopf algebras.

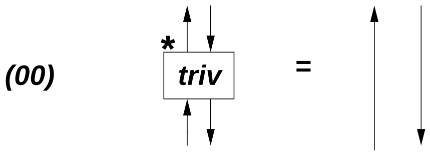
There is one obvious drawback with stating the result in the above form; namely, it would be naturally desirable to have a 'finite presentation', where both L and R are finite. We shall also describe such a finite - albeit basis-dependent - presentation, in the next result. We shall employ the orthogonal basis for H that was denoted $\{\gamma_{kl} : \gamma kl\}$ in the last section.

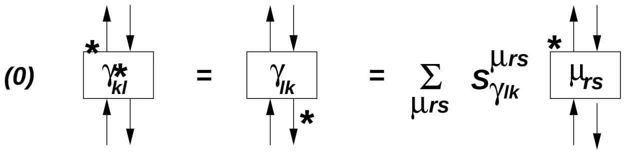
THEOREM 5.3 *Let H, M be as above. Then there is a presentation $\Phi^{\#} : P(L^{\#}; R^{\#}) \to P^{M^H \subset M}$ - which is a $*$-isomorphism - where:*

(a) $L^{\#} = L_2^{\#} = \{\gamma_{kl} : \gamma \in \hat{H}^*, 1 \leq k,l \leq d_\gamma\} \coprod \{\gamma^*_{kl} : \gamma \in \hat{H}^*, 1 \leq k,l \leq d_\gamma\}$ is equipped with the obvious involution; and

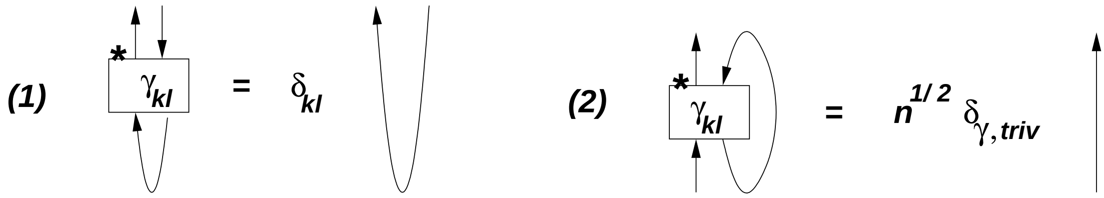
(b) $R^{\#}$ is given by the following set of relations:

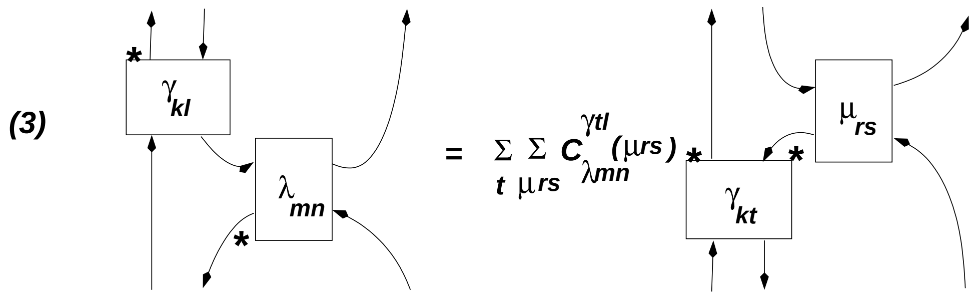
(mod)  $= n^{1/2} =$ 



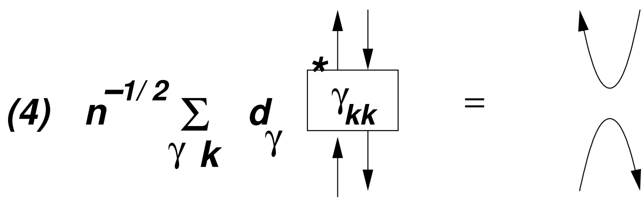


where $\sum_{\mu rs} S^{\mu rs}_{\gamma lk}\mu_{rs} = S\gamma_{lk}$, *with* γ_{lk}, μ_{rs} *are as in §4; and*





where $\sum_{\mu rs} C^{\gamma tl}_{\lambda mn}(\mu rs)\mu_{rs} = \gamma_{tl}\lambda_{mn}$; *and*



$\square$

The proof of these theorems will be a consequence of several intermediate results. We shall adopt the foregoing notation in the rest of this section - with one exception. We shall use the symbols π and $\pi^\#$ for the quotient maps $\pi : P(L) \to P(L;R)$ and $\pi^\# : P(L^\#) \to P(L^\#;R^\#)$. (We will soon, after equation (5.30) to be precise, be identifying $x \in M$ with what we were calling $\pi(x)$ in earlier sections.)

