Chapter 1

Hilbert space

1.1 Introduction

This book is about (bounded, linear) operators on (**always separable and complex**) Hilbert spaces, usually denoted by $\mathcal{H}, \mathcal{K}, \mathcal{M}$ and variants thereof. Vectors in Hilbert spaces will usually be denoted by symbols such as x, y, z and their variants, such as y_n, x'. The collection of all bounded complex-linear operators on $\mathcal{H}$ will be denoted by $B(\mathcal{H})$, whose elements will usually be denoted by symbols such as $A, B, E, F, P, Q, T, U, V, X, Y, Z$.

The only prerequisites needed for reading this book are: a nodding acquaintance with the basics of Hilbert space theory (eg: the definitions of orthonormal basis, orthogonal projection, unitary operator, etc., all of which are briefly discussed in this chapter); a first course in Functional Analysis – the spectral radius formula, the Open Mapping Theorem and the Uniform Boundedness Principle, the Riesz Representation Theorem (briefly mentioned in Appendix) which identifies $C(\Sigma)^*$ with the space $M(\Sigma)$ of finite complex measures on the compact Hausdorff space Σ, and outer and inner regularity of finite positive measures on Σ; some basic measure theory, such as the Bounded Convergence Theorem, and the not so basic Lusin's theorem (also briefly discussed in Appendix) which leads to the conclusion – see Lemma A2 in the Appendix – that any bounded measurable function on Σ is the pointwise a.e. limit of a sequence of continuous functions on Σ, and also – see Lemma A1 in the Appendix – that $C(\Sigma)$ 'is' dense in $L^2(\Sigma, \mu)$. Also, in the section on **von Neumann-Schatten ideals**, basic facts concerning the Banach sequence spaces c_0, ℓ^p and the duality relations among them will be needed/used. All the above facts may be found in [Hal], [Hal1], [Sun] and [AthSun]. Although these standard facts may also be found in other classical texts written by distinguished mathematicians, the references are limited to a very small number of books, because the author knows precisely where which fact can be found in the union of the four books mentioned above.

1.2 Inner Product spaces

While normed spaces permit us to study 'geometry of vector spaces', we are constrained to discussing those aspects which depend only upon the notion of 'distance between two points'. If we wish to discuss notions that depend upon the angles between two lines, we need something more – and that something more is the notion of an *inner product.*

The basic notion is best illustrated in the example of the space $\mathbb{R}^2$ that we are most familiar with, where the most natural norm is what we call $||\cdot||_2$. The basic fact from plane geometry that we need is the so-called *cosine law* which states that if A, B, C are the vertices of a triangle and if θ is the angle at the vertex C, then

$$2(AC)(BC)\cos\theta \;=\; (AC)^2 + (BC)^2 - (AB)^2 \;.$$

If we apply this to the case where the points A, B and C are represented by the vectors $x = (x_1, x_2),\ y = (y_1, y_2)$ and $(0,0)$ respectively, we find that

$$\begin{aligned} 2||x|| \cdot ||y|| \cdot \cos\theta &= ||x||^2 + ||y||^2 - ||x-y||^2 \\ &= 2\,(x_1 y_1 \;+\; x_2 y_2\,). \end{aligned}$$

Thus, we find that the function of two (vector) variables given by

$$\langle x, y\rangle = x_1 y_1 \;+\; x_2 y_2 \qquad (1.2.1)$$

simultaneously encodes the notion of angle as well as distance (and has the explicit interpretation $\langle x, y\rangle \;=\; ||x||\;||y||\cos\theta$). This is because the norm can be recovered from the inner product by the equation

$$||x|| \;=\; \langle x, x\rangle^{\frac{1}{2}}\;. \qquad (1.2.2)$$

The notion of an inner product is the proper abstraction of this function of two variables.

DEFINITION 1.2.1. *(a) An* **inner product** *on a (complex) vector space V is a mapping $V \times V \ni (x, y) \;\mapsto\; \langle x, y\rangle \in \mathbb{C}$ which satisfies the following conditions, for all $x, y, z \in V$ and $\alpha \in \mathbb{C}$:*
(i) (positive definiteness) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \;\Leftrightarrow\; x = 0$;
(ii) (Hermitian symmetry) $\langle x, y\rangle \;=\; \overline{\langle y, x\rangle}$;
(iii) (linearity in first variable) $\langle \alpha x + \beta z, y\rangle \;=\; \alpha\langle x, y\rangle + \beta\langle z, y\rangle$.

An **inner product space** *is a vector space equipped with a (distinguished) inner product.*

(b) An inner product space which is complete in the norm coming from the inner product (as in Equation (1.2.2)*) is called a* **Hilbert space**. *In this book, however, we shall only be concerned with Hilbert spaces which are separable when viewed as metric spaces, with the metric coming from the norm induced by the inner-product – see Proposition* 1.2.4 *and Corollary* 1.2.5.

EXAMPLE 1.2.2. (1) If $z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, define

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i} \; ; \tag{1.2.3}$$

it is easily verified that this defines an inner product on $\mathbb{C}^n$.

(2) The equation

$$\langle f, g \rangle = \int_{[0,1]} f(x)\overline{g(x)} \, dx \tag{1.2.4}$$

is easily verified to define an inner product on $C[0, 1]$. □

As in the (real) case discussed earlier of $\mathbb{R}^2$, it is generally true that any inner product gives rise to a norm on the underlying space via equation (1.2.2). Before verifying this fact, we digress with an exercise that states some easy consequences of the definitions.

EXERCISE 1.2.3. *Suppose we are given an inner product space V; for $x \in V$, define $||x||$ as in equation* (1.2.2), *and verify the following identities, for all $x, y, z \in V$, $\alpha \in \mathbb{C}$:*

(1) $\langle x, y + \alpha z \rangle = \langle x, y \rangle + \overline{\alpha} \langle x, z \rangle$;

(2) $||x + y||^2 = ||x||^2 + ||y||^2 + 2\,\mathrm{Re}\,\langle x, y \rangle$;

(3) *two vectors in an inner product space are said to be* **orthogonal** *if their inner product is* 0; *deduce from* (2) *above and an easy induction argument that if $\{x_1, x_2, \ldots, x_n\}$ is a set of pairwise orthogonal vectors, then $||\sum_{i=1}^{n} x_i||^2 = \sum_{i=1}^{n} ||x_i||^2$.*

(4) $||x + y||^2 + ||x - y||^2 = 2\,(||x||^2 + ||y||^2)$; *draw some diagrams and convince yourself as to why this identity is called the* **parallelogram identity**.

(5) *(Polarisation identity)* $4\langle x, y \rangle = \sum_{k=0}^{3} i^k \langle x + i^k y, x + i^k y \rangle$, *where, of course,* $i = \sqrt{-1}$.

The first (and very important) step towards establishing that any inner product defines a norm via equation (1.2.2) is the following celebrated inequality.

PROPOSITION 1.2.4. (**Cauchy-Schwarz inequality**)

If x, y are arbitrary vectors in an inner product space V, then

$$|\langle x, y \rangle| \leq ||x|| \cdot ||y|| \, .$$

Further, this inequality is an equality if and only if the vectors x and y are linearly dependent.

Proof. If $y = 0$, there is nothing to prove; so we may, without loss of generality, assume that $||y|| = 1$ (since the statement of the proposition is unaffected upon scaling y by a constant).

Notice now that, for arbitrary $\alpha \in \mathbb{C}$,

$$\begin{aligned} 0 &\leq ||x - \alpha y||^2 \\ &= ||x||^2 + |\alpha|^2 - 2\,\mathrm{Re}\,(\alpha\langle y, x\rangle) \ . \end{aligned}$$

A little exercise in the calculus shows that this last expression is minimised for the choice $\alpha_0 = \langle x, y\rangle$, for which choice we find, after some minor algebra, that

$$0 \leq ||x - \alpha_0 y||^2 = ||x||^2 - |\langle x, y\rangle|^2 \ ,$$

thereby establishing the desired inequality.

The above reasoning shows that the inequality becomes an equality only if $x = \alpha_0 y$, and the proof is complete. □

COROLLARY 1.2.5. *Any inner product gives rise to a norm*[1] *via Equation* (1.2.2).

Proof. Positive-definiteness and homogeneity with respect to scalar multiplication are obvious; as for the triangle inequality,

$$\begin{aligned} ||x + y||^2 &= ||x||^2 + ||y||^2 + 2\ \mathrm{Re}\ \langle x, y\rangle \\ &\leq ||x||^2 + ||y||^2 + 2||x|| \cdot ||y|| \ , \end{aligned}$$

and the proof is complete. □

EXERCISE 1.2.6. (1) *Show that*

$$\left|\sum_{i=1}^{n} z_i \overline{w_i}\right|^2 \leq \left(\sum_{i=1}^{n} |z_i|^2\right) \left(\sum_{i=1}^{n} |w_i|^2\right) , \ \forall\ z, w \in \mathbb{C}^n \ .$$

(In view of the notation used in (2) *below, we shall write* ℓ_n^2 *for* $\mathbb{C}^n$ *with the 'standard inner product' defined above.)*

(2) *Deduce from* (1) *that the series* $\sum_{i=1}^{\infty} \alpha_i \overline{\beta_i}$ *converges, for any* $\alpha, \beta \in \ell^2 = \{\gamma = (\gamma_1, \ldots, \gamma_n, \ldots) \in \mathbb{C}^{\mathbb{N}} : \sum_n |\gamma_n|^2 < \infty\}$, *and that*

$$\left|\sum_{i=1}^{\infty} \alpha_i \overline{\beta_i}\right|^2 \leq \left(\sum_{i=1}^{\infty} |\alpha_i|^2\right) \left(\sum_{i=1}^{\infty} |\beta_i|^2\right) , \ \forall\ \alpha, \beta \in \ell^2 \ ;$$

deduce that ℓ^2 *is indeed (a vector space, and in fact) an inner product space, with respect to inner product defined by*

$$\langle \alpha, \beta\rangle = \sum_{i=1}^{\infty} \alpha_i \overline{\beta}_i \ . \tag{1.2.5}$$

[1] Recall that (a) a norm on a vector space V is a function $V \ni x \mapsto ||x|| \in [0, \infty)$ which satisfies (*i*) (*positive-definiteness*) $||x|| = 0 \Leftrightarrow x = 0$; (*ii*) (*homogeneity*) $||\alpha x|| = |\alpha| ||x||$ and (*iii*) (*triangle inequality*) $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in V$ and $\alpha \in \mathbb{C}$; (b) a vector space equipped with a norm is a **normed space**; and (c) a normed space which is complete with respect to the norm is called a **Banach space**.

(3) *Write down what the Cauchy-Schwarz inequality translates into in Example* 1.2.2 (2).

(4) *Show that the inner product is continuous as a mapping from* $V \times V$ *into* $\mathbb{C}$. *(In view of Corollary* 1.2.5, *this makes sense.)*

1.3 Hilbert spaces : examples

Our first step is to arm ourselves with a reasonably adequate supply of examples of Hilbert spaces.

EXAMPLE 1.3.1. (1) $\mathbb{C}^n$ is an example of a finite-dimensional Hilbert space, and we shall soon see that these are essentially the only such examples.

(2) ℓ^2 is an infinite-dimensional Hilbert space – see Exercise 1.2.6(2). Nevertheless, this Hilbert space is not 'too big', since it is at least equipped with the pleasant feature of being a **separable** Hilbert space – i.e., it is separable as a metric space, meaning that it has a countable dense set. (Verify this assertion!)

(3) More generally, let S be an arbitrary set, and define

$$\ell^2(S) \;=\; \{x \;=\; ((x_s))_{s\in S} \in \mathbb{C}^S : \textstyle\sum_{s\in S} |x_s|^2 < \infty\} \;.$$

(The possibly uncountable sum might be interpreted as follows: a typical element of $\ell^2(S)$ is a family $x = ((x_s))$ of complex numbers which is indexed by the set S, and which has the property that $x_s = 0$ except for s coming from some countable subset of S (which depends on the element x) and which is such that the possibly non-zero x_s's, when written out as a sequence in any (equivalently, some) way, constitute a norm-square-summable sequence.)

Verify that $\ell^2(S)$, in a natural fashion, is a Hilbert space.

(4) This example will make sense to the reader who is already familiar with the theory of measure and Lebesgue integration; the reader who is not, may safely skip this example; the subsequent exercise will effectively recapture this example, at least in all cases of interest.

Suppose $(X, \mathcal{B}, \mu)$ is a measure space. Let $\mathcal{L}^2(X, \mathcal{B}, \mu)$ denote the space of $\mathcal{B}$-measurable complex-valued functions f on X such that $\int_X |f|^2 d\mu \; < \infty$. Note that $|f+g|^2 \;\leq\; 2(|f|^2 + |g|^2)$, and deduce that $\mathcal{L}^2(X, \mathcal{B}, \mu)$ is a vector space. Note next that $|f\overline{g}| \;\leq\; \frac{1}{2}(|f|^2 + |g|^2)$, and so the right-hand side of the following equation makes sense, if $f, g \in \mathcal{L}^2(X, \mathcal{B}, \mu)$:

$$\langle f, g\rangle \;=\; \int_X f\overline{g}\, d\mu \;. \tag{1.3.6}$$

It is easily verified that the above equation satisfies all the requirements of an inner product with the solitary possible exception of the positive-definiteness axiom: if $\langle f, f\rangle = 0$, it can only be concluded that $f = 0$ *a.e* – meaning that $\{x : f(x) \neq 0\}$ is a set of μ-measure 0 (which might very well be non-empty).

Observe, however, that the set $N = \{f \in \mathcal{L}^2(X, \mathcal{B}, \mu) : f = 0 \ a.e.\}$ is a vector subspace of $\mathcal{L}^2(X, \mathcal{B}, \mu)$; and a typical element of the quotient space $L^2(X, \mathcal{B}, \mu) = \mathcal{L}^2(X, \mathcal{B}, \mu)/N$ is just an equivalence class of square-integrable functions, where two functions are considered to be equivalent if they agree outside a set of μ-measure 0.

For simplicity of notation, we shall just write $L^2(X)$ or $L^2(\mu)$ for $L^2(X, \mathcal{B}, \mu)$, and we shall denote an element of $L^2(\mu)$ simply by such symbols as f, g, etc., and think of these as actual functions with the understanding that we shall identify two functions which agree μ-almost everywhere. The point of this exercise is that equation (1.3.6) now does define a genuine inner product on $L^2(X)$; most importantly, it is true that $L^2(X)$ is complete and is thus a Hilbert space. □

EXERCISE 1.3.2. (1) *Suppose X is an inner product space. Let $\overline{X}$ be a completion of X regarded as a normed space. Show that $\overline{X}$ is actually a Hilbert space. (Thus, every inner product space has a Hilbert space completion.)*

(2) *Let $X = C[0,1]$ and define*

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx .$$

Verify that this defines a genuine, i.e., positive-definite, inner product on $C[0,1]$. The completion of this inner product space is a Hilbert space – see (1) above – which may be identified with what was called $L^2([0,1], \mathcal{B}, m)$ in Example 1.3.1(4), where ($\mathcal{B}$ is the σ-algebra of Borel sets in $[0,1]$ and) m denotes the so-called Lebesgue measure on $[0,1]$.

1.4 Orthonormal bases

In the sequel, N will always denote a (possibly empty, finite or infinite) countable set.

DEFINITION 1.4.1. *A collection $\{x_n : n \in N\}$ in an inner product space is said to be* **orthonormal** *if*

$$\langle x_m, x_n\rangle = \delta_{mn} := \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \quad \forall\, m, n \in N .$$

Thus, an orthonormal set is nothing but a set of unit vectors which are pairwise orthogonal; we shall write $x \perp y$ if two vectors x, y in an inner product space are orthogonal, i.e., satisfy $\langle x, y\rangle = 0$.

EXAMPLE 1.4.2. (1) In ℓ^2_n, for $1 \leq i \leq n$, let e_i be the element whose i-th co-ordinate is 1 and all other co-ordinates are 0; then $\{e_1, \ldots, e_n\}$ is an orthonormal set in ℓ^2_n.

(2) In ℓ^2, let e_n be the element whose n-th co-ordinate is 1 and all other co-ordinates are 0, for $1 \leq n < \infty$; then $\{e_n : n = 1, 2, \ldots\}$ is an orthonormal set in ℓ^2.

(3) In the inner product space $C[0,1]$ – with inner product as described in Exercise 1.3.2 – consider the family $\{e_n : n \in \mathbb{Z}\}$ defined by $e_n(x) = \exp(2\pi i n x)$, and show that this is an orthonormal set; hence this is also an orthonormal set when regarded as a subset of $L^2([0,1], m)$ – see Exercise 1.3.2(2). □

PROPOSITION 1.4.3. *Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal set in an inner product space X, and let $x \in X$ be arbitrary. Then,*

(i) if $x = \sum_{i=1}^n \alpha_i e_i$, $\alpha_i \in \mathbb{C}$, then $\alpha_i = \langle x, e_i \rangle \; \forall i$;

(ii) $(x - \sum_{i=1}^n \langle x, e_i \rangle e_i) \perp e_j \; \forall 1 \leq j \leq n$;

(iii) **(Bessel's inequality)** $\sum_{i=1}^n |\langle x, e_i \rangle|^2 \leq ||x||^2$.

Proof. (i) If x is a linear combination of the e_j's as indicated, compute $\langle x, e_i \rangle$, and use the assumed orthonormality of the e_j's, to deduce that $\alpha_i = \langle x, e_i \rangle$.

(ii) This is an immediate consequence of (i).

(iii) Write $y = \sum_{i=1}^n \langle x, e_i \rangle e_i$, $z = x - y$, and deduce from (two applications of) Exercise 1.2.3(3) that

$$\begin{aligned} ||x||^2 &= ||y||^2 + ||z||^2 \\ &\geq ||y||^2 \\ &= \sum_{i=1}^n |\langle x, e_i \rangle|^2 . \end{aligned}$$

□

We wish to remark on a few consequences of this proposition; for one thing, (i) implies that an arbitrary orthonormal set is linearly independent; for another, if we write $\bigvee\{e_n : n \in N\}$ for the vector subspace *spanned* by $\{e_n : n \in N\}$ – this is the set of linear combinations of the e_n's, and is the smallest vector subspace containing $\{e_n : n \in N\}$ – it follows from (i) that we know how to write any element of $\bigvee\{e_n : n \in N\}$ as a linear combination of the e_n's.

We shall find the following notation convenient in the sequel: if $\mathcal{S}$ is a subset of an inner product space X, let $\bigvee \mathcal{S}$ (reps., $[\mathcal{S}]$) denote the smallest subspace (resp. closed subspace) containing $\mathcal{S}$; it should be clear that this could be described in either of the following equivalent ways: (a) $[\mathcal{S}]$ is the intersection of all closed subspaces of X which contain $\mathcal{S}$, and (b) $[\mathcal{S}] = \overline{\bigvee \mathcal{S}}$. (Verify that (a) and (b) describe the same set.)

