Operator-Valued Free Probability Theory and Block Random Matrices

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Queen's University
Kingston

I. **Operator-valued semicircular elements and block random matrices**

II. **General operator-valued free probability theory**

- Voiculescu: Asterisque 232, 1995
- Speicher: Memoir of the AMS 627 (1998)
- Rashidi Far, Oraby, Bryc, Speicher: IEEE Transactions on Information Theory 54, 2008

I. Operator-valued semicircular elements and block random matrices

Consider **Gaussian $N \times N$-random matrices**

$$A_N = \left(a_{ij}\right)_{i,j=1}^{N}$$

i.e., A_N is $N \times N$-matrix, where a_{ij} are random variables whose distribution is determined as follows:

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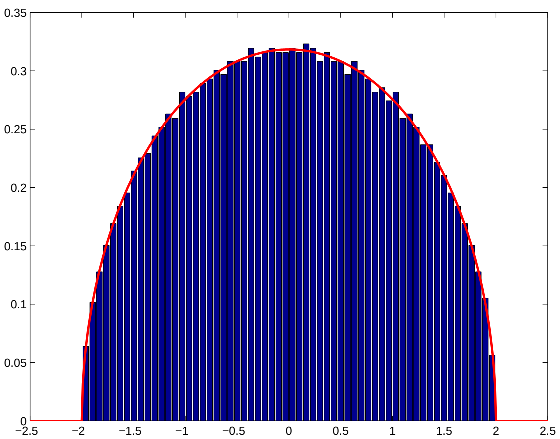
i.e., A_N is $N \times N$-matrix, where a_{ij} are random variables whose distribution is determined as follows:

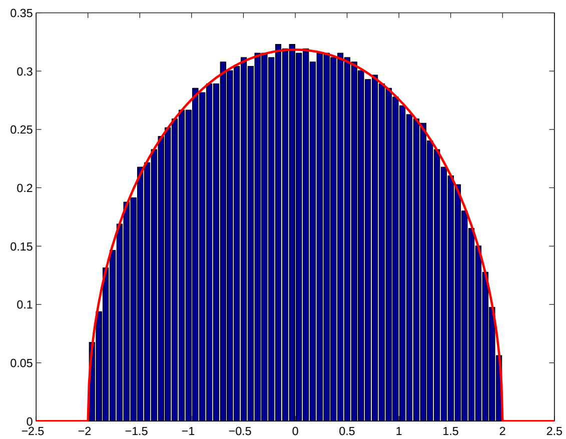
- A_N is selfadjoint, i.e., $a_{ji} = \bar{a}_{ij}$
- otherwise, all entries are independent and identically distributed with centered normal distribution of variance $1/N$

Convergence of **typical eigenvalue distribution** of Gaussian $N \times N$ random matrices to **Wigner's semicircle**

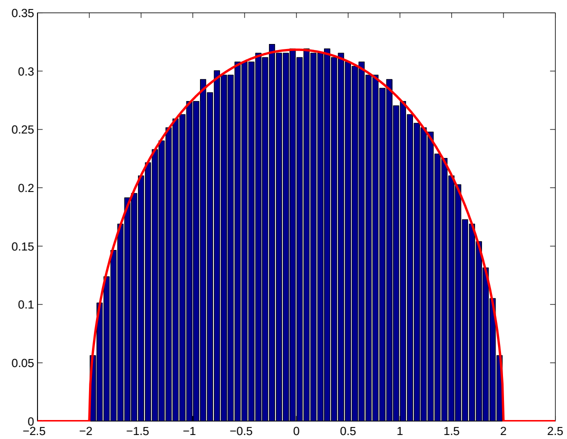
$N = 4000$



... one realization ...



... another realization ...



... yet another one ...

Consider **the empirical eigenvalue distribution of** A_N

$$\mu_{A_N}(\omega) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(\omega)}$$

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Then **Wigner's semicircle law** says that

$$\boldsymbol{\mu_{A_N} \Longrightarrow \mu_W} \qquad \textbf{almost surely,}$$

i.e., for all continuous and bounded f

$$\lim_{N\to\infty} \int_{\mathbb{R}} f(t) d\mu_{A_N}(t) = \int_{\mathbb{R}} f(t) d\mu_W(t) = \frac{1}{2\pi}\int_{-2}^{2} f(t)\sqrt{4-t^2}dt$$