LEMMA 5.4 *There exists a unique $*$-isomorphism*

$$\Phi_1 : P(L^\#;R^\#) \to P(L;R) \ , \tag{5.29}$$

such that $\Phi_1(\pi^\#(Z_{I_2^2}(a))) = \pi(Z_{I_2^2}(a)) \ , \ \forall \ a \in L_2^\#$.

Proof: Define maps $\Psi_1 : P(L^\#) \to P(L;R)$ and $\Psi_2 : P(L) \to P(L^\#;R^\#)$, as follows:

$$\begin{aligned} \Psi_1(Z_{I_2^2}(a)) &= \pi(Z_{I_2^2}(a)) \ , \ \forall a \in L_2^\# \\ \Psi_2(Z_{I_2^2}(\sum_{\gamma kl} D_{\gamma kl}\gamma_{kl})) &= \sum_{\gamma kl} D_{\gamma kl}\pi^\#(Z_{I_2^2}(\gamma_{kl})) \ , \ \forall \{D_{\gamma kl} : \gamma kl\} \subset \mathbb{C} \end{aligned}$$

We shall now verify that

(i) $R^\# \subset \ ker \ \Psi_1$ and $R \subset \ ker \ \Psi_2$, and hence conclude that Ψ_i descend to planar algebra morphisms $\Phi_1 : P(L^\#;R^\#) \to P(L;R)$ and $\Phi_2 : P(L;R) \to P(L^\#;R^\#)$; and

(ii) Φ_1 and Φ_2 are inverse to one another.

(i) The 'kernel inclusions' hold basically because of standard Hopf algebra facts (as described in Lemma 4.1) and the fact that the relations $(mod), (00), (0), (1), (2), (3)$ and (4) of Theorem 5.3 are more or less equivalent to relations $(00), (id), (4), (1), (2), (3)$ and (h) of Theorem 5.1.

(ii) The identity $\Phi_1 \circ \Phi_2(x) = x$ is obvious for $x = \pi(Z_{I_2^2}(\gamma_{kl}))$, while for general $x \in \pi(Z_{I_2^2}(H))$, this follows from the first (linearity) relation in (00) of Theorem 5.1; and the identity $\Phi_2 \circ \Phi_1(x) = x$ is obvious for $x = \pi^\#(Z_{I_2^2}(\gamma_{kl}))$, while the case $x = \pi^\#(Z_{I_2^2}(\gamma_{kl}^*))$ is a consequence of (the linearity relation in (00) of Theorem 5.1 and) the equality of the two extreme terms of relation (3) of Theorem 5.3.

Finally it is clear that the Φ_i are $*$-preserving. $\square$

We shall consider the subfactor $N = M^H \subset M$, and write

$$N \subset M \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$$

for the tower of the basic construction. We shall freely use the identifications

$$M_1 = M \rtimes H \ , \ M_2 = M_n(M) \ .$$

(Thus we identify $x \in M$ with $\pi(x) \in M_n(M)$.)

It is seen from Theorem 2.1 and Proposition 4.15(a)) that we have identifications

$$P_2^{M^H \subset M} \ = \ N' \cap M_1 \ = \ \pi(M^H)' \cap (M \rtimes H) \ = \ \lambda(H) \ ;$$

and we shall define

$$\Phi(\pi(Z_{I_2^2}(a))) \ = \ \lambda(a) \ , \ a \in H. \tag{5.30}$$

The proofs of Theorems 5.3 and 5.1 will be completed (in view of Lemma 5.4) once we have been able to establish that equation (5.30) extends uniquely to a planar algebra isomorphism $\Phi : P(L,R) \to P^{N\subset M}$.

Proof of extendability of Φ

In order to conclude that Φ does so extend to a morphism of planar algebras, we only need to verify that the $\lambda(a)$'s satisfy all the relations (in $P^{N\subset M}$) expressed by R; and this is what we shall do now.

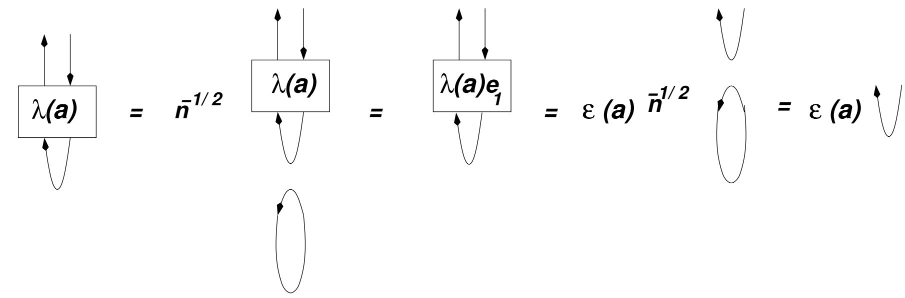
The first relation of (00) is satisfied by the $\lambda(a)$'s because of the linearity of λ; while the validity of the second relation is a consequence of Jones' theorem 2.1 and the fact that $[M_1 : M] = n$ - see Proposition 4.12, for instance.