LEMMA 1.4.4. *Suppose $\{e_n : n \in N\}$ is a countable orthonormal set in a Hilbert space $\mathcal{H}$. Then the following conditions on an arbitrary family $\{\alpha_n : n \in N\}$ of complex numbers are equivalent:*

(i) the sum $\sum_{n \in N} \alpha_n e_n$ makes sense as a finite sum in case N is a finite set, and as an 'unconditionally' norm-convergent series in $\mathcal{H}$ if N is

infinite, meaning: if $\phi : \mathbb{N} \to N$ *is any bijection, and if we define* $x(\phi)_k = \sum_{n=1}^{k} \alpha_{\phi(n)} e_{\phi(n)}$, *then the sequence* $\{x(\phi)_k : k \in \mathbb{N}\}$ *is norm-convergent and the limit of this sum is independent of the bijection* ϕ *used; the symbol of a sum of elements of a Hilbert space, and* $\mathbb{C}$, *in particular, which is indexed by arbitrary countably infinite sets (other than* $\mathbb{N}$, *when, of course, series may be conditionally convergent) will always be used to denote only such 'unconditionally convergent series'.*

(ii) $\sum_{n \in N} |\alpha_n|^2 < \infty$.

(iii) there is a vector $x \in [\{e_n : n \in N\}]$ *such that* $\langle x, e_n \rangle = \alpha_n \ \forall n \in N$.

Proof. If N is finite, the first two assertions are obvious, while the third is seen by choosing $x = \sum_{n \in N} \alpha_n e_n$.

So suppose that N is infinite, and that $\phi, x(\phi)_k$ are as above.

$(i) \Rightarrow (iii)$: Fix a bijection ϕ as in (i). Condition (i) says that $\|x(\phi)_k - x(\phi)\| \to 0$ for some $x(\phi) \in \mathcal{H}$. As $\langle x(\phi)_k, e_n \rangle = \langle x(\phi)_\ell, e_n \rangle = \alpha_n \forall k, \ell \geq \phi^{-1}(n)$, we find that $\langle x(\phi), e_n \rangle = \alpha_n \ \forall n$. Since each $x(\phi)_k \in [\{e_n : n \in N\}]$, it is clear that also $x(\phi) \in [\{e_n : n \in N\}]$.

$(iii) \Rightarrow (ii)$ is an immediate consequence of Bessel's inequality.

$(ii) \Rightarrow (i)$: Condition (ii) is seen to imply that $\{x(\phi)_k : k \in \mathbb{N}\}$ is a Cauchy sequence and hence convergent in $\mathcal{H}$. The argument given in the proof of $(i) \Rightarrow (iii)$ applies with ϕ replaced by any other bijection ψ. And $x(\phi) - x(\psi)$ would be an element of $[\{e_n : n \in N\}]$ which would be orthogonal to each e_n and hence to a dense subspce of $[\{e_n : n \in N\}]$, thereby forcing the equality $x(\phi) = x(\psi)$, as asserted. □

We are now ready to establish the fundamental proposition concerning *orthonormal bases* in a Hilbert space.

PROPOSITION 1.4.5. *The following conditions on a countable orthonormal set* $\{e_n : n \in N\}$ *in a Hilbert space* $\mathcal{H}$ *are equivalent: (in items (ii), (iii) and (iv), the sums indexed by the set N are to be understood as indicated in Lemma* 1.4.4(*i*)).

(i) $\{e_n : n \in N\}$ *is a maximal orthonormal set, meaning that it is not strictly contained in any other orthonormal set;*

(ii) $x \in \mathcal{H} \Rightarrow x = \sum_{n \in N} \langle x, e_n \rangle e_n$;

(iii) $x, y \in \mathcal{H} \Rightarrow \langle x, y \rangle = \sum_{n \in N} \langle x, e_n \rangle \langle e_n, y \rangle$;

(iv) $x \in \mathcal{H} \Rightarrow ||x||^2 = \sum_{n \in N} |\langle x, e_n \rangle|^2$.

Such an orthonormal set is called an **orthonormal basis** *of* $\mathcal{H}$.

Proof. $(i) \Rightarrow (ii)$: It is a consequence of Bessel's inequality which states that $\sum_{n \in N} |\langle x, e_n \rangle|^2 < \infty$ and (the implication $(ii) \Leftrightarrow (i)$ of) the last lemma that there exists a vector, call it $x_0 \in \mathcal{H}$, such that $x_0 = \sum_{n \in N} \langle x, e_n \rangle e_n$. If $x \neq x_0$, and if we set $e = \frac{1}{||x - x_0||}(x - x_0)$, then it is easy to see that $\{e_n : n \in N\} \cup \{e\}$ is an orthonormal set which contradicts the assumed maximality of the given orthonormal set.

$(ii) \Rightarrow (iii)$: This is obvious if N is finite, so assume without loss of generality that $N = \mathbb{N}$ For $n \in \mathbb{N}$, let $x_n = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$ and $y_n = \sum_{i=1}^{n} \langle y, e_i \rangle e_i$, and note that, by the assumption (ii), continuity of the inner-product, and the assumed orthonormality of the e_i's, we have

$$\begin{aligned} \langle x, y \rangle &= \lim_{n \to \infty} \langle x_n, y_n \rangle \\ &= \lim_{n \to \infty} \sum_{i=1}^{n} \langle x, e_i \rangle \overline{\langle y, e_i \rangle} \\ &= \lim_{n \to \infty} \sum_{i=1}^{n} \langle x, e_i \rangle \langle e_i, y \rangle \\ &= \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, y \rangle \ . \end{aligned}$$

$(iii) \Rightarrow (iv)$: Put $y = x$.

$(iv) \Rightarrow (i)$: Suppose $\{e_i : i \in I \cup J\}$ is an orthonormal set with J a non-empty index set disjoint from I; then for $j \in J$, we find, in view of (iv), that

$$1 = ||e_j||^2 = \sum_{i \in I} |\langle e_j, e_i \rangle|^2 = 0 ;$$

hence it must be that J is empty – i.e., the maximality assertion of (i) is indeed implied by (iv). □

The reason for only considering countable orthonormal sets lies in the following proposition.

PROPOSITION 1.4.6. *The following conditions on a Hilbert space $\mathcal{H}$ are equivalent:*

(i) $\mathcal{H}$ is separable;

(ii) Any orthonormal set in $\mathcal{H}$ is countable.

Proof. $(i) \Rightarrow (ii)$: Suppose D is a countable dense set in $\mathcal{H}$ and suppose $\{e_i : i \in I\}$ is an orthonormal set in $\mathcal{H}$. Notice that

$$i \neq j \ \Rightarrow \ ||e_i - e_j||^2 = 2 \ . \tag{1.4.7}$$

Since D is dense in $\mathcal{H}$, we can, for each $i \in I$, find a vector $x_i \in D$ such that $||x_i - e_i|| < \frac{\sqrt{2}}{2}$. The identity (1.4.7) shows that the map $I \ni i \mapsto x_i \in D$ is necessarily 1-1; since D is countable, we may conclude that so is I.

$(ii) \Rightarrow (i)$: If I is a countable (finite or infinite) set and if $\{e_i : i \in I\}$ is an orthonormal basis for $\mathcal{H}$, let D be the set whose typical element is of the form $\sum_{j \in J} \alpha_j e_j$, where J is a finite subset of I and α_j are complex numbers whose real and imaginary parts are both rational numbers; it can then be seen that D is a countable dense set in $\mathcal{H}$. □

COROLLARY 1.4.7. *Any orthonormal set in a Hilbert space is contained in an orthonormal basis– meaning that if $\{e_i : i \in I\}$ is any orthonormal set in a Hilbert space $\mathcal{H}$, then there exists an orthonormal set$\{e_i : i \in J\}$ such that $I \cap J = \emptyset$ and $\{e_i : i \in I \cup J\}$ is an orthonormal basis for $\mathcal{H}$. (If $\{e_i : i \in I\}$ is already an orthonormal basis, then $J = \emptyset$.)*

In particular, every Hilbert space admits an orthonormal basis.

Proof. This is an easy consequence of Zorn's lemma. □

REMARK 1.4.8. (1) Although we have formally defined an orthonormal basis only in separable Hilbert spaces, Proposition 1.4.5 is true verbatim without the countability hypothesis. The details of this generalisation – which necessitates a digression into what is meant by sums of families of vectors indexed by arbitrary, possibly uncountable, sets – may be found in [Sun], for instance. Thus, non-separable Hilbert spaces are those whose orthonormal bases are uncountable. It is probably fair to say that any true statement about a general non-separable Hilbert space can be established as soon as one knows that the statement is valid for separable Hilbert spaces; it is probably also fair to say that almost all useful Hilbert spaces are separable. So, the reader may safely assume that all Hilbert spaces in the sequel are separable; among these, the finite-dimensional ones are, in a sense, 'trivial', and one only need really worry about infinite-dimensional separable Hilbert spaces.

(2) Every separable non-zero Hilbert space is isometrically isomorphic to exactly one of the family $\{\ell_n^2 : n \in \mathbb{N}\} \cup \{\ell^2\}$, where $\mathbb{N} = \{1, 2, \ldots\}$. Thus the cardinality of an orthonormal basis is a complete invariant 'up to isometric isomorphism'. It is clear this is an invariant. For finite-dimensional spaces, the cardinality of an orthonormal basis is the usual vector space dimension, and vector spaces of differing finite dimension are not isomorphic. Also, no finite-dimensional Hilbert space can be isometrically isomorphic to ℓ^2 as the unit ball of ℓ^2 is not compact. (*Reason:* the orthonormal basis$\{e_n : n \in \mathbb{N}\}$ can have no Cauchy subsequence as $\|e_n - e_m\| = \sqrt{2}$ if $m \neq n$.)

REMARK 1.4.9. (1) It follows from Proposition 1.4.5 (ii) that if $\{e_i : i \in I\}$ is an orthonormal basis for a Hilbert space $\mathcal{H}$, then $\mathcal{H} = [\{e_i : i \in I\}]$; conversely, it is true - see Corollary 1.4.14 – that if an orthonormal set is **total** (meaning that the vector subspace spanned by the set is dense in the Hilbert space), then such an orthonormal set is necessarily an orthonormal basis. (*Reason:* apply Theorem 1.4.13(ii), with $\mathcal{M}$ as the closed subspace spanned by the orthonormal set.)

(2) Each of the three examples of an orthonormal set that is given in Example 1.4.2, is in fact an orthonormal basis for the underlying Hilbert space. This is obvious in cases (1) and (2). As for (3), it is a consequence of the Stone-Weierstrass theorem that the vector subspace of finite linear combinations of the exponential functions $\{\exp(2\pi i n x) : n \in \mathbb{Z}\}$ (usually called the set

of trigonometric polynomials) is dense in $\{f \in C[0,1] : f(0) = f(1)\}$ (with respect to the uniform norm – i.e., with respect to $|| \cdot ||_\infty$); in view of Exercise 1.3.2 (2), it is not hard to conclude that this orthonormal set is total in $L^2([0,1], m)$ and hence, by remark (1) above, this is an orthonormal basis for the Hilbert space in question.

Since $\exp(\pm 2\pi i n x) = \cos(2\pi n x) \pm i \sin(2\pi n x)$, and since it is easily verified that $\cos(2\pi m x) \perp \sin(2\pi n x)$ $\forall m, n = 1, 2, \ldots$, we find easily that

$$\{1 = e_0\} \cup \{\sqrt{2}\cos(2\pi n x),\ \sqrt{2}\sin(2\pi n x) : n = 1, 2, \ldots\}$$

is also an orthonormal basis for $L^2([0,1], m)$. (Reason: this is orthonormal, and this sequence spans the same vector subspace as is spanned by the exponential basis.) (Also, note that these are real-valued functions, and that the inner product of two real-valued functions is clearly real.) It follows, in particular, that if f is any (real-valued) continuous function defined on [0,1], then such a function admits the following **Fourier series** (with real coefficients):

$$f(x) \ = \ a_0 + \sum_{n=1}^{\infty} (a_n \cos(2\pi n x) + b_n \sin(2\pi n x)\ )$$

where the meaning of this series is that we have convergence of the sequence of the partial sums to the function f with respect to the norm in $L^2[0,1]$. Of course, the coefficients a_n, b_n are given by

$$\begin{aligned} a_0 &= \int_0^1 f(x)\, dx \\ a_n &= 2\int_0^1 f(x)\cos(2\pi n x\,)\, dx\ ,\ \forall\ n > 0, \\ b_n &= 2\int_0^1 f(x)\sin(2\pi n x)\, dx\ ,\ \forall\ n > 0 \end{aligned}$$

The theory of Fourier series was the precursor to most of modern functional analysis; it is for this reason that if $\{e_i : i \in I\}$ is any orthonormal basis of any Hilbert space, it is customary to refer to the numbers $\langle x, e_i \rangle$ as the **Fourier coefficients** of the vector x with respect to the orthonormal basis $\{e_i : i \in I\}$. □

It is a fact that any two orthonormal bases for a Hilbert space have the same cardinality, and this common cardinal number is called the **dimension** of the Hilbert space; the proof of this statement, in its full generality, requires facility with infinite cardinal numbers and arguments of a transfinite nature, and may be found in [Sun]; our interest will be confined to separable Hilbert spaces; the proof in that case of the dimension being an invariant has been outlined in Remark 1.4.9.

We next establish a lemma which will lead to the important result which is sometimes referred to as 'the projection theorem'.

LEMMA 1.4.10. *Let $\mathcal{M}$ be a closed subspace of a Hilbert space $\mathcal{H}$; (thus $\mathcal{M}$ may be regarded as a Hilbert space in its own right;) let $\{e_i : i \in I\}$ be any orthonormal basis for $\mathcal{M}$, and let $\{e_j : j \in J\}$ be any orthonormal set such that $\{e_i : i \in I \cup J\}$ is an orthonormal basis for $\mathcal{H}$, where we assume that the index sets I and J are disjoint. Then, the following conditions on a vector $x \in \mathcal{H}$ are equivalent:*

(i) $x \perp y \ \forall \ y \in \mathcal{M}$;

(ii) $x = \sum_{j \in J} \langle x, e_j \rangle e_j$.

Proof. The implication $(ii) \Rightarrow (i)$ is obvious. Conversely, it follows easily from Lemma 1.4.4 and Bessel's inequality that the 'series' $\sum_{i \in I} \langle x, e_i \rangle e_i$ and $\sum_{j \in J} \langle x, e_j \rangle e_j$ converge in $\mathcal{H}$. Let the sums of these 'series' be denoted by y and z respectively. Further, since $\{e_i : i \in I \cup J\}$ is an orthonormal basis for $\mathcal{H}$, it should be clear that $x = y + z$. Now, if x satisfies condition (i) of the lemma, it should be clear that $y = 0$ and that hence, $x = z$, thereby completing the proof of the lemma. □

We now come to the basic notion of **orthogonal complement**.

DEFINITION 1.4.11. *The orthogonal complement $S^\perp$ of a subset S of a Hilbert space is defined by*

$$S^\perp = \{x \in \mathcal{H} : x \perp y \ \forall \ y \in S\} .$$

EXERCISE 1.4.12. *If $S_0 \subset S \subset \mathcal{H}$ are arbitrary subsets, show that*

$$S_0^\perp \supset S^\perp = \left(\bigvee S\right)^\perp = ([S])^\perp .$$

Also show that $S^\perp$ is always a closed subspace of $\mathcal{H}$.

We are now ready for the basic fact concerning orthogonal complements of closed subspaces.

THEOREM 1.4.13. *Let $\mathcal{M}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Then,*

(i) $\mathcal{M}^\perp$ is also a closed subspace;

(ii) $\left(\mathcal{M}^\perp\right)^\perp = \mathcal{M}$;

(iii) any vector $x \in \mathcal{H}$ can be uniquely expressed in the form $x = y + z$, where $y \in \mathcal{M}, \ z \in \mathcal{M}^\perp$;

(iv) if x, y, z are as in (3) above, then the equation $Px = y$ defines a bounded operator $P \in B(\mathcal{H})$ with the property that

$$||Px||^2 = \langle Px, x \rangle = ||x||^2 - ||x - Px||^2 \ , \ \forall x \in \mathcal{H} .$$

Proof. (i) This is easy – see Exercise 1.4.12.

(ii) Let $I, J, \{e_i : i \in I \cup J\}$ be as in Lemma 1.4.10. We assert, to start with, that in this case, $\{e_j : j \in J\}$ is an orthonormal basis for $\mathcal{M}^\perp$. Suppose this is not true; since this is clearly an orthonormal set in $\mathcal{M}^\perp$, this means that $\{e_j : j \in J\}$ is not a maximal orthonormal set in $\mathcal{M}^\perp$, which implies the existence of a unit vector $x \in \mathcal{M}^\perp$ such that $\langle x, e_j \rangle = 0 \; \forall \; j \in J$; such an x will satisfy condition (i) of Lemma 1.4.10, but not condition (ii).

If we now reverse the roles of $\mathcal{M}, \{e_i : i \in I\}$ and $\mathcal{M}^\perp, \{e_j : j \in J\}$, we find from the conclusion of the preceding paragraph that $\{e_i : i \in I\}$ is an orthonormal basis for $\left(\mathcal{M}^\perp\right)^\perp$, from which we may conclude the validity of (ii) of this theorem.

(iii) The existence of y and z was demonstrated in the proof of Lemma 1.4.10; as for uniqueness, note that if $x = y_1 + z_1$ is another such decomposition, then we would have

$$y - y_1 \;=\; z_1 - z \;\in\; \mathcal{M} \cap \mathcal{M}^\perp \;;$$

but $w \in \mathcal{M} \cap \mathcal{M}^\perp \;\Rightarrow\; w \perp w \;\Rightarrow ||w||^2 = 0 \;\Rightarrow\; w = 0$.

(iv) The uniqueness of the decomposition in (iii) is easily seen to imply that P is a linear mapping of $\mathcal{H}$ into itself; further, in the notation of (iii), we find (since $y \perp z$) that

$$||x||^2 \;=\; ||y||^2 + ||z||^2 \;=\; ||Px||^2 + ||x - Px||^2 \;;$$

this implies that $||Px|| \leq ||x|| \; \forall \; x \in \mathcal{H}$, and hence $P \in B(\mathcal{H})$.

Also, since $y \perp z$, we find that

$$||Px||^2 \;=\; ||y||^2 \;=\; \langle y, y + z \rangle \;=\; \langle Px, x \rangle \;,$$

thereby completing the proof of the theorem. □

The following corollary to the above theorem justifies the final assertion made in Remark 1.4.9 (1).

COROLLARY 1.4.14. *The following two conditions on an orthonormal set $\{e_i : i \in I\}$ in a Hilbert space $\mathcal{H}$ are equivalent:*

(i) $\{e_i : i \in I\}$ is an orthonormal basis for $\mathcal{H}$;

(ii) $\{e_i : i \in I\}$ is total in $\mathcal{H}$ – meaning, of course, that $\mathcal{H} = [\{e_i : i \in I\}]$.

Proof. As has already been observed in Remark 1.4.9 (1), the implication $(i) \Rightarrow (ii)$ follows from Proposition 1.4.5(ii).

Conversely, suppose (i) is not satisfied; then $\{e_i : i \in I\}$ is not a maximal orthonormal set in $\mathcal{H}$; hence there exists a unit vector x such that $x \perp e_i \; \forall i \in I$; if we write $\mathcal{M} = [\{e_i : i \in I\}]$, it follows easily that $x \in \mathcal{M}^\perp$, whence $\mathcal{M}^\perp \neq \{0\}$; then, we may deduce from Theorem 1.4.13(2) that $\mathcal{M} \neq \mathcal{H}$ - i.e., (ii) is also not satisfied. □

A standard and easily proved fact is that the following conditions on a linear map $T : \mathcal{H} \to \mathcal{K}$ between Hilbert spaces are equivalent:

(1) T is continuous; i.e. $\|x_n - x\| \to 0 \Rightarrow \|Tx_n - Tx\| \to 0$;

(2) T is continuous at 0; i.e. $\|x_n\| \to 0 \Rightarrow \|Tx_n\| \to 0$;

(3) $\sup\{\|Tx\| : \|x\| \leq 1\} = \inf\{C > 0 : \|Tx\| \leq C\|x\| \ \forall x \in \mathcal{H}\} < \infty$.