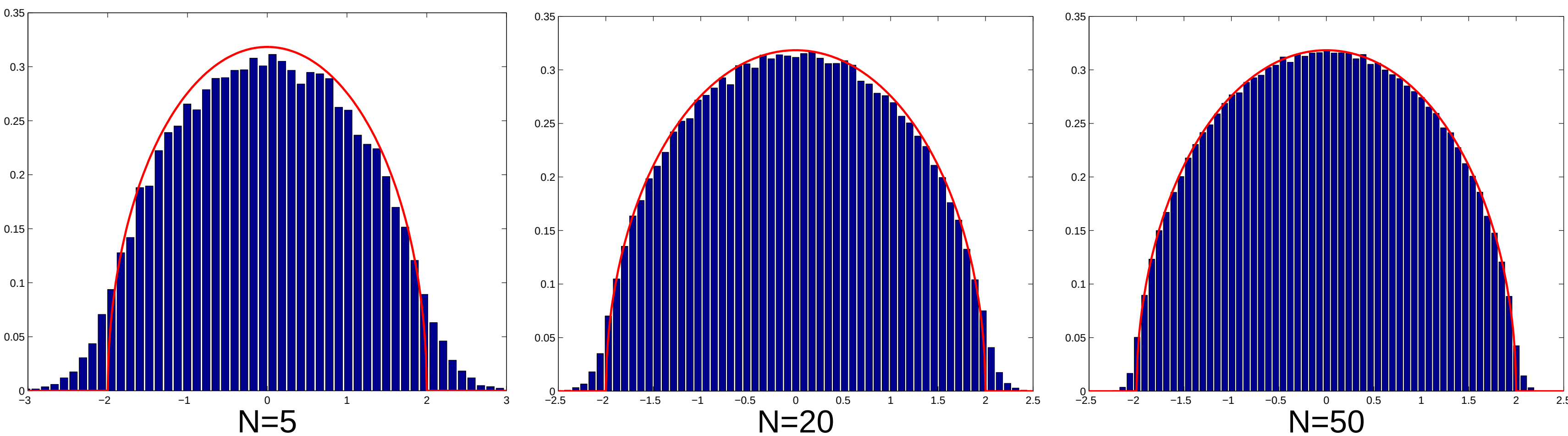
Show

$$\lim_{N\to\infty} \mu_{A_N}(f) = \mu_W(f) \qquad \text{almost surely}$$

in two steps:

- $\lim_{N\to\infty} E[\mu_{A_N}(f)] = \mu_W(f)$
- $\sum_N \mathrm{Var}[\mu_{A_N}(f)] < \infty$

Convergence of **averaged eigenvalue distribution** of Gaussian $N \times N$ random matrices to **Wigner's semicircle**



number of realizations $= 10000$

For

$$\lim_{N\to\infty} E[\mu_{A_N}(f)] = \mu_W(f)$$

it suffices to treat **convergence of all averaged moments**, i.e.,

$$\lim_{N\to\infty} E[\int t^n d\mu_{A_N}(t)] = \int t^n d\mu_W(t) \qquad \forall n \in \mathbb{N}$$

Note:

$$E[\int t^n d\mu_{A_N}(t)] = E[\frac{1}{N}\sum_{i=1}^{N} \lambda_i^n] = E[\mathrm{tr}(A_N^n)]$$

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$$E[\text{tr}(A_N^n)] = \frac{1}{N}\sum_{i_1,\dots,i_n=1}^{N} E[a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}]$$

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Asymptotically, for $N \to \infty$, only **non-crossing pairings** survive:

$$\lim_{N\to\infty} E[\mathrm{tr}(A_N^n)] = \# NC_2(n)$$

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Define limiting **semicircle element** s by

$$\varphi(s^n) := \#NC_2(n).$$

($s \in \mathcal{A}$, where $\mathcal{A}$ is some unital algebra, $\varphi : \mathcal{A} \to \mathbb{C}$)

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Then we say that our Gaussian random matrices A_N converge in distribution to the semicircle element s,

$$\boldsymbol{A_N \xrightarrow{\text{distr}} s}$$

What is distribution of s?

Claim:

$$\varphi(s^n) = \int t^n d\mu_W(t)$$

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more concretely:

$$\#NC_2(n) = \frac{1}{2\pi} \int_{-2}^{+2} t^n \sqrt{4 - t^2} dt$$

What is distribution of s?

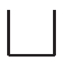
$n = 2$: $\varphi(s^2) =$

$n = 4$: $\varphi(s^4) =$

$n = 6$: $\varphi(s^6) =$

What is distribution of s?

$n = 2$: $\varphi(s^2) = 1$



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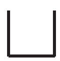

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$n = 6$: $\varphi(s^6) =$

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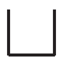
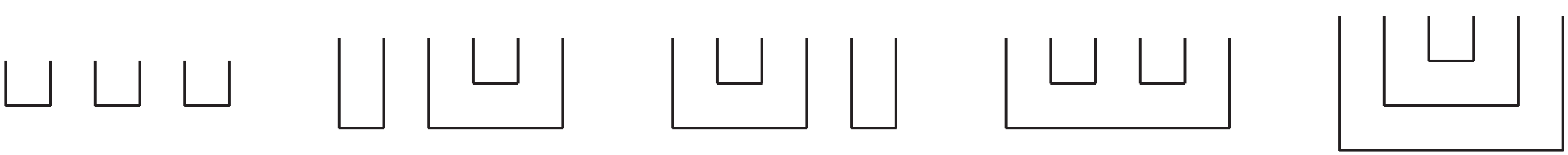


$n = 4$: $\varphi(s^4) = 2$

$n = 6$: $\varphi(s^6) = 5$



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- $C_k = \frac{1}{k+1}\binom{2k}{k}$
- C_k is determined by $C_0 = C_1 = 1$ and the recurrence relation

$$C_k = \sum_{l=1}^{k} C_{l-1} C_{k-l}.$$

$$\varphi(s^{2k}) = \sum_{l=1}^{k} \varphi(s^{2l-2})\varphi(s^{2k-2l})$$

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Put

$$M(z) := \sum_{n=0}^{\infty} \varphi(s^n) z^n = 1 + \sum_{k=1}^{\infty} \varphi(s^{2k}) z^{2k}$$

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Then

$$M(z) = 1 + z^2 \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi(s^{2l-2}) z^{2l-2} \varphi(s^{2k-2l}) z^{2k-2l}$$