The relation (id) holds because $\lambda(1_H) = 1_{M_1}$; while the truth of relation (h) is a consequence of Theorem 4.11.

For relation (1), we see that for any $a \in H$, we have

$$ah = \epsilon(a)h \ \Rightarrow \ \lambda(a)e_1 = \epsilon(a)e_1$$

and hence,



As for relation (2), notice that if $a \in H$, then

$$\begin{aligned} Z_{E_2^1}(\lambda(a)) &= n^{1/2}\ E_{N'\cap M}(\lambda(a)) \text{ (by Theorem 2.1(3))} \\ &= n^{1/2}\ tr(\lambda(a)) \text{ (since } N'\cap M = \mathbb{C}) \\ &= n^{1/2}\ \phi(a) \text{ (by Proposition 4.3(b))}\ . \end{aligned}$$

The proofs of the fact that relations (3) and (4) are satisfied by the $\lambda(a)$'s will require some preliminary lemmas.

We will use the following notation: if $\phi \in H^*$, $a \in H$, we shall write $\phi(a\cdot)$ to denote the element of H^* defined by $(\ \phi(a\cdot)\ )(b) = \phi(ab)$.

Lemma 5.5 *With the above notation, we have:*

$$\sqrt{n}\ \phi((S\widetilde{\mu_{rs}})\ \cdot)\ =\ \widetilde{e^{\mu}_{sr}} \tag{5.31}$$

and

$$tr(\Gamma(\widetilde{e^{\mu}_{sr}})\Gamma(e^{\rho}_{kl}))\ =\ \sqrt{\frac{d_\mu}{n}}\ \delta_{(\rho kl),(\mu rs)}. \tag{5.32}$$

Proof: The first assertion is verified by applying both sides to $\widetilde{\rho_{kl}}$, and appealing to Lemma 4.1(iii), the orthonormality of the $\widetilde{\mu_{rs}}$'s, and the fact that (by definition), the e^{μ}_{rs}'s and the μ_{rs}'s constitute a pair of dual bases.

The second assertion is a consequence of the fact that Γ is a homomorphism and is 'trace-preserving' - see Lemma 4.14 - and of equation (4.10). □

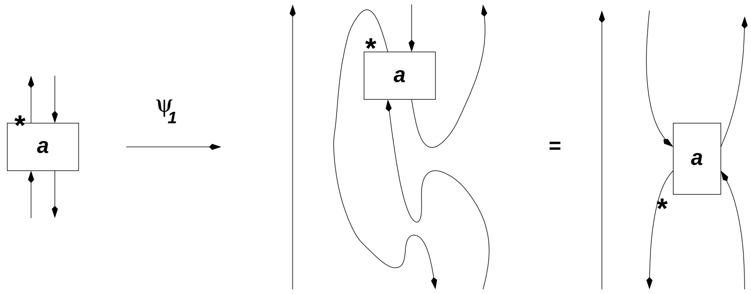
Consider the maps[7] $\psi_i : (N' \cap M_1) \to (M' \cap M_2),\ i = 1, 2$ defined by

$$\psi_1(x) \;=\; n^{3/2}\ E_{M'\cap M_2}(xe_2e_1)\ , \tag{5.33}$$

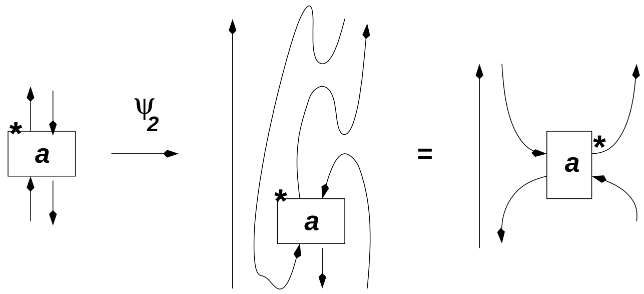
and

$$\psi_2(x) \;=\; n^{3/2}\ E_{M'\cap M_2}(e_1e_2x)\ , \tag{5.34}$$

It is known - see [BJ] - that these maps are described, pictorially, by



and



and that both the ψ_i's are linear isomorphisms which are 'isometric' in the sense that they satisfy

$$tr_{M_2}(\psi_i(x)^*\psi_i(x)) \;=\; tr_{M_2}(x^*x)\ .$$

Further, it is easily verified that

$$\psi_1 \circ Z_{R_2} \;=\; \psi_2 \;\text{ so also }\; \psi_2 \circ Z_{R_2} \;=\; \psi_1\ . \tag{5.35}$$