On account of (3) above, such continuous linear maps are called **bounded operators** and we write $B(\mathcal{H}, \mathcal{K})$ for the vector space of all bounded operators from $\mathcal{H}$ to $\mathcal{K}$. It is a standard fact that $B(\mathcal{H}, \mathcal{K})$ is a Banach space if $\|T\|$ is defined as the common value of the two expressions in item (3) above.

We write $B(\mathcal{H})$ for $B(\mathcal{H}, \mathcal{H})$, and note that $B(\mathcal{H})$ is a **Banach algebra** when equipped with composition product $AB = A \circ B$.

It is customary to write $\mathcal{H}^* = B(\mathcal{H}, \mathbb{C})$. We begin by identifying this Banach dual space $\mathcal{H}^*$.

THEOREM 1.4.15. (**Riesz lemma**)
Let $\mathcal{H}$ be a Hilbert space.
(a) If $y \in \mathcal{H}$, the equation

$$\phi_y(x) = \langle x, y \rangle \tag{1.4.8}$$

defines a bounded linear functional $\phi_y \in \mathcal{H}^*$; and further, $||\phi_y||_{\mathcal{H}^*} = ||y||_{\mathcal{H}}$.

(b) Conversely, if $\phi \in \mathcal{H}^*$, there exists a unique element $y \in \mathcal{H}$ such that $\phi = \phi_y$ as in (a) above.

Proof. (a) Linearity of the map ϕ_y is obvious, while the Cauchy-Schwarz inequality shows that ϕ_y is bounded and that $||\phi_y|| \leq ||y||$. Since $\phi_y(y) = ||y||^2$, it easily follows that we actually have equality in the preceding inequality.

(b) Suppose conversely that $\phi \in \mathcal{H}^*$. Since $||\phi_{y_1} - \phi_{y_2}|| = ||y_1 - y_2||$ for all $y_1, y_2 \in \mathcal{H}$, the uniqueness assertion is obvious; we only have to prove existence. Let $\mathcal{M} = \ker \phi$. Since existence is clear if $\phi = 0$, we may assume that $\phi \neq 0$, i.e., that $\mathcal{M} \neq \mathcal{H}$, or equivalently that $\mathcal{M}^\perp \neq 0$.

Notice that the map ϕ is 1-1 from $\mathcal{M}^\perp$ into $\mathbb{C}$; since $\mathcal{M}^\perp \neq 0$, it follows that $\mathcal{M}^\perp$ is one-dimensional. Let z be a unit vector in $\mathcal{M}^\perp$. The y that we seek – assuming it exists – must clearly be an element of $\mathcal{M}^\perp$ (since $\phi(x) = 0 \ \forall x \in \mathcal{M}$). Thus, we must have $y = \alpha z$ for some uniquely determined scalar $0 \neq \alpha \in \mathbb{C}$. With y defined thus, we find that $\phi_y(z) = \overline{\alpha}$; hence we must have $\alpha = \overline{\phi(z)}$. Since any element in $\mathcal{H}$ is uniquely expressible in the form $x + \gamma z$ for some $x \in \mathcal{M}$, and scalar $\gamma \in \mathbb{C}$, we find easily that we do indeed have $\phi = \phi_{\overline{\phi(z)}z}$. □

It must be noted that the mapping $y \mapsto \phi_y$ is not quite an isometric isomorphism of Banach spaces; it is not a linear map, since $\phi_{\alpha y} = \overline{\alpha}\phi_y$; it is only 'conjugate-linear'. The dual (*à priori Banach*) space $\mathcal{H}^*$ is actually a Hilbert space if we define

$$\langle \phi_y, \phi_z \rangle = \langle z, y \rangle ;$$

that this equation satisfies the requirements of an inner product are an easy consequence of the Riesz lemma (and the conjugate-linearity of the mapping

$y \mapsto \phi_y$ already stated); that this inner product actually gives rise to the norm on $\mathcal{H}^*$ is a consequence of the fact that $||y|| = ||\phi_y||$.

EXERCISE 1.4.16. *(1) Where is the completeness of $\mathcal{H}$ used in the proof of the Riesz lemma; more precisely, what can you say about X^* if you only know that X is an (not necessarily complete) inner product space? (Hint: Consider the completion of X.)*

(2) If $T \in B(\mathcal{H}, \mathcal{K})$, where $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, prove that

$$||T|| = \sup\{|\langle Tx, y\rangle| : x \in \mathcal{H}, y \in \mathcal{K}, \ ||x|| \leq 1, ||y|| \leq 1\} .$$

A mapping $B : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ is called a **bounded sesquilinear form** if

$$B\Big(\sum_{i=1}^{m} \alpha_i x_i, \sum_{j=1}^{n} \beta_j y_j\Big) = \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_i \bar{\beta}_j B(x_i, y_j), \ \forall \alpha_i, \beta_j \in \mathbb{C}, x_i \in \mathcal{H}, y_j \in \mathcal{K} \tag{1.4.9}$$

and

$$\|B\| := \sup\{|B(x,y)| : \|x\|, \|y\| \leq 1\} < \infty . \tag{1.4.10}$$

The following is an easy consequence of the Riesz lemma (see Theorem 1.4.15)) and so its proof is omitted.

PROPOSITION 1.4.17. *(1) $B : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ is a bounded sesquilinear form if and only if there exists a unique bounded operator $T \in B(\mathcal{H}, \mathcal{K})$ such that $B(x, y) = \langle Tx, y\rangle \ \forall x, y$; furthermore, $\|T\| = \|B\|$.*

(2) Every sesquilinear form defined on $\mathcal{H} \times \mathcal{H}$ satisfies the **polarisation identity***:*

$$4B(x,y) = \sum_{j=0}^{3} (\sqrt{-1}^{\,j} B(x + \sqrt{-1}^{\,j} y, x + \sqrt{-1}^{\,j} y)$$

It is a consequence of the open mapping theorem that the following conditions on a $T \in B(\mathcal{H}, \mathcal{K})$ are equivalent:

(1) There exists an $S \in B(\mathcal{K}, \mathcal{H})$ such that $ST = id, TS = id_{\mathcal{K}}$.

(2) T is a set-theoretic bijection, i.e., both 1-1 and onto.

We call such an operator T invertible, and write $S = T^{-1}$. It is a fact (see [Sun]) that the collection $GL(\mathcal{H}, \mathcal{K})$ of such invertible operators is open in the norm-topology of $B(\mathcal{H}, \mathcal{K})$, and that the mapping $T \mapsto T^{-1}$ is a norm-continuous map of $GL(\mathcal{H}, \mathcal{K})$ onto $GL(\mathcal{K}, \mathcal{H})$.

Recall that the **spectrum** of a $T \in B(\mathcal{H})$ is defined to be $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin GL(\mathcal{H})\}$. It follows from the previous paragraph that $\sigma(T)$ is a closed set. It is also true that $\sigma(T)$ is a non-empty compact set for any $T \in B(\mathcal{H})$.

An elementary fact about spectra that will be needed later is a special case of a more general **spectral mapping theorem**.

PROPOSITION 1.4.18. *If $p \in \mathbb{C}[t]$ is any polynomial with complex coefficients, and if $T \in B(\mathcal{H})$, then $\sigma(p(T)) = p(\sigma(T))$.*

Proof. Fix a $\lambda \in \mathbb{C}$. If p is a constant, the proposition is obvious, so we assume p is a polynomial of degree $n \geq 1$. Then the algebraic closedness of $\mathbb{C}$ permits a factorisation of the form $p(t) - \lambda = \alpha_n \prod_{i=1}^n (t - \mu_i)$. Clearly, then $p(T) - \lambda = \alpha_n \prod_{i=1}^n (T - \mu_i)$ (where the order of the product is immaterial as the factors commute pairwise). We need the fairly easy fact that if $T_1, \ldots, T_n$ are n pairwise commuting operators, then their product $T_1 \ldots T_n$ is invertible if and only if each T_i is invertible. (Verify this!) Hence conclude that

$$\lambda \notin \sigma(p(T)) \Leftrightarrow \mu_i \notin \sigma(T) \forall i$$

or equivalently, that $\lambda \in \sigma(p(T))$ if and only if there exists some i such that $\mu_i \in \sigma(T)$. This is equivalent to saying that $\lambda \in p(\sigma(T))$; and thus, indeed $\sigma(p(T)) = p(\sigma(T))$. □

It is a fact that $\lambda \in \sigma(T) \Rightarrow |\lambda| \leq \|T\|$ and that the spectrum is always compact. The non-emptiness is a more non-trivial fact. (This statement for all finite-dimensional $\mathcal{H}$ is equivalent to the fact that $\mathbb{C}$ is algebraically closed, i.e., that every complex polynomial is a product of linear factors.)

Another proof that simultaneously establishes the fact that $\sigma(T)$ is non-empty and compact is the (not surprisingly complex analytic) proof of the so-called **spectral radius formula**:

$$\operatorname{spr}(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\} = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} . \tag{1.4.11}$$

This says two things: (i) that the indicated limit exists, and (ii) that the value of the limit is as asserted. Part (ii) shows that the spectral radius is non-negative, and hence that spectrum is always non-empty. We will shortly be using part (i) to establish that $\operatorname{spr}(T) = \|T\|$ if T is 'normal', which is a key ingredient in the proof of the spectral theorem.

Most of this required background material can be found in the initial chapters of most standard books (such as [Sun]) covering the material of a first course in Functional Analysis.

1.5 Adjoints

An immediate consequence of the Riesz lemma (Lemma 1.4.15) is:

PROPOSITION 1.5.1. *If $T \in B(\mathcal{H}, \mathcal{K})$, there exists a unique operator $T^* \in B(\mathcal{K}, \mathcal{H})$ – called the* **adjoint** *of the operator T – such that*

$$\langle T^* y, x \rangle = \langle y, Tx \rangle \ \forall x \in \mathcal{H}, y \in \mathcal{K}.$$

Proof. Notice that the right side of the displayed equation above defines a bounded sesquilinear form on $\mathcal{K} \times \mathcal{H}$, and appeal to Proposition 1.4.17 to lay hands on the desired operator T^*. □

We list below some simple properties of this process of taking adjoints.

PROPOSITION 1.5.2. (1) *For all* $\alpha \in \mathbb{C}, S, S_1, S_2 \in B(\mathcal{H}, \mathcal{K}), T \in B(\mathcal{M}, \mathcal{H})$, *we have:*

$$\begin{aligned} (\alpha S_1 + S_2)^* &= \bar{\alpha} S_1^* + S_2^* \ ; \\ (S^*)^* &= S \ ; \\ (ST)^* &= T^* S^* \ ; \\ id_{\mathcal{H}}^* &= id_{\mathcal{H}} \ . \end{aligned}$$

(2) $\|T\|^2 = \|T^*T\|$ *and hence, also* $\|T^*\| = \|T\|$;

(3) $\ker(T^*) = \operatorname{ran}^{\perp}(T) := (\operatorname{ran}(T))^{\perp}$; *equivalently,* $\ker^{\perp}(T^*) = \overline{\operatorname{ran}(T)}$.

Proof. (1) Most of these identities follow from the fact that the adjoint is characterised by the equation it satisfies. Thus, for instance,

$$\begin{aligned} \langle (\alpha S_1 + S_2)^* y, x \rangle &= \langle y, (\alpha S_1 + S_2) x \rangle \\ &= \bar{\alpha} \langle y, S_1 x \rangle + \langle y, S_2 x \rangle \\ &= \bar{\alpha} \langle S_1^* y, x \rangle + \langle S_2^* y, x \rangle \\ &= \langle (\bar{\alpha} S_1^* + S_2^*) y, x \rangle . \end{aligned}$$

The other three statements are even more straight-forward to verify.

(2) On the one hand,

$$\begin{aligned} \|T\|^2 &= \sup\{\|Tx\|^2 : \|x\| \leq 1\} \\ &= \sup\{\langle T^*Tx, x \rangle : \|x\| \leq 1\} \\ &\leq \|T^*T\| \ , \end{aligned}$$

while on the other,

$$\begin{aligned} \|T^*T\| &= \sup\{|\langle T^*Tx_1, x_2 \rangle| : \|x_1\||, \|x_2\| \leq 1\} \\ &\leq \sup\{\|Tx_1\| \|Tx_2\| : \|x_1\|, \|x_2\| \leq 1\} \\ &\leq \|T\|^2 \ . \end{aligned}$$

(Observe that the Cauchy-Schwarz inequality $|\langle x, y \rangle| \leq \|x\| \ \|y\|$ has been used in the proofs of both inequalities above – in the third line of the first, and in the second line of the second.) The desired equality follows, and the sub-multiplicativity of the norm then implies that $\|T^*\| \leq \|T\|$. By interchanging the roles of T and T^*, we find that, indeed $\|T^*\| = \|T\|$.

(3)

$$\begin{aligned} y \in \ker(T^*) &\Leftrightarrow T^* y = 0 \\ &\Leftrightarrow \langle T^* y, x \rangle = 0 \ \forall x \\ &\Leftrightarrow \langle y, Tx \rangle = 0 \ \forall x \\ &\Leftrightarrow y \in \operatorname{ran}^{\perp}(T) \ . \end{aligned}$$

□

The polarisation identity has the following immediate corollaries:

COROLLARY 1.5.3. (1) *If $T \in B(\mathcal{H})$, then*

$$T = 0 \Leftrightarrow \langle Tx, x\rangle = 0 \; \forall x \in \mathcal{H}.$$

(2)

$$T = T^* \Leftrightarrow \langle Tx, x\rangle \in \mathbb{R} \; \forall x \in \mathcal{H}\ .$$

This corolllary leads to the definition of an important class of operators:

DEFINITION 1.5.4. *An operator $T \in B(\mathcal{H})$ is said to be* **self-adjoint** *(or* **Hermitian***) if $T = T^*$.*

A slightly larger class of operators, which is *the correct* class of operators for the purposes of the spectral theorem, is dealt with in our next definition.

DEFINITION 1.5.5. *An operator $Z \in B(\mathcal{H})$ is said to be* **normal** *if $Z^*Z = ZZ^*$.*

PROPOSITION 1.5.6. *Let $Z \in B(\mathcal{H})$.*

(1) *Z is normal if and only if $\|Zx\| = \|Z^*x\| \; \forall x \in \mathcal{H}$.*
(2) *If Z is normal, then $\|Z^2\| = \|Z^*Z\| = \|Z\|^2$; more generally, $\|Z^{2^n}\| = \|Z\|^{2^n}$ and consequently $\mathrm{spr}(Z) = \|Z\|$.*

Proof. (1)

$$\begin{aligned} Z^*Z = ZZ^* \quad &\Leftrightarrow \quad \langle Z^*Zx, x\rangle = \langle ZZ^*x, x\rangle \; \forall x \in \mathcal{H} \\ &\Leftrightarrow \quad \|Zx\|^2 = \|Z^*x\|^2 \; \forall x \in \mathcal{H}. \end{aligned}$$

(2) Suppose Z is normal. Then,

$$\begin{aligned} \|Z^2\| &= \sup\{\|Z^2x\| : \|x\| = 1\} \\ &= \sup\{\|Z^*Zx\| : \|x\| = 1\} \qquad \text{(by part (1) above)} \\ &= \|Z^*Z\| \\ &= \|Z\|^2 \end{aligned}$$

where we have used Proposition 1.5.2(2) in the last step; an easy induction argument now yields the statement about 2^n, which implies that $\|Z\| = \lim_{n\to\infty} \|Z^{2^n}\|^{\frac{1}{2^n}} = \mathrm{spr}(Z)$.

□

We now have the tools at hand to prove a key identity.

PROPOSITION 1.5.7. *If $X \in B(\mathcal{H})$ is self-adjoint, and $p \in \mathbb{C}[t]$ is any polynomial with complex coefficients, then*

$$\|p(X)\| = \|p\|_{\sigma(X)} := \sup\{|p(t)| : t \in \sigma(X)\}. \qquad (1.5.12)$$

Proof. Notice that $q = |p|^2 = \bar{p}p$ is a polynomial with real coefficients, and hence $q(X)$ is self-adjoint. Deduce from Proposition 1.5.6 (2) and the spectral mapping theorem (Proposition 1.4.18) that

$$\begin{aligned}
\|p(X)\|^2 &= \|p(X)^* p(X)\| \\
&= \|\bar{p}(X)p(X)\| \\
&= \|q(X)\| \\
&= \sup\{|\lambda| : \lambda \in \sigma(q(X))\} \\
&= \sup\{|q(t)| : t \in \sigma(X)\} \\
&= \|q\|_{\sigma(X)} \\
&= \|p\|^2_{\sigma(X)} ,
\end{aligned}$$

as desired. □

Just as every complex number has a unique decomposition into real and imaginary parts, it is seen that each $Z \in \mathcal{B}(H)$ has a unique **Cartesian decomposition** $Z = X + iY$, with X and Y being self-adjoint (these being necessarily given by $X = \frac{1}{2}(Z + Z^*)$ and $Y = \frac{1}{2i}(Z - Z^*)$, so that, in fact, $\langle Xx, x\rangle = \text{Re } \langle Zx, x\rangle$ and $\langle Yx, x\rangle = \text{Im } \langle Zx, x\rangle$). For this reason, we sometimes write $X = \text{Re } Z, Y = \text{Im } Z$.

For future reference, we make some observations on the Cartesian decomposition of a normal operator.

PROPOSITION 1.5.8. *Let $Z = X + iY$ be the Cartesian decomposition of an operator. Then, the following conditions are equivalent:*

(1) *Z is normal.*

(2) $\|Zx\|^2 = \|Xx\|^2 + \|Yx\|^2 \ \forall x \in \mathcal{H}$.

(3) $XY = YX$.