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$$= 1 + z^2 M(z) \cdot M(z)$$

$$M(z) = 1 + z^2 M(z) \cdot M(z)$$

Instead of moment generating series $M(z)$ consider **Cauchy transform**

$$G(z) := \varphi\left(\frac{1}{z-s}\right)$$

Note

$$G(z) = \sum_{n=0}^{\infty} \frac{\varphi(s^n)}{z^{n+1}} = \frac{1}{z} \sum_{n=0}^{\infty} \varphi(s^n) \Big(\frac{1}{z}\Big)^n = \frac{1}{z} M(1/z),$$

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thus

$$zG(z) = 1 + G(z)^2$$

For any probability measure μ on $\mathbb{R}$ corresponding Cauchy transform

$$G(z) := \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t)$$

is analytic function on complex upper half plane $\mathbb{C}^+$ and allows to recover μ via **Stieltjes inversion formula**

$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im} G(t + i\varepsilon)$$

For semicircle s:

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implies

$$G(z) = \frac{z - \sqrt{z^2 - 4}}{2}$$

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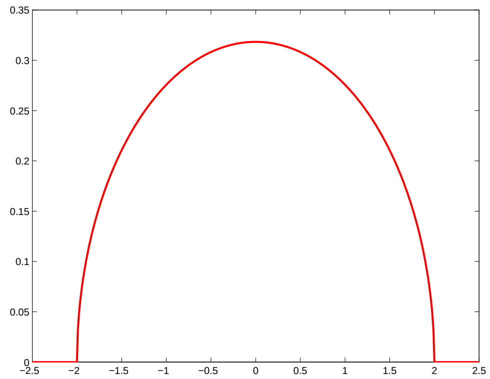
and thus

$$d\mu(t) = \frac{1}{2\pi}\sqrt{4 - t^2}dt \qquad \text{on } [-2, 2]$$



Consider now more general random matrices

- Keep the entries independent, but change distribution of entries, globally or depending on position of entry

  Arnold
  Bai und Silverstein
  Molchanov, Pastur, Khorunzhii (1996)
  Khorunzhy, Khoruzhenko, Pastur (1996)
  Shlyakhtenko (1996)
  Guionnet (2002)
  Anderson, Zeitouni (2006)

- Keep the distributions normal, but allow correlations between entries

  – for weak correlations one still gets semicircle

    Chatterjee (2006)
    Götze + Tikhomirov (2005)
    Schenker und Schulz-Baldes (2006)

  – for stronger correlations other distributions occur

    Boutet de Monvel, Khorunzhy, Vasilchuck (1996)
    Girko (2001)
    Hachem, Loubaton, Najim (2005)
    Anderson, Zeitouni (2008)
    Rashidi Far, Oraby, Bryc, Speicher (2008)

Consider **block matrix**

$$X_N = \begin{pmatrix} A_N & B_N & C_N \\ B_N & A_N & B_N \\ C_N & B_N & A_N \end{pmatrix},$$

where A_N, B_N, C_N are independent Gaussian $N \times N$-random matrices.

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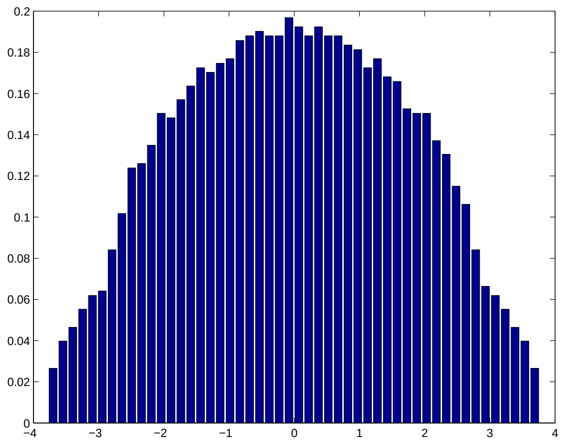
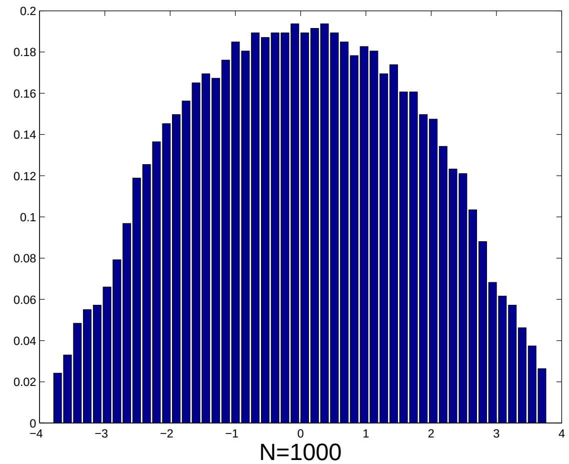
What is eigenvalue distribution of X_N for $N \to \infty$?

typical eigenvalue distribution for $N = 1000$





... one realization ...