Proposition 5.6 *For arbitrary $a, b \in H$, we have:*

$$\psi_1(\lambda(a)) \;=\; \sqrt{n}\ \Gamma(\ \phi((Sa)\ \cdot)\ ) \tag{5.36}$$

[7] sometimes also referred to as Fourier transforms

$$\psi_2(\lambda(a)) \ = \ \sqrt{n}\ \Gamma(\ \phi(a\ \cdot)\ ) \tag{5.37}$$

$$\big(\ \Theta^{-1}(\lambda(a))\ \big)(b) \ = \ ab \tag{5.38}$$

and

$$\big(\ \Theta^{-1}(\psi_1(\lambda(a)))\ \big)(b) \ = \ \sqrt{n}\ \sum \phi((Sa)b_2)\ b_1\ . \tag{5.39}$$

Proof: To prove equation (5.36), we may assume that $a = \widetilde{\mu_{rs}}$, in which case we need to check - in view of equation (5.31) - that

$$\psi_1(\lambda(\widetilde{\mu_{rs}})) \ = \ \Gamma(\widetilde{e^{\mu}_{sr}})\ . \tag{5.40}$$

Since $\{\Gamma(\widetilde{e^{\rho}_{kl}})\ :\ \rho kl\}$ linearly spans $M' \cap M_2$ (by Proposition 4.15(b)), we need to show (by the definition of ψ_1) that

$$n^{3/2} tr\,(\lambda(\widetilde{\mu_{rs}})e_2e_1\Gamma(e^{\rho}_{kl})) \ = \ tr(\Gamma(\widetilde{e^{\mu}_{rs}})\Gamma(e^{\rho}_{kl}))\ . \tag{5.41}$$

Now the left side (LHS) of equation (5.41) is seen to be given by

$$\begin{aligned}
& n^{3/2}\frac{1}{n}\sum_{\beta ij,\sigma mn} (\lambda(\widetilde{\mu_{rs}})^{\beta ij}_{triv}\lambda(h)^{triv}_{\sigma mn}(\Gamma(e^{\rho}_{kl}))^{\sigma mn}_{\beta ij} \\
&= \ \sqrt{n}\sum_{\beta ij,\sigma mn} \langle\widetilde{\mu_{rs}},\widetilde{\beta_{ij}}\rangle_H\langle h\widetilde{\sigma_{mn}}, triv\rangle_H \delta_{\rho\beta}\delta_{\rho\sigma}\delta_{jl}\delta_{im}\delta_{kn} \\
&= \ \sqrt{n}\sum_{\sigma mn}\langle\widetilde{\sigma_{mn}}, h\rangle_H\delta_{\rho\mu}\delta_{\rho\sigma}\delta_{sl}\delta_{rm}\delta_{kn} \\
&= \ \sqrt{n}\sqrt{d_\rho}\frac{1}{n}\delta_{rk}\delta_{\rho\mu}\delta_{sl} \ \text{ by equation (4.10)} \\
&= \ \sqrt{\frac{d_\mu}{n}}\delta_{(\mu rs),(\rho kl)}\ ,
\end{aligned}$$

which is equal to the RHS of equation 5.41 by equation (5.32).

As for equation (5.37), we may assume that $a = S\widetilde{\mu_{rs}}$, and as before. we find that it is enough to verify that

$$n^{3/2} tr\,(e_1e_2\lambda(S\widetilde{\mu_{rs}})\Gamma(e^{\rho}_{kl})) \ = \ \sqrt{\frac{d_\mu}{n}}\delta_{(\mu rs),(\rho kl)}\ ; \tag{5.42}$$

and we calculate thus:

$$\begin{aligned}
LHS &= n^{3/2}\frac{1}{n}\sum_{\beta ij,\sigma mn}\lambda(h)^{\beta ij}_{triv}\lambda(S\widetilde{\mu_{rs}})^{triv}_{\sigma mn}(\Gamma(e^{\rho}_{kl}))^{\sigma mn}_{\beta ij}\\
&= \sqrt{n}\sum_{\beta ij,\sigma mn}\langle h,\widetilde{\beta_{ij}}\rangle_H\langle S\widetilde{\mu_{rs}}\widetilde{\sigma_{mn}},triv\rangle_H\delta_{\rho\beta}\delta_{\rho\sigma}\delta_{jl}\delta_{im}\delta_{kn}\\
&= \sqrt{n}\sum_{\beta ij}\delta_{ij}\frac{\sqrt{d_\beta}}{n}\delta_{\rho\beta}\delta_{\rho\mu}\delta_{jl}\delta_{is}\delta_{kr}\\
&= RHS\ .
\end{aligned}$$

As for equation (5.38),

$$\begin{aligned}
\big(\ \Theta^{-1}(\lambda(a))\ \big)(\widetilde{\sigma_{mn}}) &= \sum_{\rho kl}(\lambda(a))^{\rho kl}_{\sigma mn}\widetilde{\rho_{kl}}\\
&= \sum_{\rho kl}\langle a\widetilde{\sigma_{mn}},\widetilde{\rho_{kl}}\rangle_H\widetilde{\rho_{kl}}\\
&= a\widetilde{\sigma_{mn}}\ .
\end{aligned}$$