Proof. First notice that for $Z = X + iY$, we have

$$\begin{aligned}
\|Zx\|^2 &= \|Xx + iYx\|^2 \\
&= \|Xx\|^2 + \|Yx\|^2 - 2\text{Re } (i\langle Xx, Yx\rangle)
\end{aligned}$$

while

$$\begin{aligned}
\|Z^*x\|^2 &= \|Xx - iYx\|^2 \\
&= \|Xx\|^2 + \|Yx\|^2 + 2\text{Re } (i\langle Xx, Yx\rangle),
\end{aligned}$$

so that

$$\|Z^*x\|^2 = \|Zx\|^2 \Leftrightarrow \text{Re } (i\langle Xx, Yx\rangle) = 0 \Leftrightarrow \|Zx\|^2 = \|Xx\|^2 + \|Yx\|^2.$$

Notice finally that

$$\text{Re } i\langle Xx, Yx\rangle = 0 \Leftrightarrow \langle Xx, Yx\rangle \in \mathbb{R}$$

and that (since X, Y are self-adjoint)

$$\langle Xx, Yx\rangle \in \mathbb{R}\ \forall x \in \mathcal{H} \Leftrightarrow XY = (XY)^* = YX.$$

The truth of the lemma is evident now. □

1.6 Approximate eigenvalues

DEFINITION 1.6.1. *A scalar $\lambda \in \mathbb{C}$ is said to be an* **approximate eigenvalue** *of an operator $Z \in B(\mathcal{H})$ if there exists a sequence $\{x_n : n \in \mathbb{N}\} \subset S(\mathcal{H})$ such that $\lim_{n\to\infty} \|(Z-\lambda)x_n\| = 0$. Here and in the sequel, we shall employ the symbol $S(\mathcal{H})$ to denote the unit sphere of $\mathcal{H}$; thus, $S(\mathcal{H}) := \{x \in \mathcal{H} : \|x\| = 1\}$.*

The importance – as emerges from [Hal] – of this notion in the context of the spectral theorem (equivalently, the study of self-adjoint or normal operators) lies in the following result:

THEOREM 1.6.2. *Suppose $Z \in B(\mathcal{H})$ is normal. Then:*

(1) $Z \in GL(\mathcal{H}) \Leftrightarrow Z$ is bounded below; i.e., there is an $\epsilon > 0$ such that $\|Zx\| \geq \epsilon\|x\|\ \forall x \in \mathcal{H}$, equivalently, $\inf\{\|Zx\| : x \in S(\mathcal{H})\} \geq \epsilon > 0$ *(assuming $\mathcal{H} \neq 0$).*

(2) $\lambda \in \sigma(Z)$ if and only if λ is an approximate eigenvalue of Z.

Proof. (1) If Z is invertible, then note that

$$\|x\| = \|Z^{-1}Zx\| \leq \|Z^{-1}\|\|Zx\|\ \forall x$$

which shows that $\|Zx\| \geq \|Z^{-1}\|^{-1}\|x\|\ \forall x$ and that Z is indeed bounded below.

If, conversely, Z is bounded below, deduce two consequences, *viz.*,

(a) Z^* is also bounded below (by part (1) of Proposition 1.5.6)) and hence $\ker(Z^*)(= \ker(Z)) = \{0\}$ so that $\operatorname{ran}(Z)$ is dense in $\mathcal{H}$ (by part (3) of Proposition1.5.2).

(b) Z has a closed range (*Reason*: If $Zx_n \to y$ then $\{Zx_n : n \in \mathbb{N}\}$, and consequently also $\{x_n : n \in \mathbb{N}\}$, must be a Cauchy sequence, forcing $y = Z(\lim_{n\to\infty} x_n)$.)

It follows from (a) and (b) above that Z is a bijective linear map of $\mathcal{H}$ onto itself and hence invertible.

(2) Note first that $(Z - \lambda)$ inherits normality from Z, then deduce from (1) above that $\lambda \in \sigma(Z)$ if and only if there exists a sequence $x_n \in S(\mathcal{H})$ such that $\|(Z-\lambda)x_n\| < \frac{1}{n}\ \forall n$, i.e., λ is an approximate eigenvalue of z, as desired.

□

COROLLARY 1.6.3.

$$X = X^* \;\Rightarrow\; \sigma(X) \subset \mathbb{R}.$$

Proof. If there exists a sequence $\{x_n : n \in \mathbb{N}\} \subset S(\mathcal{H})$ satisfying the condition $\|(X-\lambda)x_n\| \to 0$, then also $\langle (X-\lambda)x_n, x_n\rangle \to 0$ and hence

$$\lambda = \lim_{n\to\infty} \langle \lambda x_n, x_n\rangle = \lim_{n\to\infty} \langle X x_n, x_n\rangle \in \mathbb{R}$$

(by Corollary 1.5.3 (2)). □

For later reference, we record an immediate consequence of Theorem 1.6.2 (2) and Proposition 1.5.8 (2).

COROLLARY 1.6.4. *Suppose $\lambda = \alpha + i\beta$ and $Z = X + iY$ are the Cartesian decompositions of a scalar λ and a normal operator Z respectively. Then the following conditions are equivalent:*

(1) $\lambda \in \sigma(Z)$;

(2) *There exists a sequence $\{x_n : n \in \mathbb{N}\}$ such that $\|(X-\alpha)x_n\| \to 0$ and $\|(Y-\beta)x_n\| \to 0$.*

1.7 Important classes of operators

1.7.1 Projections

REMARK 1.7.1. The operator $P \in B(\mathcal{H})$ constructed in Theorem 1.4.13(4) is referred to as the **orthogonal projection** onto the closed subspace $\mathcal{M}$. When it is necessary to indicate the relation between the subspace $\mathcal{M}$ and the projection P, we will write $P = P_{\mathcal{M}}$ and $\mathcal{M} = \text{ran } P$ (note that $\mathcal{M}$ is indeed the range of the operator P); some other facts about closed subspaces and projections are spelt out in the following exercises. □

EXERCISE 1.7.2. (1) *Show that $\left(S^{\perp}\right)^{\perp} = [S]$, for any subset $S \subset \mathcal{H}$.*

(2) *Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$, and let $P = P_{\mathcal{M}}$;*

(a) Show that $P_{\mathcal{M}^\perp} = 1 - P_{\mathcal{M}}$;

(b) Let $x \in \mathcal{H}$; the following conditions are equivalent:

(i) $x \in \mathcal{M}$;

(ii) $x \in \text{ran}\, P$ $(:= P\mathcal{H})$;

(iii) $Px = x$;

(iv) $||Px|| = ||x||$.

(c) Show that $\mathcal{M}^\perp = \ker P = \{x \in \mathcal{H} : Px = 0\}$.

(3) *Let $\mathcal{M}$ and $\mathcal{N}$ be closed subspaces of $\mathcal{H}$, and let $P = P_{\mathcal{M}}, Q = P_{\mathcal{N}}$; show that the following conditions are equivalent:*

(i) $\mathcal{N} \subset \mathcal{M}$;

(ii) $PQ = Q$;

(i)' $\mathcal{M}^{\perp} \subset \mathcal{N}^{\perp}$;

(ii)' $(1 - Q)(1 - P) = 1 - P$;

(iii) $QP = Q$.

(4) *With $\mathcal{M}, \mathcal{N}, P, Q$ as in (3) above, show that the following conditions are equivalent:*

(i) $\mathcal{M} \perp \mathcal{N}$ – *i.e.*, $\mathcal{N} \subset \mathcal{M}^{\perp}$;

(ii) $PQ = 0$;

(iii) $QP = 0$.

(5) *When the equivalent conditions of (4) are met, show that:*

(a) $[\mathcal{M} \cup \mathcal{N}] = \mathcal{M} + \mathcal{N} = \{x + y : x \in \mathcal{M}, y \in \mathcal{N}\}$; *and that*

(c) $(P + Q)$ *is the projection onto the subspace* $\mathcal{M} + \mathcal{N}$.

(6) *Show, more generally, that*

(a) *if $\{\mathcal{M}_i : 1 \leq i \leq n\}$ is a family of closed subspaces of $\mathcal{H}$ which are pairwise orthogonal, then their 'vector sum' defined by $\sum_{i=1}^{n} \mathcal{M}_i = \{\sum_{i=1}^{n} x_i : x_i \in \mathcal{M}_i \ \forall i\}$ is a closed subspace and the projection onto this subspace is given by $\sum_{i=1}^{n} P_{\mathcal{M}_i}$; and that*

(b) *if $\{\mathcal{M}_n : n \in \mathbb{N}\}$ is a family of closed subspaces of $\mathcal{H}$ which are pairwise orthogonal, and if $\mathcal{M} = [\bigcup_{n \in \mathbb{N}} \mathcal{M}_n]$, then $P_{\mathcal{M}}$ is given by the sum of the series $\sum_{n \in \mathbb{N}} P_{\mathcal{M}_n}$ which is interpreted in the SOT-sense (see Definition 2.2.4): meaning that $(\sum_{n \in \mathbb{N}} P_{\mathcal{M}_n})x = \sum_{n \in \mathbb{N}} P_{\mathcal{M}_n} x$, with the series on the right side converging in the norm for each $x \in \mathcal{H}$.*

Self-adjoint operators are the building blocks of all operators, and they are by far the most important subclass of all bounded operators on a Hilbert space. However, in order to see their structure and usefulness, we will have to wait until after we have proved the fundamental spectral theorem. This will allow us to handle self-adjoint operators with exactly the same facility with which we handle real-valued functions.

Nevertheless, we have already seen one important special class of self-adjoint operators as shown by the next result.

PROPOSITION 1.7.3. *Let $P \in B(\mathcal{H})$. Then the following two conditions are equivalent:*
(i) $P = P_{\mathcal{M}}$ is the orthogonal projection onto some closed subspace $\mathcal{M} \subset \mathcal{H}$;
(ii) $P = P^2 = P^*$.

Proof. $(i) \Rightarrow (ii)$: If $P = P_{\mathcal{M}}$, the definition of an orthogonal projection shows that $P = P^2$; the self-adjointness of P follows from Theorem 1.4.13 (4) and Corollary 1.5.3 (2).

$(ii) \Rightarrow (i)$: Suppose (ii) is satisfied; let $\mathcal{M} = \operatorname{ran} P$, and note that

$$\begin{aligned} x \in \mathcal{M} \quad &\Rightarrow \quad \exists y \in \mathcal{H} \text{ such that } x = Py \\ &\Rightarrow \quad Px = P^2 y = Py = x \ ; \end{aligned} \tag{1.7.13}$$

on the other hand, note that

$$\begin{aligned} y \in \mathcal{M}^\perp \quad &\Leftrightarrow \quad \langle y, Pz \rangle = 0 \ \forall z \in \mathcal{H} \\ &\Leftrightarrow \quad \langle Py, z \rangle = 0 \ \forall z \in \mathcal{H} \quad (\text{since } P = P^*) \\ &\Leftrightarrow \quad Py \ = \ 0 \ ; \end{aligned} \tag{1.7.14}$$

hence, if $z \in \mathcal{H}$ and $x = P_{\mathcal{M}} z$, $y = P_{\mathcal{M}^\perp} z$, we find from equations (1.7.13) and (1.7.14) that $Pz = Px + Py = x = P_{\mathcal{M}} z$. □

Direct Sums and Operator Matrices

If $\{\mathcal{M}_n : n \in \mathbb{N}\}$ are pairwise orthogonal closed subspaces – see Exercise 1.7.2 (5)(d) – and if $\mathcal{M} = [\bigcup_{n \in \mathbb{N}} \mathcal{M}_n]$ we say that $\mathcal{M}$ is the **direct sum** of the closed subspaces $\mathcal{M}_i, 1 \leq i \leq n$, and we write

$$\mathcal{M} = \bigoplus_{n=1}^{\infty} \mathcal{M}_i \ ; \tag{1.7.15}$$

conversely, whenever we use the above symbol, it will always be tacitly assumed that the $\mathcal{M}_i$'s are closed subspaces which are pairwise orthogonal and that $\mathcal{M}$ is the (closed) subspace spanned by them.

To clarify matters, let us first consider the direct sum of two subspaces. (We are going to try and mimic the success of operators on $\mathbb{C}^2$ being identifiable with the operation of matrices acting on column vectors by multiplication.)

So suppose $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. We shall think of a typical element $x \in \mathcal{H}$ as a column vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, with $x_i \in \mathcal{H}_i$. Let $P_i = P_{\mathcal{H}_i}$ so $P_i x = x_i$ in the above notation. If we think of P_i as being an element of $B(\mathcal{H}, \mathcal{H}_i)$, then it is easily seen that its adjoint is the isometric element V_i of $B(\mathcal{H}_i, \mathcal{H})$ described thus:

$$V_1 x_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \text{ and } V_2 x_2 = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}.$$

Given a $T \in B(\mathcal{H})$, define $T_{ij} = P_i T V_j \in B(\mathcal{H}_j, \mathcal{H}_i)$ and observe that we have

$$Tx = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

If we refer to $((T_{ij}))$ as the matrix corresponding to T, then the matrices corresponding to P_1 and P_2 are seen to be

$$\begin{bmatrix} \mathrm{id}_{\mathcal{H}_1} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{id}_{\mathcal{H}_2} \end{bmatrix}.$$

More generally, if $\mathcal{H} = \bigoplus_{j\in\mathbb{N}} \mathcal{H}_j, \mathcal{K} = \bigoplus_{i\in\mathbb{N}} \mathcal{K}_i$, there exists a unique matrix $((T_{ij}))$ with $T_{ij} \in B(\mathcal{H}_j, \mathcal{K}_i)$ such that whenever $\xi_j \in \mathcal{H}_j$ satisfy $\sum_{j\in\mathbb{N}} \|\xi_j\|^2 < \infty$ (so that the series $\sum_{j\in\mathbb{N}} \xi_j$ converges in $\mathcal{H}$ (to ξ, say), then $T\xi = \sum_{i\in\mathbb{N}} \left(\sum_{j\in\mathbb{N}} T_{ij}\xi_j\right)$ – with the inner series converging in $\mathcal{K}_i$ for each $i \in \mathbb{N}$ to η_i, say, with $\sum_{i\in\mathbb{N}} \|\eta_i\|^2 < \infty$ and $T\xi = \sum_{i\in\mathbb{N}} \eta_i$. In the special case when each $\mathcal{H}_j$ and $\mathcal{K}_i$ is one-dimensional, this reduces to saying that if $T \in B(\mathcal{H}, \mathcal{K})$ and if $\{x_j : j \in \mathbb{N}\}$ (resp., $\{y_i : i \in \mathbb{N}\}$) is an orthonormal basis in $\mathcal{H}$ (resp., $\mathcal{K}$), then the operator T can be described by matrix multiplication in the following sense: if the vector $x \in \mathcal{H}$ (resp., $y \in \mathcal{K}$) is thought of as the countably infinite column matrix $[x] = [\beta_j]$ with $\beta_j = \langle x, x_j\rangle$ (resp., $[y] = [\alpha_i]$ with $\alpha_i = \langle y, y_i\rangle$), and if $[T]$ is the matrix $((t_{ij}))$ with countably infinitely many rows and columns with $t_{ij} = \langle Tx_j, y_i\rangle$, then $Tx = y \Leftrightarrow \alpha_i = \sum_j t_{ij}\beta_j \ \forall i$.

EXERCISE 1.7.4. *(1) Verify the assertions of the previous paragraphs. (*Hint: *The computation in the case of finite direct sums will show what needs to be done in the infinite case.)*

(2) With the notation of the paragraph preceding this exercise, verify that the familiar E_{ij} matrix whose only non-zero entry is a 1 in the (i,j)-th spot is the matrix of the operator denoted by $(\bar{x}_j \otimes y_i)$ in Exercise 3.2.11, *and defined in the paragraph preceding that exercise.*

(3) Verify the following fundamental rules concerning the **system** $\{E_{ij}\}$:

(1) $E_{ij}^* = E_{ji}$;

(2) $E_{ij}E_{kl} = \delta_{jk}E_{il}$

where the Kronecker symbol is defined by

$$\delta_{pq} = \begin{cases} 1 & if\ p = q \\ 0 & otherwise \end{cases}.$$

1.7.2 Isometric versus Unitary

The two propositions given below identify two important classes of operators between Hilbert spaces.

PROPOSITION 1.7.5. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces; the following conditions on an operator $U \in B(\mathcal{H}, \mathcal{K})$ are equivalent:*

(*i*) *if $\{e_i : i \in I\}$ is any orthonormal set in $\mathcal{H}$, then also $\{Ue_i : i \in I\}$ is an orthonormal set in $\mathcal{K}$;*

(*ii*) *there is an orthonormal basis $\{e_i : i \in I\}$ for $\mathcal{H}$ such that $\{Ue_i : i \in I\}$ is an orthonormal set in $\mathcal{K}$;*

(*iii*) $\langle Ux, Uy\rangle = \langle x, y\rangle \ \forall\, x, y \in \mathcal{H}$;

(*iv*) $||Ux|| = ||x|| \ \forall\, x \in \mathcal{H}$;

(*v*) $U^*U = 1_{\mathcal{H}}$.

An operator satisfying these equivalent conditions is called an **isometry**.

Proof. (*i*) $\Rightarrow$ (*ii*) : There exists an orthonormal basis for $\mathcal{H}$.

(*ii*) $\Rightarrow$ (*iii*) : If $x, y \in \mathcal{H}$ and if $\{e_i : i \in I\}$ is as in (*ii*), then

$$\begin{aligned}\langle Ux, Uy\rangle &= \left\langle U\Big(\sum_{i\in I}\langle x, e_i\rangle e_i\Big)\ ,\ U\Big(\sum_{j\in I}\langle y, e_j\rangle e_j\Big)\right\rangle \\ &= \sum_{i,j\in I}\langle x, e_i\rangle\langle e_j, y\rangle\langle Ue_i, Ue_j\rangle \\ &= \sum_{i\in I}\langle x, e_i\rangle\langle e_i, y\rangle \\ &= \langle x, y\rangle\ .\end{aligned}$$

(*iii*) $\Rightarrow$ (*iv*) : Put $y = x$.

(*iv*) $\Rightarrow$ (*v*) : If $x \in \mathcal{H}$, note that

$$\langle U^*Ux, x\rangle = ||Ux||^2 = ||x||^2 = \langle 1_{\mathcal{H}}x, x\rangle\ ,$$

and appeal to the fact that a bounded operator is determined by its quadratic form – see Corollary 1.5.3.

(*v*) $\Rightarrow$ (*i*) : If $\{e_i : i \in I\}$ is any orthonormal set in $\mathcal{H}$, then

$$\langle Ue_i, Ue_j\rangle = \langle U^*Ue_i, e_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}\ .$$

□

PROPOSITION 1.7.6. *The following conditions on an isometry $U \in B(\mathcal{H}, \mathcal{K})$ are equivalent:*

(*i*) *if $\{e_i : i \in I\}$ is any orthonormal basis for $\mathcal{H}$, then $\{Ue_i : i \in I\}$ is an orthonormal basis for $\mathcal{K}$;*

(*ii*) *there is an orthonormal set $\{e_i : i \in I\}$ in $\mathcal{H}$ such that $\{Ue_i : i \in I\}$ is an orthonormal basis for $\mathcal{K}$;*

(*iii*) $UU^* = 1_{\mathcal{K}}$;

(*iv*) *U is invertible;*

(v) U maps $\mathcal{H}$ onto $\mathcal{K}$.

An isometry which satisfies the above equivalent conditions is said to be **unitary**.

Proof. $(i) \Rightarrow (ii)$: Obvious.

$(ii) \Rightarrow (iii)$: If $\{e_i : i \in I\}$ is as in (ii), and if $x \in \mathcal{K}$, observe that

$$\begin{aligned} UU^*x &= UU^*\Big(\sum_{i\in I}\langle x, Ue_i\rangle Ue_i\Big) \\ &= \sum_{i\in I}\langle x, Ue_i\rangle UU^*Ue_i \\ &= \sum_{i\in I}\langle x, Ue_i\rangle Ue_i \quad \text{(since } U \text{ is an isometry)} \\ &= x\ . \end{aligned}$$

$(iii) \Rightarrow (iv)$: The assumption that U is an isometry, in conjunction with the hypothesis (iii), says that $U^* = U^{-1}$.

$(iv) \Rightarrow (v)$: Obvious.