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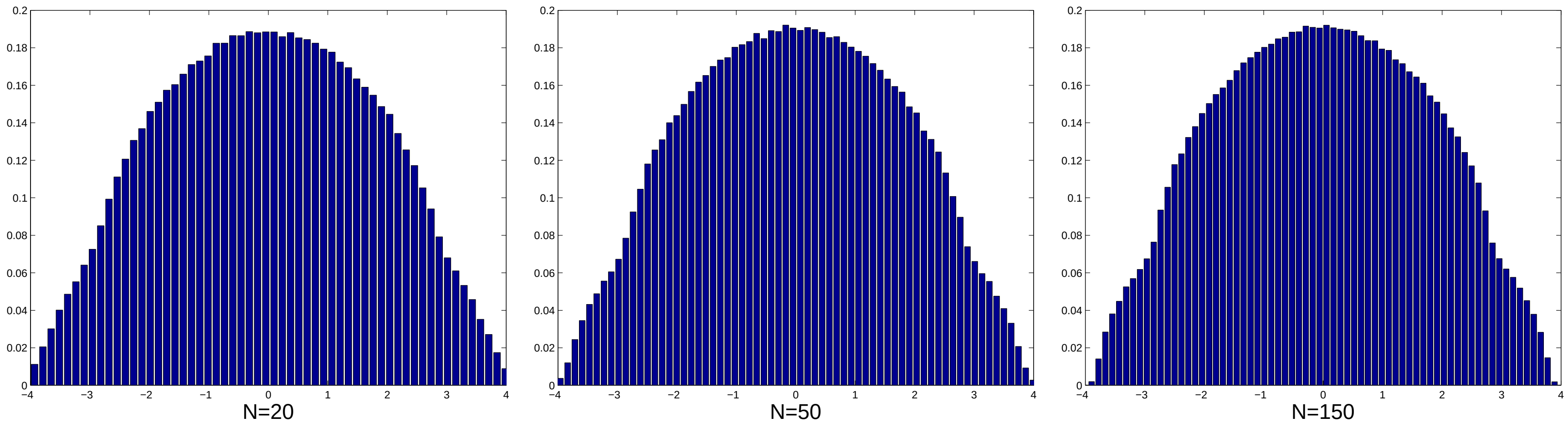
averaged eigenvalue distributions



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However, it fits well into the frame of

operator-valued free probability theory!

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$\mathbb{C}$		$\mathcal{B}$

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		$E[b_1 a b_2] = b_1 E[a] b_2$
moments	$\longrightarrow$	operator-valued moments
$\varphi(a^n)$		$E[a b_1 a b_2 a \cdots a b_{n-1} a]$

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Consider an operator-valued probability space

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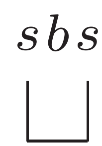
- higher moments of s are given in terms of second moments by summing over non-crossing pairings

$$E[sbs] = \eta(b)$$



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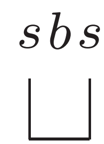
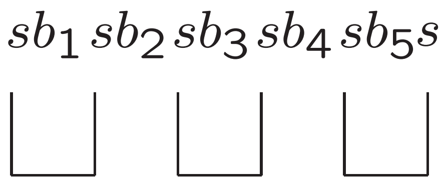
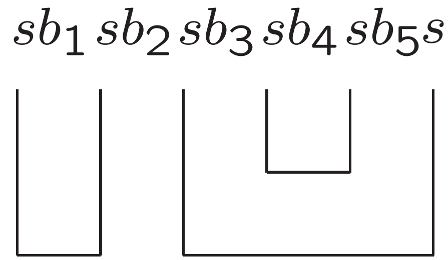
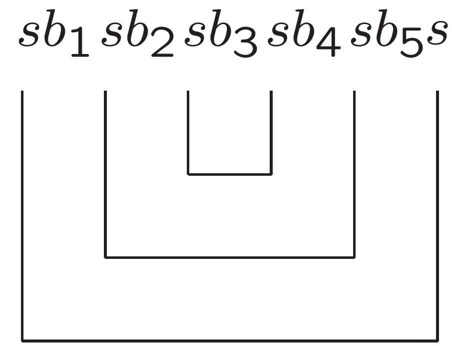


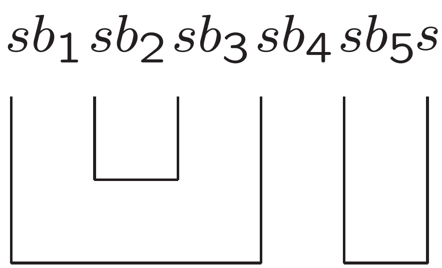
$$E[sb_1sb_2s\cdots sb_{n-1}s] = \sum_{\pi\in NC_2(n)} \Big(\text{iterated application of } \eta \text{ according to } \pi\Big)$$

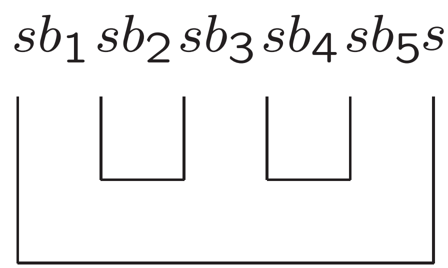






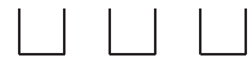
$$\eta(b_1)\cdot b_2\cdot\eta(b_3)\cdot b_4\cdot\eta(b_5) \qquad \eta(b_1)\cdot b_2\cdot\eta\Big(b_3\cdot\eta(b_4)\cdot b_5\Big) \qquad \eta\Big(b_1\cdot\eta\Big(b_2\cdot\eta(b_3)\cdot b_4\Big)\cdot b_5\Big)$$



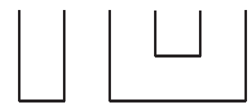


$$\eta\Big(b_1\cdot\eta(b_2)\cdot b_3\Big)\cdot b_4\cdot\eta(b_5) \qquad \eta\Big(b_1\cdot\eta(b_2)\cdot b_3\cdot\eta(b_4)\cdot b_5\Big)$$