Finally, for equation (5.39), we shall verify that $\psi_1(\lambda(a)) = \Theta(A)$, where $A \in End_{\mathbb{C}}(H)$ is given by $A(b) = \sqrt{n}\sum\phi((Sa)b_2)\,b_1$; and we may assume that $a = \widetilde{\mu_{rs}}$. Observe that - see equation (5.40) -

$$(\psi_1(\lambda(\widetilde{\mu_{rs}})))^{\rho kl}_{\sigma mn} = \Big(\Gamma(e^{\mu}_{sr})\Big)^{\rho kl}_{\sigma mn} = \sqrt{\frac{n}{d_\mu}}\ \delta_{\mu\sigma}\delta_{\mu\rho}\delta_{km}\delta_{ls}\delta_{nr}\ ,$$

whereas

$$(\Theta(A))^{\rho kl}_{\sigma mn} = \sqrt{n}\sum_t\phi(\widetilde{\mu_{sr}}^*\widetilde{\sigma_{tn}})\frac{1}{\sqrt{d_\sigma}}\langle\widetilde{\sigma_{mt}},\widetilde{\rho_{kl}}\rangle_H = \sqrt{\frac{n}{d_\mu}}\ \delta_{\mu\sigma}\delta_{\mu\rho}\delta_{km}\delta_{ls}\delta_{nr}\ ,$$

and the proof is complete. $\square$

Verification of relation (3)*:*

Let us write π_l for the left-regular representation of H. We need to show that

$$\lambda(a)\psi_1(\lambda(b)) = \sum \psi_1(\lambda(a_2 b))\lambda(a_1) \ .$$

Since Θ is an isomorphism, it suffices - by equation (5.38) - to check that

$$\pi_l(a)\Theta^{-1}(\psi_1(\lambda(b))) = \sum \Theta^{-1}(\psi_1(\lambda(a_2 b)))\pi_l(a_1)$$

in $End_{\mathbb{C}}(H)$. If $c \in H$, we have

$$\begin{aligned}
\left(\sum \Theta^{-1}(\psi_1(\lambda(a_2 b)))\pi_l(a_1)\right)(c) &= \left(\sum \Theta^{-1}(\psi_1(\lambda(a_2 b)))\right)(a_1 c) \\
&= \sqrt{n}\ \sum \phi\left((Sb)(Sa_3)a_2 c_2\right)\ a_1 c_1 \\
&= \sqrt{n}\ \sum \phi\left((Sb)\epsilon(a_2)c_2\right)\ a_1 c_1 \\
&= \sqrt{n}\ \sum \phi((Sb)c_2)\ a c_1 \\
&= \pi_l(a)\left(\sqrt{n}\ \sum \phi((Sb)c_2)\ c_1\right) \\
&= \left(\pi_l(a)\Theta^{-1}(\psi_1(\lambda(b)))\right)(c)\ ,
\end{aligned}$$

where we have used equation (5.39) at the second and fourth equalities.

Verification of relation (4)*:*

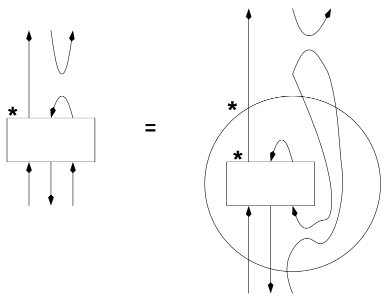
We need to verify that $Z_{R_2}(\lambda(a)) = \lambda(Sa)\ ,\ \forall a \in H$. In view of the first relation of (00), and the linearity of λ and S, it suffices to verify the relation for $a = S(\widetilde{\mu_{rs}})$; and by virtue of equation (5.35), we need to check that $\psi_1\left(\lambda(\widetilde{\mu_{rs}})\right) = \psi_2\left(\lambda(S\widetilde{\mu_{rs}})\right)$, which is guaranteed by equations (5.36) and (5.37). $\square$

Proof that Φ *is an isomorphism*

We need to prove that Φ - which has been now shown to define a planar algebra morphism - is an isomorphism. We prepare for this with a couple of lemmas, the first of which is about abstract planar algebras (where, of course, the e_k's are as in Remark 2.2(a)).

LEMMA 5.7 *In any planar algebra* $P = \{P_k : k \in Col\}$, *we have:*
(a) $e_k P_k = e_k P_{k+1}$, *and* $P_k e_k = P_{k+1} e_k$ *for any* $k \geq 1$;
(b) $P_k e_k P_k$ *is an ideal in* P_{k+1} *for any* $k \geq 1$; *and*
(c) if P *has non-zero modulus, say* δ, *and if* $P_k e_k P_k = P_{k+1}$ *for some* $k \geq 1$, *then,*
(i) $P_l e_l P_l = P_{l+1}$ *for all* $l \geq k$; *and*
(ii) $P_k e_k e_{k+1} \cdots e_l P_l = P_{l+1}\ \forall\ l \geq k$.