$(v) \Rightarrow (i)$: If $\{e_i : i \in I\}$ is an orthonormal basis for $\mathcal{H}$, then $\{Ue_i : i \in I\}$ is an orthonormal set in $\mathcal{H}$, since U is isometric. Now, if $z \in \mathcal{K}$, pick $x \in \mathcal{H}$ such that $z = Ux$, and observe that

$$\begin{aligned} \|z\|^2 &= \|Ux\|^2 \\ &= \|x\|^2 \\ &= \sum_{i\in I}|\langle x, e_i\rangle|^2 \\ &= \sum_{i\in I}|\langle z, Ue_i\rangle|^2\ , \end{aligned}$$

and since z was arbitrary, this shows that $\{Ue_i : i \in I\}$ is an orthonormal basis for $\mathcal{K}$. □

Thus, unitary operators are the natural isomorphisms in the context of Hilbert spaces. The collection of unitary operators from $\mathcal{H}$ to $\mathcal{K}$ will be denoted by $\mathcal{U}(\mathcal{H}, \mathcal{K})$; when $\mathcal{H} = \mathcal{K}$, we shall write $\mathcal{U}(\mathcal{H}) = \mathcal{U}(\mathcal{H}, \mathcal{H})$. We list some elementary properties of unitary and isometric operators in the next exercise.

EXERCISE 1.7.7. (1) *Suppose that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and suppose $\{e_i : i \in I\}$ (resp., $\{f_i : i \in I\}$) is an orthonormal basis (resp., orthonormal set) in $\mathcal{H}$ (resp., $\mathcal{K}$), for some index set I. Show that:*

(a) $\dim \mathcal{H} \leq \dim \mathcal{K}$; *and*

(b) there exists a unique isometry $U \in B(\mathcal{H}, \mathcal{K})$ such that $Ue_i = f_i\ \forall i \in I$.

(2) *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Show that:*

(a) there exists an isometry $U \in B(\mathcal{H}, \mathcal{K})$ if and only if dim $\mathcal{H} \leq \dim \mathcal{K}$;

(b) there exists a unitary $U \in B(\mathcal{H}, \mathcal{K})$ *if and only if* $\dim \mathcal{H} = \dim \mathcal{K}$.

(3) *Show that* $\mathcal{U}(\mathcal{H})$ *is a group under multiplication, which is a (norm-) closed subset of the Banach space* $B(\mathcal{H})$.

(4) *Suppose* $U \in \mathcal{U}(\mathcal{H}, \mathcal{K})$*; show that the association*

$$B(\mathcal{H}) \ni T \overset{adU}{\mapsto} UTU^* \in B(\mathcal{K}) \tag{1.7.16}$$

defines a mapping $(ad\,U) : B(\mathcal{H}) \to B(\mathcal{K})$ *which is an 'isometric isomorphism of Banach *-algebras', meaning that:*

(a) $ad\,U$ *is an isometric isomorphism of Banach spaces: i.e.,* $ad\ U$ *is a linear mapping which is 1-1, onto, and is norm-preserving; (Hint: verify that it is linear and preserves norm and that its inverse is given by* $ad\,U^*$*.)*

(b) $ad\,U$ *is a product-preserving map between Banach algebras; i.e.,* $(ad\,U)(T_1T_2) = \left((ad\,U)\,(T_1)\right)\left((ad\,U)(T_2)\right)$*, for all* $T_1, T_2 \in B(\mathcal{H})$*;*

(c) $ad\,U$ *is a *-preserving map between *-algebras; i.e.,*

$$\left((ad\,U)(T)\right)^* = (ad\,U)(T^*)\ \forall T \in B(\mathcal{H}).$$

(5) *Show that the map* $U \mapsto (ad\,U)$ *is a homomorphism from the group* $\mathcal{U}(\mathcal{H})$ *into the group* $Aut\,B(\mathcal{H})$ *of all automorphisms (= isometric isomorphisms of the Banach *-algebra* $B(\mathcal{H})$ *onto itself); further, verify that if* $U_n \to U$ *in* $\mathcal{U}(\mathcal{H}, \mathcal{K})$*, then* $(ad\,U_n)(T) \to (ad\,U)(T)$ *in* $B(\mathcal{K})$ *for all* $T \in B(\mathcal{H})$.

A unitary operator between Hilbert spaces should be viewed as 'implementing an inessential variation'; thus, if $U \in \mathcal{U}(\mathcal{H}, \mathcal{K})$ and if $T \in B(\mathcal{H})$, then the operator $UTU^* \in B(\mathcal{K})$ should be thought of as being 'essentially the same as T', except that it is probably being viewed from a different observer's perspective. All this is made precise in the following definition.

DEFINITION 1.7.8. *Two operators* $T \in B(\mathcal{H})$ *and* $S \in B(\mathcal{K})$ *(on two possibly different Hilbert spaces) are said to be* **unitarily equivalent** *if there exists a unitary operator* $U \in \mathcal{U}(\mathcal{H}, \mathcal{K})$ *such that* $S = UTU^*$.

We conclude this section with a discussion of some examples of isometric operators, which will illustrate the preceding notions quite nicely.

EXAMPLE 1.7.9. To start with, notice that if $\mathcal{H}$ is a finite-dimensional Hilbert space, then an isometry $U \in B(\mathcal{H})$ is necessarily unitary. (Prove this!) Hence, the notion of non-unitary isometries of a Hilbert space into itself makes sense only in infinite-dimensional Hilbert spaces. We discuss some examples of a non-unitary isometry in a separable Hilbert space.

(1) Let $\mathcal{H} = \ell^2$ $(= \ell^2(\mathbb{N})$). Let $\{e_n : n \in \mathbb{N}\}$ denote the standard orthonormal basis of $\mathcal{H}$ (consisting of sequences with a 1 in one co-ordinate and 0 in all other co-ordinates). In view of Exercise 1.7.7(1)(b), there exists a unique isometry $S \in B(H)$ such that $Se_n = e_{n+1}\ \forall n \in \mathbb{N}$; equivalently, we have

$$S(\alpha_1, \alpha_2, \ldots) = (0, \alpha_1, \alpha_2, \ldots).$$

For obvious reasons, this operator is referred to as a 'shift' operator; in order to distinguish it from a near relative, we shall refer to it as the **unilateral shift**. It should be clear that S is an isometry whose range is the proper subspace $\mathcal{M} = \{e_1\}^\perp$, and consequently, S is not unitary.

A minor computation shows that the adjoint S^* is the 'backward shift':

$$S^*(\alpha_1, \alpha_2, \ldots) = (\alpha_2, \alpha_3, \ldots)$$

and that $SS^* = P_{\mathcal{M}}$ (which is another way of seeing that S is not unitary). Thus S^* is a left-inverse, but not a right-inverse, for S. (This, of course, is typical of a non-unitary isometry.)

Further – as is true for any non-unitary isometry – each power $S^n, n \geq 1$, is a non-unitary isometry.

(2) The 'near-relative' of the unilateral shift, which was referred to earlier, is the so-called **bilateral shift**, which is defined as follows: consider the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$ with its standard basis $\{e_n : n \in \mathbb{Z}\}$ for $\mathcal{H}$. The bilateral shift is the unique isometry B on $\mathcal{H}$ such that $Be_n = e_{n+1} \; \forall n \in \mathbb{Z}$. This time, however, since B maps the standard basis onto itself, we find that B is unitary. The reason for the terminology 'bilateral shift' is this: denote a typical element of $\mathcal{H}$ as a 'bilateral' sequence (or a sequence extending to infinity in both directions); in order to keep things straight, let us underline the 0-th co-ordinate of such a sequence; thus, if $x = \sum_{n=-\infty}^{\infty} \alpha_n e_n$, then we write $x = (\ldots, \alpha_{-1}, \underline{\alpha_0}, \alpha_1, \ldots)$; we then find that

$$B(\ldots, \alpha_{-1}, \underline{\alpha_0}, \alpha_1, \ldots) = (\ldots, \alpha_{-2}, \underline{\alpha_{-1}}, \alpha_0, \ldots) .$$

(3) Consider the Hilbert space $\mathcal{H} = L^2([0,1], m)$ (where, of course, m denotes 'Lebesgue measure') – see Remark 1.4.9(2) – and let $\{e_n : n \in \mathbb{Z}\}$ denote the exponential basis of this Hilbert space. Notice that $|e_n(x)|$ is identically equal to 1, and conclude that the operator defined by

$$(Wf)(x) = e_1(x)f(x) \; \forall f \in \mathcal{H}$$

is necessarily isometric; it should be clear that this is actually unitary, since its inverse is given by the operator of multiplication by e_{-1}.

It is easily seen that $We_n = e_{n+1} \; \forall n \in \mathbb{Z}$. If $U : \ell^2(\mathbb{Z}) \to \mathcal{H}$ is the unique unitary operator such that U maps the n-th standard basis vector to e_n, for each $n \in \mathbb{Z}$, it follows easily that $W = UBU^*$. Thus, the operator W of this example is unitarily equivalent to the bilateral shift (of the previous example).

More is true; let $\mathcal{M}$ denote the closed subspace $\mathcal{M} = [\{e_n : n \geq 1\}]$; then $\mathcal{M}$ is invariant under W – meaning that $W(\mathcal{M}) \subset \mathcal{M}$; and it should be clear that the restricted operator $W|_{\mathcal{M}} \in B(\mathcal{M})$ is unitarily equivalent to the unilateral shift.

(4) More generally, if $(X, \mathcal{B}, \mu)$ is any measure space and if $\phi : X \to \mathbb{C}$ is any measurable function such that $|\phi| = 1$ μ-a.e., then the equation

$$M_\phi f = \phi f \ , f \in L^2(X, \mathcal{B}, \mu)$$

defines a unitary operator on $L^2(X, \mathcal{B}, \mu)$ (with inverse given by $M_{\bar{\phi}}$). □

Chapter 2

The Spectral Theorem

2.1 C^*-algebras

It will be convenient, indeed desirable, to use the language of C^*-algebras.

DEFINITION 2.1.1. *A C^*-algebra is a Banach algebra $\mathcal{A}$ equipped with an adjoint operation $\mathcal{A} \ni S \mapsto S^* \in \mathcal{A}$ which satisfies the following conditions for all $S, T \in \mathcal{A}$*

$$\begin{aligned} (\alpha S_1 + S_2)^* &= \bar{\alpha} S_1^* + S_2^* \\ (S^*)^* &= S \\ (ST)^* &= T^* S^* \\ \|T\|^2 &= \|T^*T\| \ (\mathbf{C}^* - \textit{identity}). \end{aligned}$$

All our C^*-algebras will be assumed to have a multiplicative identity, which is necessarily self-adjoint (as 1^* is also a multiplicative identity), and has norm one – thanks to the C^*-identity ($\|1\|^2 = \|1^*1\| = \|1\|$). (We ignore the trivial possibility $1 = 0$, i.e., $\mathcal{A} = \{0\}$.)

EXAMPLE 2.1.2. (1) $B(\mathcal{H})$ is a C^*-algebra, and in particular $M_n(\mathbb{C})$ $\forall n$, so also $\mathbb{C} = M_1(\mathbb{C})$.

(2) Any norm-closed unital $*$-subalgebra of a C^*-algebra is also a C^*-algebra with the induced structure from the ambient C^*-algebra.

(3) For any subset S of a C^*-algebra, there is a smallest C^*-subalgebra of $\mathcal{A}$, denoted by $C^*(S)$, which contains S. (*Reason:* $C^*(S)$ may be defined somewhat uninformatively as the intersection of all C^*-subalgebras that contain S, and described more constructively as the norm-closure of the linear span of all 'words' in the alphabet $\{1\} \cup S \cup S^* := \{1\} \cup \{x : x \in S \textit{ or } x^* \in S\}$.) The latter description in the previous sentence shows that $C^*(\{x\})$ is a *commutative* 'singly generated' C^*-subalgebra if and only if x satisfies $x^*x = xx^*$; such an element of a C^*-algebra, which commutes with its adjoint, is said to be **normal**.

(4) If Σ is any compact space, then $C(\Sigma)$ is a commutative C^*-algebra – with respect to pointwise algebraic operations, $f^* = \bar{f}$ and $\|f\| = \sup\{|f(x)| : x \in \Sigma\}$. If $\Sigma \subset \mathbb{R}$ (resp., $\mathbb{C}$), then the Weierstrass polynomial approximation theorem (resp., the Stone-Weierstrass theorem) shows that $C(\Sigma)$ is a commutative unital C^*-algebra which is singly generated – with generator given by $f_0(t) = t \; \forall t \in \Sigma$.

DEFINITION 2.1.3. *A* **representation** *of a C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is just a *-preserving unital algebra homomorphism of $\mathcal{A}$ into $B(\mathcal{H})$.*

Representations $\pi_i : \mathcal{A} \to B(\mathcal{H}_i)$, $i = 1, 2$, are said to be **equivalent** *if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_2(a) = U\pi_1(a)U^* \; \forall a \in \mathcal{A}$.*

REMARK 2.1.4. It is true that any representation – and more generally, any unital *-algebra homomorphism between C^*-algebras – is contractive. This is essentially a consequence of (a) the C^*-identity, which shows that it suffices to check that $\|\pi(x)\| \leq \|x\| \; \forall x = x^*$ (b) the fact that the norm of a self-adjoint operator is its spectral radius, (see the last part of Proposition 1.5.6 (2)), and (c) the obvious fact that a unital homomorphism preserves invertibility and hence 'shrinks spectra'. Thus,

$$\|\pi(x)\|^2 = \|\pi(x)^*\pi(x)\| = \|\pi(x^*x)\|$$
$$= \mathrm{spr}(\pi(x^*x)) \leq \mathrm{spr}(x^*x) \leq \|x^*x\| = \|x\|^2.$$

But we will not need this fact in this generality, so we shall say no more about it.

The observation that sets the ball rolling for us is Proposition 1.5.7.

PROPOSITION 2.1.5. *Let $\Sigma \subset \mathbb{R}$ be a compact set and let $f_0 \in C(\Sigma)$ be given by $f_0(t) = t \; \forall t \in \Sigma$. If $X \in B(\mathcal{H})$ is a self-adjoint operator such that $\sigma(X) \subset \Sigma$, then there exists a unique representation $\pi : C(\Sigma) \to B(\mathcal{H})$ such that $\pi(f_0) = X$. Conversely given any representation $\pi : C(\Sigma) \to B(\mathcal{H})$, it is the case that $\pi(f_0)$ is a self-adjoint operator X satisfying $\sigma(X) \subset \Sigma$.*

Proof. To begin with, if $X \in B(\mathcal{H})$ is a self-adjoint operator such that $\sigma(X) \subset \Sigma$, then it follows from the inequality (1.5.12) that $\|p(X)\|_{B(\mathcal{H})} \leq \|p\|_{C(\sigma(X))} \leq \|p\|_{C(\Sigma)}$ for any polynomial p. It is easily deduced now, from Weierstrass' theorem, that this mapping $\mathbb{C}[t] \ni p \mapsto p(X) \in B(\mathcal{H})$ extends uniquely to the desired *-homomorphism from $C(\Sigma)$ to $B(\mathcal{H})$.

Conversely, it is easily seen that $f_0 - \lambda$ is not invertible in $C(\Sigma)$ if and only if $\lambda \in \Sigma$ and as π preserves invertibility, we find that

$$\sigma(X) = \sigma(\pi(f_0)) \subset \sigma(f_0) = \Sigma$$

as desired. (Strictly speaking, we have only defined spectra of operators, while we are here talking of the spectra of elements of unital Banach algebras – $C(\Sigma)$, to be precise – but the definition is more or less the same.) □

REMARK 2.1.6. Representations $\pi_i : C(\Sigma) \to B(\mathcal{H}_i)$ are equivalent if and only if the operators $\pi_i(f_0), i = 1, 2$ are unitarily equivalent. This is because a representation of a singly generated C^*-algebra is uniquely determined by the image of the generator.

2.2 Cyclic representations and measures

Assume, for the rest of this book, that Σ is a separable compact metric space. Suppose $\pi : C(\Sigma) \to B(\mathcal{H})$ is a representation of $C(\Sigma)$ on a separable Hilbert space.

DEFINITION 2.2.1. *A representation $\pi : C(\Sigma) \to B(\mathcal{H})$ is said to be* **cyclic** *if there exits a vector $x \in \mathcal{H}$ such that $\pi(C(\Sigma))x$ is a dense subspace of $\mathcal{H}$. In such a case, the vector x is called a* **cyclic vector** *for the representation. If such a vector exists, one can always find a unit vector which is cyclic for the representation.*

Before proceeding, it will be wise to spell out a trivial, but nevertheless very useful, observation.

LEMMA 2.2.2. *If $\mathcal{S}_i = \{x_j^{(i)} : j \in \Lambda\}$ is a set which linearly spans a dense subspace of a Hilbert space $\mathcal{H}_i$ for $i = 1, 2$, and if $\langle x_j^{(1)}, x_k^{(1)} \rangle = \langle x_j^{(2)}, x_k^{(2)} \rangle$ for all $j, k \in \Lambda$, then there exists a unique unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $Ux_j^{(1)} = x_j^{(2)}\ \forall j \in \Lambda$.*

Proof. The hypotheses guarantee that the equation

$$U_0\Big(\sum_{\ell=1}^{n} \alpha_\ell x_{j_\ell}^{(1)}\Big) = \sum_{\ell=1}^{n} \alpha_\ell x_{j_\ell}^{(2)}$$

unambiguously defines a linear bijection U_0 between dense linear subspaces of the two Hilbert spaces preserving inner product, and hence extends uniquely to a unitary operator U with the desired property. Uniqueness of such a U follows from the fact that the difference between two such U's would have a dense linear subspace in its kernel. □

PROPOSITION 2.2.3. *(1) If μ is a finite positive measure defined on the Borel subsets of Σ, then the equation*

$$(\pi_\mu(f))\,(g) = fg\ \forall f \in C(\Sigma),\ g \in L^2(\Sigma, \mu)$$

defines a cyclic representation π_μ of $C(\Sigma)$ with cyclic vector $g_0 \equiv 1$.

(2) Conversely, if $\pi : C(\Sigma) \to B(\mathcal{H})$ is a representation with a cyclic vector x, then there exists a finite positive measure μ defined on the Borel subsets of Σ and a unitary operator $U : \mathcal{H} \to L^2(\Sigma, \mu)$ such that $Ux = g_0$ and $U\pi(f)U^* = \pi_\mu(f)\ \forall f \in C(\Sigma)$.

(3) *In the setting of (1) above, there exists a unique representation* $\widetilde{\pi_\mu} : L^\infty(\Sigma, \mu) \to B(L^2(\Sigma, \mu))$ *such that (i)* $\widetilde{\pi_\mu}|_{C(\Sigma)} = \pi_\mu$*, and (ii) if* $\{f_n : n \in \mathbb{N}\}$ *is such that* $\sup_n \|f_n\|_{L^\infty(\Sigma,\mu)} < \infty$ *and* $f_n \to f$ *μ-a.e., then* $\|\widetilde{\pi_\mu}(f_n)g - \widetilde{\pi_\mu}(f)g\|_{L^2(\Sigma,\mu)} \to 0 \ \forall g \in L^2(\Sigma, \mu)$.