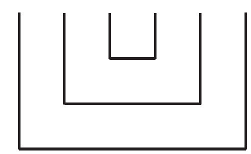
$$E[sb_1sb_2sb_3sb_4sb_5s] = \eta(b_1)\cdot b_2\cdot\eta(b_3)\cdot b_4\cdot\eta(b_5)$$



$$+\,\eta(b_1)\cdot b_2\cdot\eta\Big(b_3\cdot\eta(b_4)\cdot b_5\Big)$$



$$+\,\eta\Big(b_1\cdot\eta\Big(b_2\cdot\eta(b_3)\cdot b_4\Big)\cdot b_5\Big)$$



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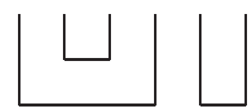

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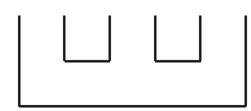
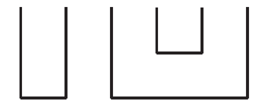
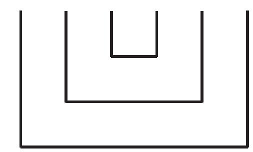


$$E[ssssss] = \eta(1) \cdot \eta(1) \cdot \eta(1)$$

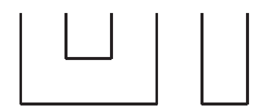
$$+ \eta(1) \cdot \eta\Big(\eta(1)\Big)$$



$$+ \eta\Bigg(\eta\Big(\eta(1)\Big)\Bigg)$$



$$+ \eta\Big(\eta(1)\Big) \cdot \eta(1)$$



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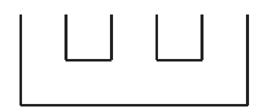

We have the recurrence relation

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Put

$$M(z) := \sum_{n=0}^{\infty} E[s^n] z^n = 1 + \sum_{k=1}^{\infty} E[s^{2k}] z^{2k},$$

thus

$$M(z) = 1 + z\eta\Big(M(z)\Big) \cdot M(z)$$

Consider the **operator-valued Cauchy transform**

$$G(z) := E[\frac{1}{z-s}]$$

for $z \in \mathbb{C}^+$.

Note

$$G(z) = E[\frac{1}{z} \cdot \frac{1}{1 - sz^{-1}}] = \frac{1}{z} M(sz^{-1}),$$

thus

$$zG(z) = 1 + \eta\Big(G(z)\Big) \cdot G(z)$$

Thus: operator-valued Cauchy-transform of s

$$G : \mathbb{C}^+ \to \mathcal{B}$$

satisfies

- G analytic

- G solution of

$$zG(z) = 1 + \eta\Big(G(z)\Big) \cdot G(z)$$

- $G(z) \sim \frac{1}{z}1$ for $z \to \infty$

back to random matrices

special classes of random matrices are asymptotically described by operator-valued semicircular elements, e.g.

- band matrices (Shlyakhtenko 1996)

- block matrices (Rashidi Far, Oraby, Bryc, Speicher 2006)

Example:

$$X_N = \begin{pmatrix} A_N & B_N & C_N \\ B_N & A_N & B_N \\ C_N & B_N & A_N \end{pmatrix},$$

where A_N, B_N, C_N are independent Gaussian $N \times N$ random matrices

For $N \to \infty$, X_N converges to

$$s = \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & s_1 & s_2 \\ s_3 & s_2 & s_1 \end{pmatrix},$$

where s_1, s_2, s_3 is free semicircular family.

$$s = \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & s_1 & s_2 \\ s_3 & s_2 & s_1 \end{pmatrix}, \qquad s_1, s_2, s_3 \in (\tilde{\mathcal{A}}, \varphi)$$

s is an operator-valued semicircular element over $M_3(\mathbb{C})$ with respect to

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- $\mathcal{A} = M_3(\tilde{\mathcal{A}}), \qquad \mathcal{B} = M_3(\mathbb{C})$

- $E = \mathrm{id} \otimes \varphi : M_3(\tilde{\mathcal{A}}) \to M_3(\mathbb{C}), \quad \Big(a_{ij}\Big)_{i,j=1}^{3} \mapsto \Big(\varphi(a_{ij})\Big)_{i,j=1}^{3}$

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- $\eta : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ given by $\eta(D) = E[sDs]$

$$s = \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & s_1 & s_2 \\ s_3 & s_2 & s_1 \end{pmatrix}, \qquad s_1, s_2, s_3 \in (\tilde{\mathcal{A}}, \varphi)$$

Asymptotic eigenvalue distribution μ of X_N is given by distribution of s with respect to $\mathrm{tr}_3 \otimes \varphi$:

$$H(z) = \int \frac{1}{z-t} d\mu(t) = \mathrm{tr}_3 \otimes \varphi\Big(\frac{1}{z-s}\Big)$$

$$s = \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & s_1 & s_2 \\ s_3 & s_2 & s_1 \end{pmatrix}, \qquad s_1, s_2, s_3 \in (\tilde{\mathcal{A}}, \varphi)$$

Asymptotic eigenvalue distribution μ of X_N is given by distribution of s with respect to $\text{tr}_3 \otimes \varphi$:

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and $G(z) = E[\frac{1}{z-s}]$ is solution of

$$zG(z) = 1 + \eta\Big(G(z)\Big) \cdot G(z)$$

$$X = \begin{pmatrix} A & B & C \\ B & A & B \\ C & B & A \end{pmatrix}:$$

$$X = \begin{pmatrix} A & B & C \\ B & A & B \\ C & B & A \end{pmatrix} : \qquad G(z) = \begin{pmatrix} f(z) & 0 & h(z) \\ 0 & g(z) & 0 \\ h(z) & 0 & f(z) \end{pmatrix}$$

$$X = \begin{pmatrix} A & B & C \\ B & A & B \\ C & B & A \end{pmatrix} : \qquad G(z) = \begin{pmatrix} f(z) & 0 & h(z) \\ 0 & g(z) & 0 \\ h(z) & 0 & f(z) \end{pmatrix}$$

$$\eta\Big(G(z)\Big) = \frac{1}{3}\begin{pmatrix} 2\,f(z) + g(z) & 0 & g(z) + 2\,h(z) \\ 0 & 2\,f(z) + g(z) + 2\,h(z) & 0 \\ g(z) + 2\,h(z) & 0 & 2\,f(z) + g(z) \end{pmatrix},$$

$$X = \begin{pmatrix} A & B & C \\ B & A & B \\ C & B & A \end{pmatrix} : \qquad G(z) = \begin{pmatrix} f(z) & 0 & h(z) \\ 0 & g(z) & 0 \\ h(z) & 0 & f(z) \end{pmatrix}$$

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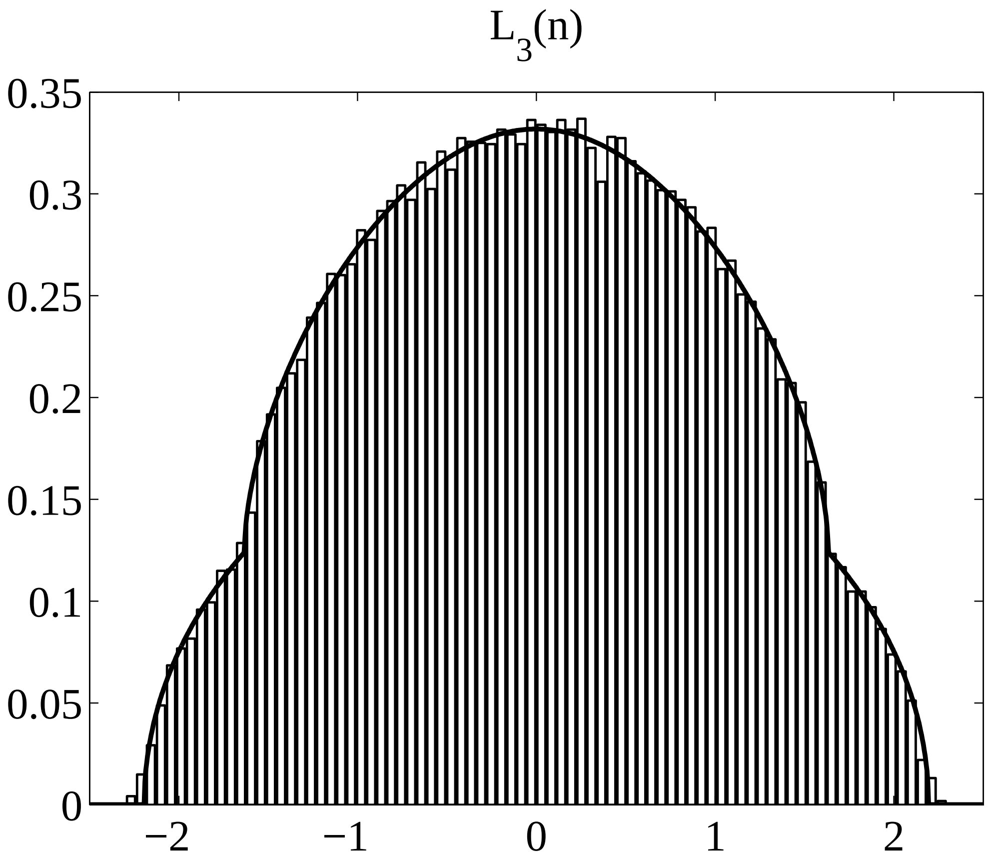
$$\eta\Big(G(z)\Big) = \frac{1}{3}\begin{pmatrix} 2\,f(z) + g(z) & 0 & g(z) + 2\,h(z) \\ 0 & 2\,f(z) + g(z) + 2\,h(z) & 0 \\ g(z) + 2\,h(z) & 0 & 2\,f(z) + g(z) \end{pmatrix},$$

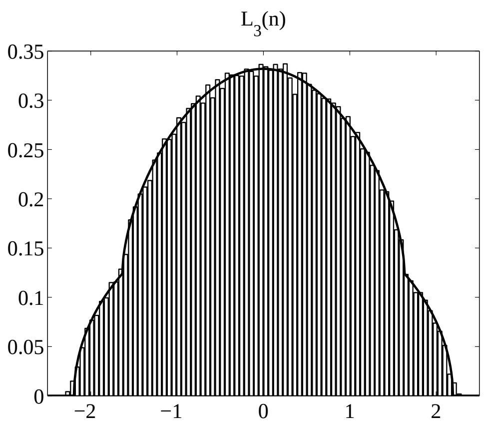
$$zG(z) = 1 + \eta\Big(G(z)\Big) \cdot G(z)$$