Proof: (a) Putting exactly one cap on a '$(k+1)$-box' results in a k-box. (We illustrate the case $k = 2$ below.)



(b) follows from (a).

(c) Both assertions are proved by induction on $l \geq k$. Both assertions are clearly valid for $k = l$. Suppose they have been proved for some $l \geq k$. Then, by induction hypothesis, we can find $a_i, b_i \in P_l$ such that $1 (= 1_{P_{l+1}}) = \sum_i a_i e_l b_i$. Hence,

$$\begin{aligned} 1_{P_{l+2}} (= 1_{P_{l+1}}) &= \sum_i a_i e_l b_i \\ &= \delta^2 \sum_i a_i e_l e_{l+1} e_l b_i \\ &\in P_{l+1} e_{l+1} P_{l+1} \ ; \end{aligned}$$

and we may deduce from (b) above that $P_{l+1} e_{l+1} P_{l+1} = P_{l+2}$, thereby establishing the inductive step in (i). As for (ii), the induction hypothesis is that $P_k e_k e_{k+1} \cdots e_l P_l = P_{l+1}$; and we find that

$$\begin{aligned} P_{l+2} &= P_{l+1} e_{l+1} P_{l+1} \quad \text{by (c)(i) above} \\ &= P_k e_k e_{k+1} \cdots e_l P_l e_{l+1} P_{l+1} \\ &= P_k e_k e_{k+1} \cdots e_l e_{l+1} P_l P_{l+1} \\ &= P_k e_k e_{k+1} \cdots e_l e_{l+1} P_{l+1} \ , \end{aligned}$$

and the proof is complete. $\square$

Lemma 5.8 *(a)*

$$dim_{\mathbb{C}}\ P_2(L^{\#}, R^{\#}) \ = \ dim_{\mathbb{C}}\ P_2^{N\subset M}\ ; \tag{5.43}$$

and

(b)

$$P_3(L^{\#}, R^{\#}) = P_2(L^{\#}, R^{\#}) e_2 P_2(L^{\#}, R^{\#})\ . \tag{5.44}$$

Proof: We shall write $\Phi^{\#} = \Phi \circ \Phi_1$ - with Φ as in equation (5.30) and Φ_1 as in Lemma 5.4; and so we now know that $\Phi^{\#}$ and Φ are morphisms of planar algebras.

For (a), note that

$$dim_{\mathbb{C}}\ P_2^{N\subset M} \ = \ dim_{\mathbb{C}}\lambda(H) = dim_{\mathbb{C}}H \ = \ n\ .$$

On the other hand, begin by observing that $P(L^{\#}; R^{\#})$ is an exchange relation planar algebra. (*Reason:* Condition (0) of Definition 3.2 is met because of relation (mod). Condition (1) of Definition 3.2 amounts to the requirement that any way of 'putting one cap' on a labelled 2-box results in a scalar; in view of relation (0), we may restrict ourselves to boxes with labels from $\{\gamma_{kl} : \gamma kl\}$; and relations (1) and (2) give us the desired condition (1) of Definition 3.2. Thanks to equation (3.8), we see that condition (2) of Definition 3.2 amounts to verifying that for any one $1 \leq i \leq 3$, the range of Z_{H_i} is contained in the sum of the ranges of Z_{H_j}, for $j \neq i$. However, we see from relations (0) and (3) that the range of Z_{H_3} is contained in that of Z_{H_1}.)

Hence, Proposition 3.3 applies; and according to Proposition 3.3 (iii), we know that $P_2(L^{\#}, R^{\#})$ is linearly spanned by the image under the quotient map $\pi^{\#}$ of the set

$$\{Z_{I_2^2}(\gamma_{kl}) : \gamma kl\} \cup \{Z_{I_2^2}(\gamma_{kl}^*) : \gamma kl\} \cup \{1, e_1\}\ ,$$

while it is seen from relations $(0), (00)$ and (4) that the above set is contained in the linear span of $\pi^{\#}\left(\{Z_{I_2^2}(\gamma_{kl}) : \gamma kl\}\right)$; and hence, $dim_{\mathbb{C}} P_2(L^{\#}, R^{\#}) \leq n$. Finally the map $\Phi_2^{\#}$ is a linear surjection since $\{\lambda(\gamma_{kl}) = \Phi^{\#}(\pi^{\#}(Z_{A_2}(\gamma_{kl}))) : \gamma kl\}$ is a basis for $P_2^{N\subset M} = \lambda(H)$, so that

$$n \geq dim_{\mathbb{C}} P_2(L^{\#}, R^{\#}) \geq dim_{\mathbb{C}}\ P_2^{N\subset M} = n\ ,$$

and equation (5.43) is proved.