Further, the measure μ *is a probability measure precisely when the cyclic vector* x *is a unit vector.*

Proof. (1) It is fairly clear that $C(\Sigma) \ni f \mapsto \pi_\mu(f) \in B(\mathcal{H})$ is a representation of $C(\Sigma)$ and $\|\pi_\mu(f)\|_{B(L^2(\Sigma,\mu))} \leq \|f\|_{L^\infty(\Sigma,\mu)}$. Clearly each $\pi_\mu(f)$ is normal, and it follows from Theorem 1.6.2 (2) that

$$\lambda \in \mathrm{sp}\,(\pi_\mu(f)) \Leftrightarrow \mu(\{w \in \Sigma : |f(w) - \lambda| < \epsilon\}) > 0 \ \forall \epsilon > 0,$$

and in particular, $\|\pi_\mu(f)\| = spr(\pi_\mu(f)) = \|f\|_{L^\infty(\Sigma,\mu)} \leq \|f\|_{C(\Sigma)}$. it is a basic fact from measure theory – see Lemma A1 in the Appendix – that g_0 is indeed a cyclic vector for the representation π_μ.

(2) Consider the functional $\phi : C(\Sigma) \to \mathbb{C}$ defined by $\phi(f) = \langle \pi(f)x, x\rangle$. It is clear that if $f \in C(\Sigma)$ is non-negative, then also $f^{\frac{1}{2}} \in C(\Sigma)$ is non-negative and, in particular, real-valued, and hence

$$\phi(f) = \langle \pi(f^{\frac{1}{2}})x, \pi(f^{\frac{1}{2}})x\rangle \geq 0.$$

Thus ϕ is a positive – and clearly bounded – linear functional on $C(\Sigma)$, and the **Riesz representation theorem** – which identifies the dual space of $C(\mathbf{\Sigma})$ with the set $M(\Sigma)$ of finite complex measures – guarantees the existence of a positive measure μ defined on the Borel sets of Σ such that $\phi(f) = \int f \, d\mu$. It follows that for arbitrary $f, g \in C(\Sigma)$, we have

$$\begin{aligned} \langle \pi(f)x, \pi(g)x\rangle &= \langle \pi(\bar{g}f)x, x\rangle \\ &= \phi(\bar{g}f) \\ &= \int \bar{g}f \, d\mu \\ &= \langle \pi_\mu(f)g_0, \pi_\mu(g)g_0\rangle \ . \end{aligned}$$

An appeal to Lemma 2.2.2 now shows that there exists a unitary operator $U : \mathcal{H} \to L^2(\Sigma, \mu)$ such that $U\pi(f)x = \pi_\mu(f)g_0 \ \forall f \in C(\Sigma)$. Setting $f = 1$, we find that $Ux = g_0$. And for all $g \in C(\Sigma)$, we see that $U\pi(f)U^*\pi_\mu(g)g_0 = U\pi(f)\pi(g)x = \pi_\mu(f)\pi_\mu(g)x_0$ with the result that, indeed, $U\pi(f)U^* = \pi_\mu(f)$, completing the proof of (2).

(3) Simply define $\widetilde{\pi_\mu}(\phi)g = \phi g \ \forall g \in L^2(\Sigma, \mu)$. Then (i) is clearly true, while (ii) is just a restatement of the bounded convergence theorem of measure theory. The uniqueness assertion regarding $\widetilde{\pi_\mu}$ follows from the demanded (i) and Lemma A2 in the Appendix.

□

It would make sense to introduce a definition and a notation for a notion that has already been encountered more than once.

DEFINITION 2.2.4. *A sequence* $\{X_n : n \in \mathbb{N}\}$ *in* $B(\mathcal{H})$ *is said to converge in the* **strong operator topology** – *henceforth abbreviated to SOT – if* $\{X_n x : n \in \mathbb{N}\}$ *converges in the norm of* $\mathcal{H}$ *for every* $x \in \mathcal{H}$*. It is a consequence of the 'uniform boundedness principle' that in this case, the equation*

$$Xx = \lim_{n \to \infty} X_n x$$

defines a bounded operator $X \in B(\mathcal{H})$*. We shall abbreviate all this by writing* $X_n \stackrel{SOT}{\to} X$*.*

We record a couple of simple but very useful facts concerning SOT convergence. But first, recall that a set $\mathcal{S} \subset \mathcal{H}$ is said to be **total** if the linear subspace spanned by $\mathcal{S}$ is dense in $\mathcal{H}$. (eg: any orthonormal basis (onb) is total.)

LEMMA 2.2.5. *(1) The following conditions on a sequence* $\{X_n : n \in \mathbb{N}\} \subset B(\mathcal{H})$ *are equivalent:*

(a) $X_n \stackrel{SOT}{\to} X$ *for some* $X \in B(\mathcal{H})$*;*

(b) $\sup_n \|X_n\| < \infty$*, and there exists some total set* $\mathcal{S} \subset \mathcal{H}$ *such that* $X_n x$ *converges for all* $x \in \mathcal{S}$*;*

(c) $\sup_n \|X_n\| < \infty$*, and there exists a dense subspace* $\mathcal{M} \subset \mathcal{H}$ *such that* $X_n x$ *converges for all* $x \in \mathcal{M}$*.*

(2) If sequences $X_n \stackrel{SOT}{\to} X$ *and* $Y_n \stackrel{SOT}{\to} Y$ *in* $B(\mathcal{H})$*, then also* $X_n Y_n \stackrel{SOT}{\to} XY$*.*

Proof. (1) The implication $(a) \Rightarrow (b)$ follows from the uniform boundedness principle, while $(b) \Rightarrow (c)$ is seen on setting $\mathcal{M} = \bigvee \mathcal{S}$, the vector subspace spanned by $\mathcal{S}$. As for $(c) \Rightarrow (a)$, if $\sup_n \|X_n\| < K (> 0)$, note that the equation $Xx = \lim_n X_n x$ defines a linear map $X : \mathcal{M} \to \mathcal{H}$ with $\|Xx\| \leq K\|x\| \ \forall x \in \mathcal{M}$; the assumed density of $\mathcal{M}$ ensures that X admits a unique extension to an element of $B(\mathcal{H})$, also denoted by X, with $\|X\| \leq K$ and $X_n x \to Xx \ \forall x \in \mathcal{M}$. Now, if $x \in \mathcal{H}$ and $\epsilon > 0$, choose $x' \in \mathcal{M}$ such that $\|x - x'\| < \epsilon/3K$, then choose an $n_0 \in \mathbb{N}$ such that $\|(X_n - X)x'\| < \epsilon/3 \ \forall n \geq n_0$ and compute thus, for $n \geq n_0$:

$$\begin{aligned} \|(X_n - X)x\| &\leq \|(X_n - X)(x - x')\| + \|(X_n - X)x'\| \\ &< (2K)\frac{\epsilon}{3K} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

(2) Begin by deducing from the uniform boundedness principle that there exists a constant $K > 0$ such that $\|X_n\| \leq K$ and $\|Y_n\| \leq K$ for all n. Fix $x \in \mathcal{H}$ and an $\epsilon > 0$. Under the hypotheses, we can find an $n_0 \in \mathbb{N}$ such

that $\|(Y_n - Y)x\| < \epsilon/2K$ and $\|(X_n - X)Yx\| < \epsilon/2$ for all $n \geq n_0$. We then see that for every $n \geq n_0$

$$\begin{aligned} \|(X_nY_n - XY)x\| &= \|(X_nY_n - X_nY + X_nY - XY)x\| \\ &\leq \|X_n(Y_n - Y)x\| + \|(X_n - X)Yx\| \\ &< \epsilon, \end{aligned}$$

thus proving that indeed $X_nY_n \stackrel{SOT}{\longrightarrow} XY$.

□

The following important consequence of Proposition 2.2.3 is 'one half' of the celebrated Hahn-Hellinger classification of separable representations of $C(\Sigma)$. (See Remark 2.3.3.)

THEOREM 2.2.6. *If $\pi : C(\Sigma) \to B(\mathcal{H})$ is a representation on a separable Hilbert space $\mathcal{H}$, there exists a countable collection $\{\mu_n : n \in N\}$ (for some countable set N) of probability measures defined on the Borel-σ-algebra $\mathcal{B}_\Sigma$ such that π is (unitarily) equivalent to $\oplus\pi_{\mu_n} : C(\Sigma) \to B(\bigoplus L^2(\Sigma, \mu_n))$.*

Proof. Note that $\mathcal{H}$ is separable, as is the Hilbert space underlying any cyclic representation of $C(\Sigma)$ (since the latter is separable). Also observe that $\pi(C(\Sigma))$ is closed under adjoints, as a consequence of which, if a subspace of $\mathcal{M} \subset \mathcal{H}$ is left invariant by the entire *-algebra $\pi(C(\Sigma))$, then so is $\mathcal{M}^\perp$. It follows from the previous sentence and a simple use of Zorn's lemma, that there exists a countable (possibly finite) collection $\{x_n : n \in N\}$ (for some countable set N) of unit vectors such that $\mathcal{H} = \bigoplus_{n\in N} \overline{(\pi(C(\Sigma))x_n)}$. Clearly each $\mathcal{M}_n = \overline{(\pi(C(\Sigma))x_n)}$ is a closed subspace that is invariant under the algebra $\pi(C(\Sigma))$ and yields a cyclic subrepresentation $\pi_n(\cdot) = \pi(\cdot)|_{\mathcal{M}_n}$. It follows from Proposition 2.2.3 (2) that

$$\pi = \bigoplus\nolimits_{n\in N} \pi_n \sim \bigoplus \pi_{\mu_n} ,$$

for the probability measures given by

$$\int_\Sigma f \, d\mu_n = \langle \pi(f)x_n, x_n \rangle .$$

□

LEMMA 2.2.7. *In the notation of Proposition 2.2.3(3), the following conditions on a bounded sequence $\{f_n\}$ in $L^\infty(\mu)$ are equivalent:*

(1) *the sequence $\{f_n\}$ converges in (μ-) measure to 0;*

(2) $\widetilde{\pi_\mu}(f_n) \stackrel{SOT}{\longrightarrow} 0$.

Proof. (1) $\Rightarrow$ (2): This is an immediate consequence of a version of the dominated convergence theorem.

(2) $\Rightarrow$ (1): Since the constant function $g_0 \equiv 1$ belongs to $L^2(\Sigma, \mu)$, it follows from the inequality

$$\begin{aligned}\mu(\{|f_n - f| \geq \epsilon\}) &\leq \epsilon^{-2} \int_{\{|f_n - f| \geq \epsilon\}} |f_n - f|^2 d\mu \\ &\leq \epsilon^{-2} \int |f_n - f|^2 d\mu\end{aligned}$$

that indeed $\mu(\{|f_n - f| \geq \epsilon\}) \to 0 \ \forall \epsilon > 0$. □

THEOREM 2.2.8. *Let $\pi : C(\Sigma) \to B(\mathcal{H})$ and $\{\mu_n : n \in N\}$ be as given in Theorem 2.2.6. Choose some set $\{\epsilon_n : n \in N\}$ of strictly positive numbers such that $\sum_{n \in N} \epsilon_n = 1$, and define the probability measure μ on $(\Sigma, \mathcal{B}_\Sigma)$ by $\mu = \sum_{n \in N} \epsilon_n \mu_n$. Then, we have:*

(1) For $E \in \mathcal{B}_\Sigma$ we have $\mu(E) = 0 \Leftrightarrow \mu_n(E) = 0 \ \forall n \in N$. Further, $\phi \in L^\infty(\Sigma, \mu) \Rightarrow \phi \in L^\infty(\Sigma, \mu_n) \ \forall n \in N$ and $\sup_n \|\phi\|_{L^\infty(\mu_n)} = \|\phi\|_{L^\infty(\mu)}$.

(2) The equation $\tilde{\pi} = \bigoplus_{m \in N} \widetilde{\pi_{\mu_m}}$ defines an isometric representation $\tilde{\pi} : L^\infty(\Sigma, \mu) \to B(\mathcal{H})$ such that the following conditions on a uniformly norm-bounded sequence $\{\phi_n : n \in \mathbb{N}\}$ in $L^\infty(\mu)$ are equivalent:

(a) $\phi_n \to 0$ in measure w.r.t. μ.

(b) $\phi_n \to 0$ in measure w.r.t. μ_m for all m.

(c) $\tilde{\pi}(\phi_n) \overset{SOT}{\longrightarrow} 0$.

Proof. Before proceeding with the proof, we wish to underline the (so far unwritten) convention that we use throughout this book: we treat elements of different L^p-spaces as if they were functions (rather than equivalence classes of functions agreeing almost everywhere.).

(1) Since $\epsilon_n > 0 \ \forall n \in N$, it follows that $\mu(E) = 0 \Leftrightarrow \mu_n(E) = 0 \ \forall n \in N$.

Since a countable union of null sets is also a null set, it is clear that if $\phi \in L^\infty(\mu)$, we may find a μ-null set F which will satisfy the condition $\|\phi\|_{L^\infty(\mu)} = \sup\{|\phi(\lambda)| : \lambda \in \Sigma \setminus F\}$. For $E \in \mathcal{B}_\Sigma$ we have $\mu(E) = 0 \Leftrightarrow \mu_n(E) = 0 \ \forall n \in N$. Let $F_n = F \cup \{\frac{d\mu_n}{d\mu} = 0\}$. Clearly,

$$\mu_n(\textstyle\bigcap_{m \in N} F_m) \leq \mu_n(F_n) = 0 \ \forall n$$

so, also $\mu(\bigcap_{m \in N} F_m) = 0$. Since $(\bigcap_{m \in N} F_m) \supset F$, we see thus that

$$\begin{aligned}\|\phi\|_{L^\infty(\mu)} &= \sup\left\{|\phi(\lambda)| : \lambda \in \Sigma \setminus \textstyle\bigcap_{m \in N} F_m\right\} \\ &= \sup\left\{|\phi(\lambda)| : \lambda \in \textstyle\bigcup_{m \in N}(\Sigma \setminus F_m)\right\} \\ &= \sup_m \ \sup\{|\phi(\lambda)| : \lambda \in \Sigma \setminus F_m\} \\ &= \sup_m \|\phi\|_{L^\infty(\mu_m)} .\end{aligned}$$

(2) If $\phi \in L^\infty(\mu)$, then

$$\begin{aligned}\|\tilde{\pi}(\phi)\| &= \sup_{m\in N} \|\widetilde{\pi_{\mu_m}}(\phi)\| \\ &= \sup_{m\in N} \|\phi\|_{L^\infty(\mu_m)} \\ &= \|\phi\|_{L^\infty(\mu)} \qquad \text{by part (1) of this Theorem}\end{aligned}$$

so $\tilde{\pi}$ is indeed an isometry.

Suppose $\sup_{n\in\mathbb{N}} \|\phi_n\|_{L^\infty(\mu)} \leq C < \infty$.

$(a) \Rightarrow (b)$: This follows immediately from $\mu_m \leq \epsilon_m^{-1}\mu$.

$(b) \Rightarrow (a)$: Let $\delta, \epsilon > 0$. We assume, for this proof, that the index set N is the whole of $\mathbb{N}$; the case of finite N is trivially proved. First choose $N' \in N$ such that $\sum_{m=N'+1}^\infty \epsilon_m < \epsilon/2$. Then choose an n_0 so large that $n \geq n_0 \Rightarrow \mu_m(\{|\phi_n| > \delta\}) < \epsilon/2N'\epsilon_m$; and conclude that for an $n \geq n_0$, we have

$$\begin{aligned}\mu(\{|\phi_n| > \delta\}) &\leq \sum_{m=1}^{N'} \epsilon_m \mu_m(\{|\phi_n| > \delta\}) + \sum_{m=N'+1}^{\infty} \epsilon_m \\ &< \sum_{m=1}^{N'} \epsilon_m \frac{\epsilon}{2N'\epsilon_m} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

$(b) \Rightarrow (c)$: As $\|\tilde{\pi}(\phi_n)\| \leq C \; \forall n$, using Lemma 2.2.5 it is enough to prove that $\lim_{n\to\infty} \tilde{\pi}(\phi_n)x = 0$ whenever $x = ((x_m)) \in \bigoplus_{m=1}^\infty L^2(\mu_m)$ is such that $x_m = 0 \; \forall \; m \neq k$ for some one k. By Lemma 2.2.7, the condition (b) is seen to imply that $\|\widetilde{\pi_{\mu_k}}(\phi_n)x_k\| \to 0$; but $\|\tilde{\pi}(\phi_n)x\| = \|\widetilde{\pi_{\mu_k}}(\phi_n)x_k\|$ and we are done.

$(c) \Rightarrow (b)$: If condition (c) holds, it can be seen by restricting to the subspace $L^2(\mu_m)$ that $\widetilde{\pi_{\mu_m}}(\phi_n) \stackrel{SOT}{\longrightarrow} 0$, and it now follows by applying Lemma 2.2.7 that the sequence $\phi_n \to 0$ in measure with respect to μ_m for each $m \in \mathbb{N}$.

□

2.3 Spectral Theorem for self-adjoint operators

Throughout this section, we shall assume that $X \in B(\mathcal{H})$ is a self-adjoint operator and that $\Sigma = \sigma(X)$. In the interest of minimising parentheses, we shall simply write $C^*(X)$ rather than $C^*(\{X\})$ for the (unital) C^*-algebra generated by X. As advertised in the preface, we shall prove the following formulation of what we would like to think of as the spectral theorem.

THEOREM 2.3.1. *[Spectral theorem for self-adjoint operators]*

(1) **(Continuous Functional Calculus)** *There exists a unique isometric *-algebra isomorphism*

$$C(\Sigma) \ni f \mapsto f(X) \in C^*(X)$$

of $C(\Sigma)$ *onto* $C^*(X)$ *such that* $f_0(X) = X$.[1]

(2) **(Measurable Functional Calculus)** *There exists a measure* μ *defined on* $\mathcal{B}_\Sigma$ *and a unique isometric *-algebra homomorphism*

$$L^\infty(\Sigma, \mu) \ni f \mapsto f(X) \in B(\mathcal{H})$$

of $L^\infty(\Sigma, \mu)$ *into* $B(\mathcal{H})$ *such that (i)* $f_0(X) = X$, *and (ii) a norm-bounded sequence* $\{f_n : n \in \mathbb{N}\}$ *in* $L^\infty(\Sigma, \mu)$ *converges in measure w.r.t.* μ *(to* f*, say) if and only if the sequence* $\{f_n(X) : n \in \mathbb{N}\}$ *SOT converges (to* $f(X)$*).*

Proof. (1) It follows from Proposition 2.1.5 that there exists a unique representation $\pi : C(\Sigma) \to B(\mathcal{H})$ such that $\pi(f_0) = X$. As for the 'isometry' assertion, observe that for any $p \in \mathbb{C}[t]$, the spectral mapping theorem ensures that

$$\|\pi(p)\|_{B(\mathcal{H})} = \operatorname{spr}(p(X)) = \|p\|_\Sigma = \|p\|_{C(\Sigma)}$$

and the Weierstrass approximation theorem now guarantees that

$$\|\pi(f)\|_{B(\mathcal{H})} = \|f\|_{C(\Sigma)} \; \forall f \in C(\Sigma)$$

as desired.

(2) As $f \mapsto f(X)$ is a representation, say π, of $C(\Sigma)$, if μ and $\tilde{\pi}$ are as in Theorem 2.2.8 (2), the equation $\tilde{\pi}(\phi) = \phi(X)$ defines a measurable functional calculus with the desired properties. Thanks to Lemma A2 in the Appendix, there can be at most one isometric (unital) *-homomorphism of $L^\infty(\Sigma, \mu)$ into $B(\mathcal{H})$, i.e., a measurable functional calculus, which (i) extends the continuous functional calculus π and (ii) maps uniformly bounded sequences converging in measure (w.r.t. μ) to SOT convergent sequences.' So we see that there indeed exists a unique *-homomorphism from $L^\infty(\Sigma, \mu)$ into $B(\mathcal{H})$ with the desired property.