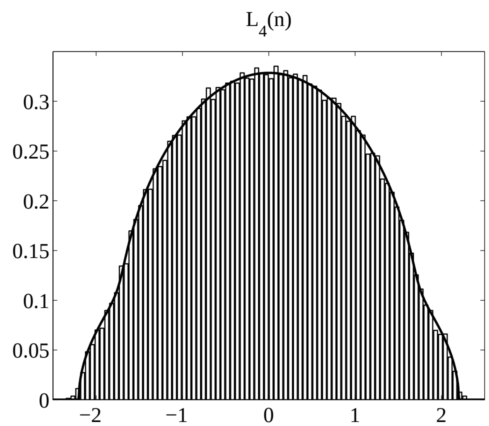
$$H(z) = \mathrm{tr}_3(G(z)) = \frac{1}{3}(2f(z) + g(z))$$

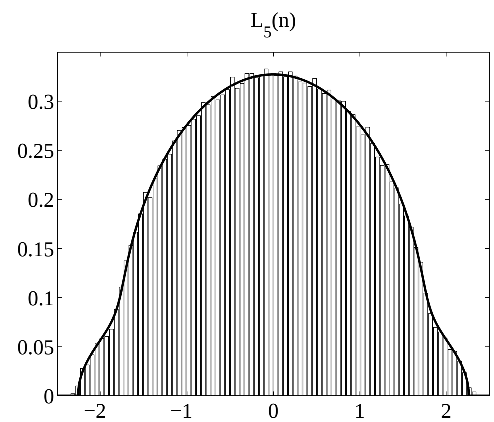
Comparison of this solution with simulations



some more examples







$$\begin{pmatrix} A & B & C \\ B & A & B \\ C & B & A \end{pmatrix} \qquad \begin{pmatrix} A & B & C & D \\ B & A & B & C \\ C & B & A & B \\ D & C & B & A \end{pmatrix} \qquad \begin{pmatrix} A & B & C & D & E \\ B & A & B & C & D \\ C & B & A & B & C \\ D & C & B & A & B \\ E & D & C & B & A \end{pmatrix},$$

II. General operator-valued free probability theory

What can we say about the relation between two matrices, when we know that the entries of the matrices are free?

$$X = (x_{ij})_{i,j=1}^{N} \qquad Y = (y_{kl})_{k,l=1}^{N}$$

with

$\{x_{ij}\}$ and $\{y_{kl}\}$ free w.r.t. φ

$$\Downarrow$$

X and Y are not free w.r.t. $\text{tr} \otimes \varphi$ in general

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$$\Downarrow$$

X and Y are not free w.r.t. $\text{tr} \otimes \varphi$ in general

however: relation between X and Y is more complicated, but still treatable in terms of

operator-valued freeness

Let $(\mathcal{C}, \varphi)$ be non-commutative probability space.

Consider $N \times N$ matrices over $\mathcal{C}$:

$$M_N(\mathcal{C}) := \{(a_{ij})_{i,j=1}^N \mid a_{ij} \in \mathcal{C}\}$$

$$M_N(\mathcal{C}) = M_N(\mathbb{C}) \otimes \mathcal{C}$$

is a non-commutative probability space with respect to

$$\mathrm{tr} \otimes \varphi : M_N(\mathcal{C}) \to \mathbb{C}$$

but there is also an intermediate level

Instead of

$$M_N(\mathcal{C})$$

$$\downarrow \mathrm{tr} \otimes \varphi$$

$$\mathbb{C}$$

consider ...

$$
\begin{array}{cc}
M_N(\mathcal{C}) = M_N(\mathbb{C}) \otimes \mathcal{C} & \\
\downarrow \mathsf{id} \otimes \varphi & \Big| \\
M_N(\mathbb{C}) & \Big| \ \mathsf{tr} \otimes \varphi \\
\downarrow \mathsf{tr} & \downarrow \\
\mathbb{C} &
\end{array}
$$

$$
\begin{array}{ccc}
M_N(\mathcal{C}) = M_N(\mathbb{C}) \otimes \mathcal{C} =: \mathcal{A} & & \\
\downarrow \mathrm{id} \otimes \varphi =: E & & | \\
M_N(\mathbb{C}) =: \mathcal{B} & & | \; \mathrm{tr} \otimes \varphi \\
\downarrow \mathrm{tr} & & \downarrow \\
\mathbb{C} & &
\end{array}
$$

Let $\mathcal{B} \subset \mathcal{A}$ be a unital subalgebra. A linear map

$$E : \mathcal{A} \to \mathcal{B}$$

is a **conditional expectation** if

$$E[b] = b \qquad \forall b \in \mathcal{B}$$

and

$$E[b_1 a b_2] = b_1 E[a] b_2 \qquad \forall a \in \mathcal{A}, \quad \forall b_1, b_2 \in \mathcal{B}$$

An **operator-valued probability space** consists of $\mathcal{B} \subset \mathcal{A}$ and a conditional expectation $E : \mathcal{A} \to \mathcal{B}$

Consider an operator-valued probability space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$. The **operator-valued distribution** of $a \in \mathcal{A}$ is given by all operator-valued moments

$$E[ab_1ab_2 \cdots b_{n-1}a] \in \mathcal{B} \qquad (n \in \mathbb{N}, b_1, \dots, b_{n-1} \in \mathcal{B})$$

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Random variables $x, y \in \mathcal{A}$ are **free with respect to E** (or **free with amalgamation over $\mathcal{B}$)** if

$$E[p_1(x)q_1(y)p_2(x)q_2(y) \cdots] = 0$$

whenever p_i, q_j are polynomials with coefficients from $\mathcal{B}$ and

$$E[p_i(x)] = 0 \quad \forall i \qquad \text{and} \qquad E[q_j(y)] = 0 \quad \forall j.$$

Note: polynomials in x with coefficients from $\mathcal{B}$ are of the form

- x^2
- $b_0 x^2$

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- x^2
- $b_0 x^2$
- $b_1 x b_2 x b_3$

b's and x do not commute in general!