As for (b), it suffices, by Lemma 5.4, to prove that

$$P_3(L : R) = P_2(L : R) e_2 P_2(L : R)\ .$$

For this, it suffices, in view of Lemma 5.7(b), to verify that $1 \in P_2(L; R) e_2 P_2(L; R)$. In terms of the tangles H_i defined earlier (cf. equation (3.8)), this is shown to translate into the requirement that $1(= 1_{P_3(L;R)}) \in\ ran(Z_{H_2})$. For this, note, from Theorem 5.1(3), that

$$\pi(Z_{H_3}(h, 1_H))\ =\ \sum \pi(Z_{H_1}(h_1, h_2))\ ; \qquad (5.45)$$

apply equation (3.8) twice (and use the fact that π is a morphism of planar algebras), to find that

$$\pi(Z_{H_1}(h, 1_H))\ =\ \sum \pi(Z_{H_2}(h_1, h_2))\ ; \qquad (5.46)$$

Notice however, thanks to the relations (id) and (h) of Theorem 5.1, that the left side of equation (5.46) is nothing but $n^{-1/2}\ 1_{P_3(L;R)}$, and conclude that

$$\begin{aligned} 1_{P_3(L;R)} &= \sqrt{n}\ \sum \pi(Z_{H_2}(h_1, h_2)) \\ &= \sqrt{n}\ \sum Z_{H_2}(\pi(Z_{I_2^2}(h_1)) \otimes \pi(Z_{I_2^2}(h_2)))\ . \end{aligned}$$

□

We are now ready to prove that $\Phi^{\#}$, and hence also Φ, is an isomorphism of planar algebras. Since $N \subset M$ has depth 2 (by Proposition 4.15), it is true that $P^{N\subset M}$ is generated, as a planar algebra, by $P_2^{N\subset M}$; since the image of $\Phi_2^{\#}$ linearly spans $P_2^{N\subset M}$, we find that $\Phi^{\#}$ is surjective. To complete the proof, we shall prove that $\Phi_l^{\#}$ is injective, for each $l \geq 2$, or equivalently (in view of the already established surjectivity of each $\Phi_l^{\#}$) that

$$dim_{\mathbb{C}}\ P_l(L^{\#}, R^{\#})\ \leq\ dim_{\mathbb{C}}\ P_l^{N\subset M}\ ,\ \forall l \geq 2\ . \qquad (5.47)$$

In other words, we need[8] to show that

$$dim_{\mathbb{C}}\ P_l(L^{\#}, R^{\#})\ \leq\ n^{l-1}\ ,\ \forall l \geq 2\ . \tag{5.48}$$

We prove this last inequality by induction. For $l = 2$ this follows from equation (5.43). Also, equation (5.44) says that the hypotheses for Lemma 5.7(c) are satisfied with $P = P(L^{\#}, R^{\#})$ and $k = 2$; and Lemma 5.7(c)(ii) then says that

$$dim_{\mathbb{C}} P_{l+1} \leq (dim_{\mathbb{C}} P_2) \times (dim_{\mathbb{C}} P_l)\ \ \forall\ l \geq 2\ ,$$

and the inductive step is seen to follow. □

6 Concluding remarks

We wish to conclude with a few remarks.

(a) If α is any outer action of the finite-dimensional Kac algebra H on M (which will always denote the hyperfinite II_1 factor), it is seen from Theorem 5.1 that the isomorphism-type of the planar algebra $P^{M^H \subset M}$ is independent of α; and since a finite-depth subfactor is uniquely determined by its planar algebra, we recover the fact that the isomorphism-type of the subfactor $(M^H \subset M)$ is independent of the outer action α.

(b) It might be worth observing that an equivalent form of Theorem 5.1 is obtained if we replace condition (3) by the following two conditions:

(3.1) which is just condition (3) with b replaced by 1_H; and

(3.2) $Z_{M_2}(a, b) = Z_{I_2^2}(ab)$.

(The advantage with this formulation is that it seems to split the condition (3) neatly into two components, one corresponding to comultiplication and one corresponding to multiplication.)

(*Reason:* On writing $b = 1_H b$, it is easy to see that the new version - with (3) replaced by (3.1) and (3.2) - implies the old version (with (3)). Conversely, it is clear that (3.1) is a

[8]The fact that $dim_{\mathbb{C}}\ P_l^{N\subset M} = n^{l-1}$ is a standard fact from 'subfactor theory' and can be deduced from Proposition 4.15 and facts about 'the basic construction'.

consequence of (3) and relation (id). Observe next that if the bottoms of the two left-most strands are 'capped', the L.H.S of (3) reduces to the L.H.S. of (3.2), while the R.H.S. of (3) reduces to the R.H.S. of (3.2), thanks to relations $(1), (00)$ and the fact that $\sum \epsilon(a_1)a_2 = a \; \forall \; a \in H$.)