□

COROLLARY 2.3.2. *If* $\mu_i, i = 1, 2$ *are two probability measures satisfying the conditions imposed on* μ *in Theorem* 2.3.1, *then* μ_1 *and* μ_2 *are mutually absolutely continuous. In particular, the Banach algebra* $L^\infty(\Sigma, \mu)$ *featuring in Theorem* 2.3.1 (2) *is uniquely determined by the operator* X, *even if* μ *itself is not.*

[1]Recall that $\Sigma \subset \mathbb{R}$ – see Corollary 1.6.3 – and that f_0 denotes the function $f_0 : \Sigma \to \mathbb{R}$ defined by $f_0(t) = t$.

Proof. Suppose $\pi_i : L^\infty(\Sigma, \mu_i) \to B(\mathcal{H}), i = 1, 2$ are isometric *-isomorphisms which (*i*) extend the continuous functional calculus (call it $\pi : C(\Sigma) \to C^*(\{X\})$), and (*ii*) satisfy the convergence in measure – sequential SOT convergence homeomorphism property as in part (2) of Theorem 2.3.1. Define $\nu = (\mu_1 + \mu_2)/2$. Then convergence in measure w.r.t ν implies convergence in measure w.r.t. μ_i for $1 = 1, 2$ since $\mu_i \leq 2\nu$.

Suppose $\mu_1(E) = 0$ for some $E \in \mathcal{B}_{\mathbb{C}}$.

Then appeal to Lemma A2 of the Appendix to find a sequence $\{f_n : n \in \mathbb{N}\} \subset C(\Sigma)$ such that $\|f_n\| \leq 1$ ν-a.e. and such that $f_n \to 1_E$ in measure w.r.t. ν. Then also $f_n \to 1_E$ in measure w.r.t. $\mu_i, 1 = 1, 2$. Then the assumptions imply that

$$\begin{aligned} \pi_2(1_E) &= SOT - \lim_n \pi_2(f_n) \\ &= SOT - \lim_n \pi_1(f_n) \\ &= \pi_1(1_E) \\ &= 0 \end{aligned}$$

and hence $\mu_2(E) = 0$. By toggling the roles of 1 and 2, we find that μ_1 and μ_2 are mutually absolutely continuous, thereby proving the corollary.

The last assertion is an off-shoot of the statement that the 'identity map' is an isometric isomorphism between L^∞ spaces of mutually absolutely continuous probability measures. □

REMARK 2.3.3. (1) Our proof of the spectral theorem, for self-adjoint operators, actually shows that if Σ is a compact metric space and $\pi : C(\Sigma) \to B(\mathcal{H})$ is a representation, i.e., a unital *-homomorphism, *on a separable Hilbert space*, there exists a probability measure μ defined on $\mathcal{B}_\Sigma$ – which is unique up to mutual absolute continuity – and a representation $\tilde{\pi} : L^\infty(\mu) \to B(\mathcal{H})$ which is uniquely determined by (*i*) $\tilde{\pi}$ 'extends' π, and (*ii*) a norm-bounded sequence $\{f_n : n \in \mathbb{N}\} \subset L^\infty(\mu)$ converges to 0 in (μ) measure if and only if $\tilde{\pi}(f_n)$ SOT-converges to 0.

(2) Further, if π is isometric, so is $\tilde{\pi}$ and in particular, if U is a non-empty open set in Σ, then $\mu(U) \neq 0$, or equivalently $\tilde{\pi}(1_U) \neq 0$.

(3) All this is part of the celebrated **Hahn-Hellinger theorem** which says: the representation π is determined up to unitary equivalence by the measure class (w.r.t. mutual absolute continuity) of μ and a measurable **spectral multiplicity function** $m : \Sigma \to (\{0\} \cup \bar{\mathbb{N}}) := \{0, 1, 2, \cdots, \aleph_0\}$, which is determined uniquely up to sets of μ measure zero; in fact if $E_n = m^{-1}(n)$, $n \in \{0\} \cup \bar{\mathbb{N}}$, then π is unitarily equivalent to the representation on $\bigoplus_{n \in \bar{\mathbb{N}}} L^2(E_n, \mu|_{E_n}) \otimes \mathcal{H}_n$ given by $\bigoplus_{n \in \bar{\mathbb{N}}} \pi_{\mu|_{E_n}} \otimes \mathrm{id}_{\mathcal{H}_n}$, where $\mathcal{H}_n$ is some (multiplicity) Hilbert space of dimension n.

2.4 The spectral subspace for an interval

This section is devoted to a pretty and useful characterisation, from [Hal], of the spectral subspace for the unit interval. We first list some simple facts concerning spectral subspaces (= ranges of spectral projections). We use the following notation below: for a self-adjoint operator X, let $\mathcal{M}_X(E) = \text{ran}\, 1_E(X)$. We also use the as yet undefined notions (but only the definition) of order and positivity – see Proposition 2.8.12, especially part (b) and the final paragraph in it – in the following Proposition.

PROPOSITION 2.4.1. *Let $X \in B(\mathcal{H})$ be self-adjoint. Then,*

(1) $a\|x\|^2 \leq \langle Xx, x\rangle \leq b\|x\|^2 \ \forall x \in \mathcal{M}_X([a,b])$;

(2) $X1_{[0,\infty)}(X) \geq 0$;

(3) $\epsilon > 0, x \in \mathcal{M}_X(\mathbb{R} \setminus (t_0 - \epsilon, t_0 + \epsilon)) \Rightarrow \|(X - t_0)x\| \geq \epsilon\|x\|$;

(4) $t_0 \in \sigma(X) \Leftrightarrow \mathcal{M}_X((t_0 - \epsilon, t_0 + \epsilon)) \neq \{0\} \ \forall \epsilon > 0$; *and*

(5) $\mathcal{M}_X(\{t_0\}) = \ker(X - t_0)$.

Proof. (1) Notice first that *-homomomorphisms of C^*-algebras are *order-preserving* since

$$\begin{aligned} x \leq y \quad &\Rightarrow \quad y - x \geq 0 \ \ (i.e., \exists z \text{ such that } y - x = z^*z) \\ &\Rightarrow \quad \pi(y) - \pi(x) = \pi(y - x) = \pi(z)^*\pi(z) \geq 0 \\ &\Rightarrow \quad \pi(x) \leq \pi(y) \ . \end{aligned}$$

Hence

$$a1_{[a,b]}(t) \leq t1_{[a,b]}(t) \leq b1_{[a,b]}(t) \Rightarrow a1_{[a,b]}(X) \leq X1_{[a,b]}(X) \leq b1_{[a,b]}(X)$$

and the desired result follows from the fact that $1_{[a,b]}(X)x = x \ \forall x \in \mathcal{M}_X([a,b])$.

(2) This follows from (1) since $1_{[0,\infty)}(X) = 1_{[0,\|X\|]}(X)$.

(3) It follows from (1) that if $x \in \mathcal{M}_X(\mathbb{R} \setminus (t_0 - \epsilon, t_0 + \epsilon) = \mathcal{M}_{(X-t_0)^2}([\epsilon^2, \infty))$ (by the spectral mapping theorem), then $\epsilon^2\|x\|^2 \leq \langle (X - t_0)^2 x, x\rangle = \|(X - t_0)x\|^2$.

(4) If μ is as in Theorem 2.3.1 (2), observe that

$$\begin{aligned} t_0 \notin \sigma(X) \quad &\Leftrightarrow \quad (X - t_0) \in GL(\mathcal{H}) \\ &\Leftrightarrow \quad (f_0 - t_0) \text{ is invertible in } L^\infty(\sigma(X), \mu) \\ &\Leftrightarrow \quad \exists \epsilon > 0 \text{ such that } |f_0 - t_0| \geq \epsilon \ \mu - a.e. \\ &\Leftrightarrow \quad \exists \epsilon > 0 \text{ such that } \mu((t_0 - \epsilon, t_0 + \epsilon)) = 0 \\ &\Leftrightarrow \quad \exists \epsilon > 0 \text{ such that } \mathcal{M}_X((t_0 - \epsilon, t_0 + \epsilon)) = \{0\} \ . \end{aligned}$$

(5) Clearly X commutes with $1_E(X)$ $\forall X$ and hence the subspace $\mathcal{M}_X(E)$ is invariant under X for all E, X. It follows from (1) above that $\langle X_0 x, x\rangle = t_0\|x\|^2$ $\forall x \in \mathcal{M}_X\{t_0\}$ where $X_0 = X|_{\mathcal{M}_X(\{t_0\})}$ and hence $\ker(X - t_0) \supset \mathcal{M}_X(\{t_0\})$. Conversely, if $x \in$ ker $(X - t_0)$, then for any $\epsilon > 0$, we have $(X - t_0)1_{\mathbb{R}\setminus(t_0-\epsilon,t_0+\epsilon)}(X)(x) = 1_{\mathbb{R}\setminus(t_0-\epsilon,t_0+\epsilon)}(X)(X - t_0)(x) = 0$, whence $1_{\mathbb{R}\setminus(t_0-\epsilon,t_0+\epsilon)}(X)x = 0$ by (3) above. So $x \in M_X(t_0 - \epsilon, t_0 + \epsilon)$; since ϵ was arbitrary, we have $x \in \bigcap_{\epsilon>0} M_X(t_0 - \epsilon, t_0 + \epsilon) = M_X(\{t_0\})$, so indeed ker $(X - t_0) = M_X(\{t_0\})$.

□

Now we come to the much advertised pretty description by Halmos of $\mathcal{M}_X([-1,1])$.

PROPOSITION 2.4.2. *Let $X = X^*$ be as above, and let $x \in \mathcal{H}$. The following conditions are equivalent:*

(1) $x \in \mathcal{M}_X([-1,1])$.

(2) $\|X^n x\| \leq \|x\|$ $\forall n \in \mathbb{N}$.

(3) $\{\|X^n x\| : n \in \mathbb{N}\}$ *is a bounded set.*

Proof. (1) ⇒ (2): The operator X leaves the subspace $\mathcal{M}_X([-1,1])$ invariant, and its restriction X_1 to this spectral subspace satisfies $-1 \leq X_1 \leq 1$ (by Proposition 2.4.1(1) and hence $\|X_1\| = \text{spr}(X_1) \leq 1$ whence also $\|X_1^n\| \leq 1$, as desired.

(2) ⇒ (3) is obvious.

(3) ⇒ (1): If we let $x_1 = 1_{[-1,1]}(X)x$, we need to show that $x = x_1$; for this, note that

$$\begin{aligned} x - x_1 &= (1 - 1_{[-1,1]}(X))x \\ &= 1_{\mathbb{R}\setminus[-1,1]}(X)x \\ &= \lim_{n\to\infty} 1_{\mathbb{R}\setminus I_n}(X)x \end{aligned}$$

where we write the symbol I_n to denote the interval $(-1 - \frac{1}{n}, 1 + \frac{1}{n})$; so it suffices to show that $1_{\mathbb{R}\setminus I_n}(X)x = 0$ $\forall n$. Indeed, if there exists some n such that $y_n = 1_{\mathbb{R}\setminus I_n}(X)x \neq 0$, it would follow from Proposition 2.4.1 (3) that $\|Xy_n\| \geq (1+\frac{1}{n})\|y_n\|$ and that hence $\|X^m x\| \geq \|1_{\mathbb{R}\setminus I_n}(X)X^m x\| = \|X^m y_n\| \geq (1 + \frac{1}{n})^m\|y_n\|$. So the sequence $\{\|X^m x\| : m \in \mathbb{N}\}$ is not a bounded set if any $y_n \neq 0$. □

COROLLARY 2.4.3. *(1) $x \in \mathcal{M}_X([t_0 - \epsilon, t_0 + \epsilon]) \Leftrightarrow \{(\frac{X-t_0}{\epsilon})^n x : n \in \mathbb{N}\}$ is bounded.*

(2) If a $Y \in B(\mathcal{H})$ commutes with X, ie., $YX = XY$, then Y leaves $\mathcal{M}_X(I)$ invariant for every bounded interval I.

Proof. (1) This follows by applying Proposition 2.4.2 to $(X - t_0)/\epsilon$ rather than to the operator X.

(2) If $YX = XY$ and if I is a compact interval (which can always be written in the form $[t_0 - \epsilon, t_0 + \epsilon]$), it follows from (1) above that

$$\begin{aligned} x \in \mathcal{M}_X([t_0 - \epsilon, t_0 + \epsilon]) &\Rightarrow \left\{ \left(\frac{X - t_0}{\epsilon}\right)^n x : n \in \mathbb{N} \right\} \text{ is bounded} \\ &\Rightarrow \left\{ Y \left(\frac{X - t_0}{\epsilon}\right)^n x : n \in \mathbb{N} \right\} \text{ is bounded} \\ &\Rightarrow \left\{ \left(\frac{X - t_0}{\epsilon}\right)^n Yx : n \in \mathbb{N} \right\} \text{ is bounded} \\ &\Rightarrow Yx \in \mathcal{M}_X([t_0 - \epsilon, t_0 + \epsilon]), \end{aligned}$$

so Y leaves the spectral subspaces corresponding to compact intervals invariant.

If I is an open interval, there exist an increasing sequence $\{I_n : n \in \mathbb{N}\}$ of compact intervals such that $I = \bigcup_{n \in \mathbb{N}} I_n$. But then $1_I(X) = SOT - \lim_{n\to\infty} 1_{I_n}(X)$ and $\mathcal{M}_X(I) = \overline{(\bigcup_n \mathcal{M}_X(I_n))}$. The previous paragraph shows that Y leaves each $\mathcal{M}_X(I_n)$, and hence also $\mathcal{M}(I)$, invariant.

Similar approximation arguments can be conjured up if I is of the form $[a, b)$ or $(a, b]$. (For example, $[a, b - \frac{1}{n}] \uparrow [a, b)$ and $[a + \frac{1}{n}, b] \uparrow (a, b]$.) □

2.5 Finitely many commuting self-adjoint operators

We assume in the rest of this chapter that $X_1, \ldots, X_n, \ldots$ are commuting self-adjoint operators on $\mathcal{H}$.

DEFINITION 2.5.1. *Consider the set* $\mathbf{\Sigma_k} = \Sigma(X_1, \ldots, X_k)$ *consisting of those* $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ *for which there exists a sequence* $\{x_n : n \in \mathbb{N}\}$ *of unit vectors in* $\mathcal{H}$ *such that* $\lim_{n\to\infty} \|(X_i - \lambda_i)x_n\| = 0$ *for* $1 \leq i \leq k$. *Thus* $\mathbf{\Sigma_k}$ *consists of k-tuples of scalars which admit a sequence of 'simultaneous approximate eigenvectors' of the* X_i*'s, and will be referred to simply as the* **joint spectrum of** $X_1, \ldots, X_k$.

If $(\lambda_1, \ldots, \lambda_k) \in \mathbf{\Sigma_k}$, it is clear that $\lambda_i \in \sigma(X_i)$ for $1 \leq i \leq k$, and in particular $\mathbf{\Sigma_k} \subset \prod_{i=1}^k \sigma(X_i)$ and is hence bounded.

LEMMA 2.5.2. *(1)* $\mathbf{\Sigma_k}$ *is a compact set for* $k > 0$*; and*

(2) If $k > 0$ *then* $\mathrm{pr}_k(\mathbf{\Sigma_k}) = \Sigma(X_k)$, *where* $\mathrm{pr}_k : \mathbb{R}^k \to \mathbb{R}$ *denotes the projection onto the k-th coordinate; in particular,* $\mathbf{\Sigma_k} \neq \emptyset$.

Proof. (1) We have already seen above that $\mathbf{\Sigma_k}$ is bounded, so we only need to prove that it is closed. So suppose $(\lambda_1^{(n)}, \ldots, \lambda_k^{(n)}) \in \mathbf{\Sigma_k}$ for each $n \in \mathbb{N}$ and $\lambda_j^{(n)} \to \lambda_j$ for each $1 \leq j \leq k$. Pick any $\epsilon > 0$. Then $(\lambda_1^{(n)}, \ldots, \lambda_k^{(n)}) \in$

$\mathbf{\Sigma_k} \Rightarrow \exists x \in S(\mathcal{H})$ such that $\|(X_j - \lambda_j^{(n)})x\| < \epsilon/2$ for $1 \leq j \leq k$ (and for all n). Next, $\lambda_j^{(n)} \to \lambda_j \Rightarrow \exists n$ such that $|\lambda_j^{(n)} - \lambda_j| < \epsilon/2$. Thus, for any $\epsilon > 0$, we have shown that $\exists x \in S(\mathcal{H})$ such that

$$\|(X_j - \lambda_j)x\| \leq \|(X_j - \lambda_j^{(n)})x\| + |\lambda_j^{(n)} - \lambda_j| < \epsilon \text{ for } 1 \leq j \leq k$$

and indeed $(\lambda_1, \ldots, \lambda_k) \in \mathbf{\Sigma_k}$ and $\mathbf{\Sigma_k}$ is closed.

(2) We shall prove the result by induction on k. For $k = 1$, assertion (2) follows from Theorem 1.6.2 (2) and the non-emptiness of $\sigma(X_1)$.

Suppose now that the Theorem is valid for k, and suppose we are given commuting self-adjoint operators $X_1, \ldots, X_k, X_{k+1}$. Let us prove that $\lambda_{k+1} \in \sigma(X_{k+1})$ implies that there exists $(\lambda_1, \ldots, \lambda_k) \in \Sigma(X_1, \ldots, X_k)$ such that $(\lambda_1, \ldots, \lambda_k, \lambda_{k+1}) \in \Sigma(X_1, \cdots, X_k, X_{k+1})$.

For each $n \in \mathbb{N}$, let $\mathcal{M}_n = \mathcal{M}_{X_{k+1}}(\lambda_{k+1} - \frac{1}{n}, \lambda_{k+1} + \frac{1}{n})$, where we continue to use the notation $\mathcal{M}_X(E) := 1_E(X)$ of the last section. By Proposition 2.4.1 (4), we see that $\mathcal{M}_n \neq \{0\}\ \forall n$. By Corollary 2.4.3 (2), each X_i leaves $\mathcal{M}_n$ invariant. Define $X_i(n) = X_i|_{\mathcal{M}_n}\ \forall 1 \leq i \leq k, n \in \mathbb{N}$. Deduce by induction hypothesis that $\mathbf{\Sigma_k}(n) := \mathbf{\Sigma_k}(X_1(n), \ldots, X_k(n)) \neq \emptyset\ \forall n$. Since $\{\mathcal{M}_n : n \in \mathbb{N}\}$ is a decreasing sequence of subspaces, it is clear that also $\{\mathbf{\Sigma_k}(n) : n \in \mathbb{N}\}$ is a decreasing sequence of non-empty compact sets. The finite intersection property then assures us that we can find a $(\lambda_1, \ldots, \lambda_k)$ in the non-empty set $\bigcap_{n \in \mathbb{N}} \mathbf{\Sigma_k}(n)$. Hence, by definition of the joint spectrum of commuting self-adjoint operators, we can find unit vectors $x_n \in \mathcal{M}_n$ such that $\|(X_i - \lambda_i)x_n\| = \|(X_i(n) - \lambda_i)x_n\| < \frac{1}{n}$ for $1 \leq i \leq k$, and $n \in \mathbb{N}$. On the other hand, it follows from the definition of $\mathcal{M}_n$ that $\|(X_{k+1} - \lambda_{k+1})x_n\| < \frac{1}{n}$. Thus, $\|(X_i - \lambda_i)x_n\| < \frac{1}{n}\ \forall 1 \leq i \leq k+1$ for every $n \in \mathbb{N}$; in other words, $(\lambda_1, \ldots, \lambda_k, \lambda_{k+1}) \in \Sigma(X_1, \ldots, X_k, X_{k+1})$. Since $\Sigma(X_{k+1}) = \sigma(X_{k+1}) \neq \emptyset$ the proof is complete.