Note: polynomials in x with coefficients from $\mathcal{B}$ are of the form

- x^2
- b_0x^2
- $b_1xb_2xb_3$
- $b_1xb_2xb_3 + b_4xb_5xb_6 + \cdots$
- etc.

b's and x do not commute in general!

Operator-valued freeness works mostly like ordinary freeness, one only has to take care of the order of the variables; in all expressions they have to appear in their original order!

Example: If x and $\{y_1, y_2\}$ are free, then one has

$$E[y_1 x y_2] = E\Big[y_1 E[x] y_2\Big];$$

and more general

$$E[y_1 b_1 x b_2 y_2] = E\Big[y_1 b_1 E[x] b_2 y_2\Big].$$

Consider $E : \mathcal{A} \to \mathcal{B}$.

Define **free cumulants**

$$\kappa_n^{\mathcal{B}} : \mathcal{A}^n \to \mathcal{B}$$

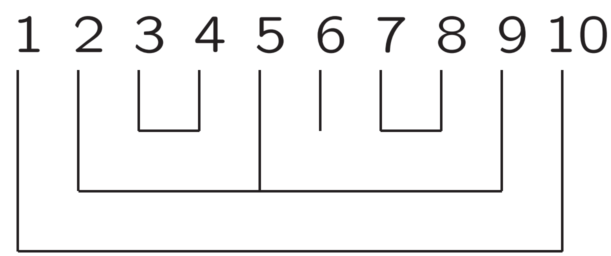
by

$$E[a_1 \cdots a_n] = \sum_{\pi \in NC(n)} \kappa_\pi^{\mathcal{B}}[a_1, \dots, a_n]$$

- arguments of $\kappa_\pi^{\mathcal{B}}$ are distributed according to blocks of π
- but now: cumulants are nested inside each other according to nesting of blocks of π

Example:

$$\pi = \Big\{\{1,10\},\{2,5,9\},\{3,4\},\{6\},\{7,8\}\Big\} \in NC(10),$$



$$\kappa_\pi^{\mathcal{B}}[a_1,\dots,a_{10}]$$

$$= \kappa_2^{\mathcal{B}}\Big(a_1 \cdot \kappa_3^{\mathcal{B}}\big(a_2 \cdot \kappa_2^{\mathcal{B}}(a_3,a_4), a_5 \cdot \kappa_1^{\mathcal{B}}(a_6) \cdot \kappa_2^{\mathcal{B}}(a_7,a_8), a_9\big), a_{10}\Big)$$

For $a \in \mathcal{A}$ define its **operator-valued Cauchy transform**

$$G_a(b) := E[\frac{1}{b-a}] = \sum_{n \geq 0} E[b^{-1}(ab^{-1})^n]$$

and **operator-valued R-transform**

$$\begin{aligned} R_a(b) :&= \sum_{n \geq 0} \kappa_{n+1}^{\mathcal{B}}(ab, ab, \dots, ab, a) \\ &= \kappa_1^{\mathcal{B}}(a) + \kappa_2^{\mathcal{B}}(ab, a) + \kappa_3^{\mathcal{B}}(ab, ab, a) + \cdots \end{aligned}$$

Then

$$bG(b) = 1 + R(G(b)) \cdot G(b) \qquad \text{or} \qquad G(b) = \frac{1}{b - R(G(b))}$$

If x and y are free over $\mathcal{B}$, then

- mixed $\mathcal{B}$-valued cumulants in x and y vanish

- $R_{x+y}(b) = R_x(b) + R_y(b)$

- $G_{x+y}(b) = G_x\Big[b - R_y\Big(G_{x+y}(b)\Big)\Big]$ subordination

If s is a semicircle over $\mathcal{B}$ then

$$R_s(b) = \eta(b)$$

where $\eta : \mathcal{B} \to \mathcal{B}$ is a linear map given by

$$\eta(b) = E[sbs].$$

Back to our matrix example

Proposition and Exercise:

Assume $\{x_{ij}\}$ and $\{y_{kl}\}$ are free in the n.c.p.s. $(\mathcal{C}, \varphi)$.

Then the matrices

$$X = (x_{ij})_{i,j=1}^{N}, \qquad Y = (y_{kl})_{k,l=1}^{N} \qquad \in \qquad M_N(\mathcal{C})$$

are free with amalgamation over $M_N(\mathbb{C}) \subset M_N(\mathcal{C})$,

i.e., with respect to $E = \text{id} \otimes \varphi$.

A final remark

The analytic theory of operator-valued free probability lacks at the moment some of the deeper statements of the scalar-valued theory (like Nevalinna-like characterizations of operator-valued Cauchy-transforms);

developing a reasonable analogue of complex function theory on an operator-valued level is an active area in free probability at the moment, see

Voiculescu: Free Analysis Questions, IMRN 2004

(see also recent work of Helton, Vinnikov, Belinschi, Popa)