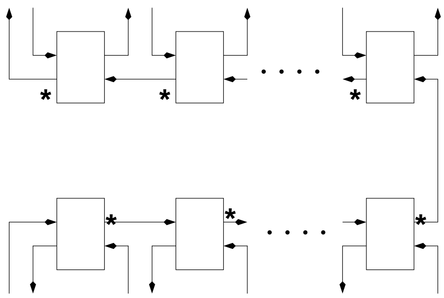
(c) We wish to observe here that the dual of the subfactor $(M^H \subset M)$ is isomorphic to $(M^{H^*} \subset M)$. One way of seeing this is to verify that the equation

$$\tilde{\alpha}_f(\pi(x)\lambda(a)) = \sum f(a_2)\pi(x)\lambda(a_1)$$

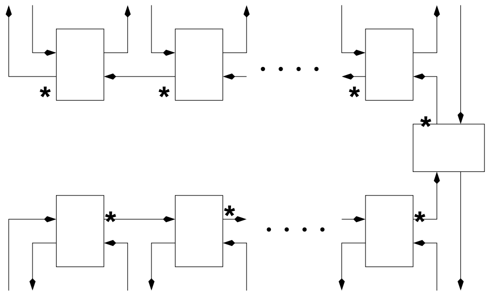
extends uniquely to an outer action of H^* on $M \rtimes H$ such that $(M \rtimes H)^{H^*} = M$. We omit the straightforward verifications. (This is just 'the dual action', and it follows from Theorem 4.11 and standard facts about 'duality for subfactors' (see [PP] or [JS], for instance) that $(M \rtimes_\alpha H) \rtimes_{\tilde{\alpha}} H^* \cong M \otimes End_{\mathbb{C}}(H)$.)

(d) Suppose that $H = \mathbb{C}G$, with G a finite group. Then, the minimal (central) projections of H^* are given by the 'point-evaluations' $\{ev_g : g \in G\}$ and the associated dual basis is $\{g : g \in G\}$; so the presentation given by Theorem 5.3 is equivalent to the one given in [L].

(e) As in [L], we find that for each $k \geq 2$, an orthonormal basis for $P_k^{M^H \subset M}$ is given by $\{Z_{B_k}(g_1, \cdots, g_{k-1}) : g_i \in \{\widetilde{\gamma_{kl}} : \gamma kl\}, 1 \leq i \leq k-1\}$, where the B_k's are the k-tangles given as follows:



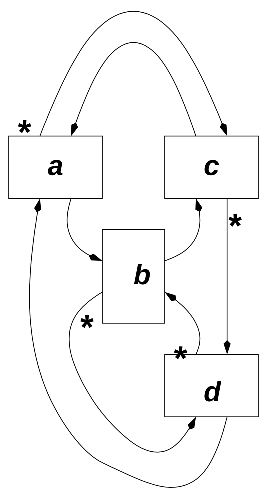
if k is odd, and



if k is even. The proof is very similar to the group case.

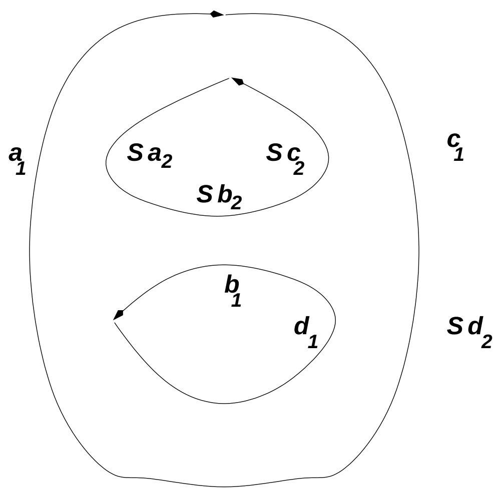
(f) Again, as in [L], the partition function - of what we have called $P(L;R)$ - can seen to be obtained according to the following prescription, which we illustrate in an example, rather than give the abstract prescription (which is the same for tangles of colour 0_+ and 0_-).

Suppose we want to compute the partition function of the following 0_+-tangle:



1. Replace each labelled 2-box - with label l (say) - by a pair of parallel strands and insert a symbol l_1 close to the strand through the distinguished point ($*$) and a symbol Sl_2 ($= S(l_2)$) close to the other strand; thus, in our example, we would arrive

at the following:



2. Then arbitrarily pick a base point on each component of the resulting figure, read the labels on that component in the order opposite to that prescribed by the orientation of the loop, evaluate $\sqrt{n}\phi$ on each resulting product, multiply the answers, and form the summations indicated. Thus, in our example, we would obtain

$$\sum_{(a)}\sum_{(b)}\sum_{(c)}\sum_{(d)} \sqrt{n}\phi(a_1(Sd_2)c_1)\,\sqrt{n}\phi((Sc_2)(Sb_2)(Sa_2))\,\sqrt{n}\phi(b_1d_1)\,,$$

and this is the value of the partition function of the labelled 0-tangle that we started with. (The answer is independent of the choices of base-points since ϕ is a trace.)

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