□

PROPOSITION 2.5.3. *For any polynomial $p \in \mathbb{C}[t_1, \ldots, t_k]$, the operator $Z = p(X_1, \ldots, X_k)$ is normal, and*

(1) $\sigma(Z) = p(\mathbf{\Sigma_k})$; and

(2) $\|p(X_1, \ldots, X_k)\| = \|p\|_{\mathbf{\Sigma_k}}$, where the p on the right is the evaluation function on $\mathbf{\Sigma_k}$ given by the polynomial p.

Proof. (1) Let $q = \frac{1}{2}(p + \bar{p}), r = \frac{1}{2i}(p - \bar{p})$ and $X_{k+1} = q(X_1, \ldots, X_k), Y_{k+1} = r(X_1, \ldots, X_k)$. Then clearly $q, r \in \mathbb{R}[t_1, \ldots, t_k]$, so that X_{k+1} and Y_{k+1} are self-adjoint operators commuting with $X_1, \ldots, X_k$ and with each other as well (so Z is indeed normal). Since it follows from Corollary 1.6.4 that $\lambda = \alpha + i\beta \in \sigma(Z) \Leftrightarrow \alpha \in \sigma(X_{k+1})$ and $\beta \in \sigma(Y_{k+1})$, we see that it suffices to prove the case when $p = q$ is real-valued and $Z = X_{k+1}$ is a self-adjoint operator which is a real polynomial in $X_1, \ldots, X_k$ (and hence commutes with each X_i).

Suppose $\lambda_{k+1} \in \sigma(X_{k+1})$. It then follows from Lemma 2.5.2 that there exists $(\lambda_1, \ldots, \lambda_k) \in \mathbf{\Sigma_k}$ such that $(\lambda_1, \ldots, \lambda_k, \lambda_{k+1}) \in \Sigma(X_1, \ldots, X_k, X_{k+1})$. Thus there exists a sequence $\{x_n : n \in \mathbb{N}\}$ of unit vectors in $\mathcal{H}$ such that $\|(X_i - \lambda_i)x_n\| \to 0 \; \forall 1 \leq i \leq k+1$. It follows easily from this requirement for the first k i's that then, necessarily, we must have $\|[p(X_1, \cdots, X_k) - p(\lambda_1, \cdots, \lambda_k)]x_n\| \to 0$ while also $\|(X_{k+1} - \lambda_{k+1})x_n\| \to 0$, which forces $\lambda_{k+1} = p(\lambda_1, \cdots, \lambda_k)$; in view of the arbitrariness of λ_{k+1}, this shows that $\sigma(X_{k+1}) \subset p(\mathbf{\Sigma_k})$. Conversely, it must be clear that if $(\lambda_1, \ldots, \lambda_k) \in \mathbf{\Sigma_k}$, then $p((\lambda_1, \ldots, \lambda_k))$ is an approximate eigenvalue of $p(X_1, \cdots, X_k)$ and thus, indeed, $\sigma(p(X_1, \cdots, X_k)) = p(\Sigma(X_1, \cdots, X_k))$.

(2) This follows immediately from (1) above and Proposition 1.5.6 (2).

□

COROLLARY 2.5.4. *With the notation of Proposition 2.5.3, we have:*

*(1) The 'polynomial functional calculus' extends uniquely to a isometric *-algebra isomorphism*

$$C(\Sigma) \ni f \stackrel{\pi}{\mapsto} f(X_1, \cdots, X_k) \in C^*(\{X_1, \ldots, X_k\}) \; ;$$

*(2) There exists a probability measure μ on $\mathcal{B}_\Sigma$ and an isometric *-algebra monomorphism $\tilde{\pi} : L^\infty(\mu) \to B(\mathcal{H})$ such that (i) $\tilde{\pi}$ 'extends' π, and (ii) a norm-bounded sequence $\{f_n : n \in \mathbb{N}\}$ in $L^\infty(\mu)$ converges to the constant function 0 in (μ) measure if and only if $\tilde{\pi}(f_n)$ SOT-converges to 0.*

Proof. (1) This follows from Proposition 2.5.3(2) and a routine application of the Stone-Weierstrass theorem, to show that the collection of complex polynomial functions on a compact subset $\mathbf{\Sigma}$ of $\mathbb{R}^k$, by virtue of being a self-adjoint unital subalgebra of functions which separates points of $\mathbf{\Sigma}$, is dense in $C(\mathbf{\Sigma})$.

(2) This is a consequence of item (1) above and Remark 2.3.3.

□

2.6 The Spectral Theorem for a normal operator

We are now ready to generalise Theorem 2.3.1 to the case of a normal operator. This is essentially just the specialisation of Corollary 2.5.4 for $k = 2$.

Thus, assume that $Z = X + iY \in B(\mathcal{H})$ is the Cartesian decomposition of a normal operator and that $\Sigma = \sigma(Z)$. In view of Proposition 2.5.3 (1), we see that $\Sigma = \{s + it : (s,t) \in \Sigma(X,Y)\}$, and we may and will identify $\Sigma \subset \mathbb{C}$ with $\Sigma(X,Y) \subset \mathbb{R}^2$.

In the following formulation of the spectral theorem for the normal operator Z (as above), the functions $f_i, i = 1, 2$ denote the functions $f_i : \Sigma \to \mathbb{R}$ defined by $f_1(z) = \text{Re } z, f_2(z) = \text{Im } z$. We omit the proof as it is just Corollary 2.5.4 for $k = 2$.

THEOREM 2.6.1. *(1)* **(Continuous Functional Calculus)** *There exists a unique isometric *-algebra isomorphism*

$$C(\Sigma) \ni f \mapsto f(Z) \in C^*(Z)$$

of $C(\Sigma)$ onto $C^*(Z)$ such that $f_1(Z) = X, f_2(Z) = Y$.

(2) **(Measurable Functional Calculus)** *There exists a measure μ defined on $\mathcal{B}_\Sigma$ and a unique isometric *-algebra homomorphism*

$$L^\infty(\Sigma, \mu) \ni f \mapsto f(Z) \in B(\mathcal{H})$$

of $L^\infty(\Sigma, \mu)$ into $B(\mathcal{H})$ such that (i) $f_1(Z) = X, f_2(Z) = Y$, and (ii) a norm-bounded sequence $\{f_n : n \in \mathbb{N}\}$ in $L^\infty(\Sigma, \mu)$ converges in (μ)-measure to f if and only if the sequence $\{f_n(Z) : n \in \mathbb{N}\}$ SOT-converges to $f(Z)$.

Now we proceed to the conventional formulation of the spectral theorem in terms of spectral or projection-valued measures $P : \mathcal{B}_\mathbb{C} \to B(\mathcal{H})$.

THEOREM 2.6.2. *Let N be a normal operator on a separable Hilbert space $\mathcal{H}$. Then there exists a unique mapping $P := P_N : \mathcal{B}_\mathbb{C} \to B(\mathcal{H})$ such that:*

(1) $P(E)$ is an orthogonal projection for all $E \in \mathcal{B}_\mathbb{C}$;

(2) $E \mapsto P(E)$ is a projection-valued measure; i.e., whenever $\{E_n : n \in \mathbb{N}\} \subset \mathcal{B}_\mathbb{C}$ is a sequence of pairwise disjoint Borel sets, and $E = \coprod_{n\in\mathbb{N}} E_n$, then $P(E) = \sum_{n\in\mathbb{N}} P(E_n)$, the series being interpreted as the SOT-limit of the sequence of partial sums;

(3) for $x \in \mathcal{H}$, the equation $P_{x,x}(E) = \langle P(E)x, x\rangle$ defines a finite positive scalar measure with $P_{x,x}(\mathbb{C}) = \|x\|^2$;

(4) for $x, y \in \mathcal{H}$, the equation $P_{x,y}(E) = \langle P(E)x, y\rangle$ defines a finite complex measure, with the property that

$$\langle Nx, y\rangle = \int_\mathbb{C} \lambda \, dP_{x,y}(\lambda) \; ; \tag{2.6.1}$$

more generally for any bounded measurable function $f : \mathbb{C} \to \mathbb{C}$, we have

$$\langle f(N)x, y\rangle = \int_\mathbb{C} f(\lambda) \, dP_{x,y}(\lambda) \; ; \tag{2.6.2}$$

(5) the **spectral measure** *P is 'supported' on the spectrum of N in the sense that $P(U) \neq 0$ for all open sets U that have non-empty intersection with $\Sigma := \sigma(N)$ – or equivalently Σ is the smallest closed set with $P(\Sigma) = 1$.*

Proof. Existence: Use the measurable functional calculus to define $P(E) = 1_N(E)$. As $1_E = \overline{1_E} = 1_E^2$, we see immediately that $P(E) = P(E)^* = P(E)^2$, and hence (1) is proved. As for (2), note that the pairwise disjointness assumption ensures that $1_{\coprod_{k=1}^n E_k} = \sum_{k=1}^n 1_{E_k}$, while $\coprod_{k=1}^n E_k \uparrow \coprod_{k\in\mathbb{N}} E_k$ implies $P(\coprod_{k=1}^\infty E_k) = SOT\text{-}\lim_{n\to\infty} P(\coprod_{k=1}^n E_k)$, thus establishing (2).

Since $\langle Qx, x\rangle = \|Qx\|^2 \geq 0$ for any projection Q, item (3) follows immediately from item (2). The polarisation identity and the definitions show that $P_{x,y} = \frac{1}{4}\sum_{j=0}^3 i^j P_{x+i^j y, x+i^j y}$, thereby demonstrating that $P_{x,y}$ is a complex linear combination of four finite positive measures, and is hence a finite complex measure. To complete the proof of item (4), it suffices to prove equation (2.6.2) since equation (2.6.1) is a special case (with $f(z) = 1_\Sigma(z)z$). Equation (2.6.2) is, by definition, valid when f is of the form 1_E, and hence by linearity, also valid for any simple function. For a general bounded measurable function f, and an $\epsilon > 0$, choose a simple function s such that $\|s - f\| < \epsilon$ uniformly. Then,

$$|\langle f(N)x, y\rangle - \langle s(N)x, y\rangle| \leq \epsilon\|x\|\ \|y\|$$

and

$$\left|\int f\,dP_{x,y} - \int s\,dP_{x,y}\right| \leq \epsilon\|P_{x,y}\|$$

so

$$\left|\langle f(N)x, y\rangle - \int f\,dP_{x,y}\right| \leq \epsilon(\|x\|\ \|y\| + \|P_{x,y}\|)\ .$$

As ϵ was arbitrary, we find that equation (2.6.2) indeed holds for any bounded measurable f.

As for (5), suppose $P(U) = 0$ for some open U, and $z_0 \in U$. Pick $\epsilon > 0$ such that $D = \{z \in \mathbb{C} : |z - z_0| < \epsilon\} \subset U$. Then $P(U) = 0 \Rightarrow P(D) = 0 \Rightarrow \|1_D\|_{L^\infty(\mu)} = 0 \Rightarrow \mu(D) = 0 \Rightarrow \frac{1}{f_0 - z_0} \in L^\infty(\mu) \Rightarrow z_0 \notin \sigma(N)$, so, indeed $P(U) = 0, U$ open $\Rightarrow U \cap \Sigma = \emptyset$.

Uniqueness: If, conversely $\tilde{P}$ is another such spectral measure satisfying the conditions (1)–(5) of the theorem, it follows from equation (2.6.2) that

$$\int z^m \bar{z}^n\, d\tilde{P}_{x,y}(z) = \langle N^m N^{*n} x, y\rangle = \int z^m \bar{z}^n\, dP_{x,y}(z)\ \forall m, n \in \mathbb{Z}_+\ .$$

Since functions of the form $z \mapsto z^m \bar{z}^n$ span a dense subspace of $C(\Sigma)$, thanks to the Stone-Weierstrass theorem, it now follows from the Riesz representation theorem that $\tilde{P}_{x,y} = P_{x,y}$. The validity of this equality for all $x, y \in \mathcal{H}$ shows, finally, that indeed $\tilde{P} = P$, as desired. □

REMARK 2.6.3. Now that we have the uniqueness assertion of Theorem 2.6.2, we can re-connect with a way to produce probability measures in the measure class of the mysterious μ appearing in the measurable functional calculus. If P denotes **the** spectral measure of X, the following conditions on an $E \in \mathcal{B}_\Sigma$ are equivalent:

(1) $1_E(X)(= P(E)) = 0$.

(2) $\mu(E) = 0$.

(3) $P_{x,x}(E) = 0$ for all x in a total set $\mathcal{S} \subset \mathcal{H}$.

Hence, a possible choice for μ is $\sum_{n\in\mathbb{N}} 2^{-n} P_{e_n,e_n}$ where $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}$.

Incidentally, a measure of the form $P_{x,x}$ is sometimes called a **scalar spectral measure** for N.

Reason: (1) $\Leftrightarrow$ (2) This is because $L^\infty(\mu) \ni f \mapsto f(X) \in B(\mathcal{H})$ is isometric by Theorem 2.3.1 (2).

(1) $\Leftrightarrow$ (3) This is because (i) for a projection P – in this case, $P(E)$ – $\langle Px, x\rangle = 0 \Leftrightarrow Px = 0$, and (ii) a bounded operator is the zero operator if and only if its kernel contains a total set.

REMARK 2.6.4. To tie a loose-end, we wish to observe that $\|P_{x,y}\| \leq \|x\| \, \|y\|$. This is because

$$\|P_{x,y}\| = \inf\{K > 0 : \left|\int f\, dP_{x,y}\right| \leq K\|f\|_{C(\Sigma)} \ \forall f \in C(\Sigma)\}$$

and

$$\begin{aligned} \left|\int f\, dP_{x,y}\right| &= |\langle f(N)x, y\rangle| \\ &\leq \|f(N)\| \, \|x\| \, \|y\| \\ &\leq \|f\|_{C(\Sigma)} \, \|x\| \, \|y\|. \end{aligned}$$

REMARK 2.6.5. This final remark is an advertising pitch for my formulation of the spectral theorem in terms of functional calculi, in comparison with the conventional version in terms of spectral measures: the difference is between having some statement for all bounded measurable functions and only having it for indicator functions and having to go through the exercise of integration every time one wants to get to the former situation!

EXERCISE 2.6.6. *Let $\pi_\mu : L^\infty(\mu) \to B(L^2(\mu))$ be the 'multiplication representation' as in Proposition 2.2.3. Can you identify the spectral measure P_N where $N = \pi_\mu(f)$?* (Hint: *Consider the cases $\Sigma = \{z \in \mathbb{C} : |z| = 1\}$ and $f(z) = z^n$ with $n = 1, 2, \ldots$, in increasing order of difficulty as n varies.*)

2.7 Several commuting normal operators

2.7.1 The Fuglede Theorem

THEOREM 2.7.1. *[Fuglede] If an operator T commutes with a normal operator N, then it necessarily also commutes with N^*.*

Proof. When $\mathcal{H}$ is finite-dimensional, the spectral theorem says that N admits the decomposition $N = \sum_{i=1}^{k} \lambda_i P_i$ where $\sigma(N) = \{\lambda_1, \ldots, \lambda_k\}$ and $P_i = 1_{\{\lambda_i\}}(N)$; observe that $P_i = p_i(N)$ for appropriate polynomials $p_1, \ldots, p_k$, and deduce that T commutes with each P_i and hence also with $f(N)$ for any function $f : \sigma(N) \to \mathbb{C}$, and in particular with $N^* = \bar{f}_0$ where $f_0(z) = z$.

We shall similarly prove that T commutes with each spectral projection $1_E(N), E \in \mathcal{B}_{\mathbb{C}}$ and hence also with $f(N)$ for each (simple, and hence each) bounded measurable function f, and in particular, for $f(z) = 1_{\sigma(N)}(z)\bar{z}$. Note that T commutes with a projection P if and only if T leaves both $\mathcal{M}$ and $\mathcal{M}^\perp$ invariant, where $\mathcal{M} = \text{ran}(P)$.

We shall write $\mathcal{M}(E) = \text{ran } 1_E(N)$. Since $\mathcal{M}(E)^\perp = \mathcal{M}(E')$ (where we write $E' = \mathbb{C} \setminus E$), we see from the previous paragraph that Fuglede's theorem is equivalent to the assertion that if T commutes with a normal N, then T leaves each $\mathcal{M}(E)$ invariant – which is what we shall accomplish in a sequence of simple steps:

Define $\mathcal{F} = \{E \in \mathcal{B}_{\mathbb{C}} : T \text{ leaves } \mathcal{M}(E) \text{ invariant}\}$, so we need to prove that $\mathcal{F} = \mathcal{B}_{\mathbb{C}}$.

(1) Write $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ and simply $\mathbb{D} = D(0,1)$, so the closure $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. We shall need the following analogue of Proposition 2.4.2 for normal operators: The following conditions on an $x \in \mathcal{H}$ are equivalent:

   (1) $x \in \mathcal{M}(\bar{\mathbb{D}})$.

   (2) $\|N^n x\| \leq \|x\| \; \forall n \in \mathbb{N}$.

   (3) $\{\|N^n x\| : n \in \mathbb{N}\}$ is a bounded set.

   Reason: $(a) \Rightarrow (b)$:

$$\bar{z}z 1_{\bar{\mathbb{D}}}(z) \leq 1 \Rightarrow N^*N 1_{\bar{\mathbb{D}}}(N) \leq id_{\mathcal{H}} \Rightarrow \|Nx\|^2 \leq 1 \; \forall x \in \mathcal{M}(\bar{\mathbb{D}}) \; .$$

   $(b) \Rightarrow (c)$ is obvious.

   $(c) \Rightarrow (a)$ Let $x_m := 1_{\{z:|z| \geq 1+\frac{1}{m}\}}(N)x \; \forall m \in \mathbb{N}$; then, for all $n \in \mathbb{N}$, we have, by Proposition 2.4.2,

$$\begin{aligned} \|N^{2n}x\| = \|(N^*N)^n x\| &\geq \|1_{\{[(1+\frac{1}{m})^2,\infty)}(N * N)(N^*N)^n x\| \\ &\geq \left(1 + \frac{1}{m}\right)^{2n} \|x_m\| \; ; \end{aligned}$$

and now, the assumed boundedness condition (c) implies that we must have $x_m = 0 \; \forall m$ and hence that $x = x - \lim_{m \to \infty} x_m \in \mathcal{M}(\bar{\mathbb{D}})$; and the proof of the normal analogue of Proposition 2.4.2 is complete.

Since

$$\|N^n Tx\| = \|TN^n x\| \leq \|T\| \|N^n x\|,$$

condition (c) above implies that if $x \in \mathcal{M}(\bar{\mathbb{D}})$, then also $Tx \in \mathcal{M}(\bar{\mathbb{D}})$; so $\bar{\mathbb{D}} \in \mathcal{F}$.

(2) $D(z,r) \in \mathcal{F} \ \forall z \in \mathbb{C}, r > 0$.

Reason: This follows by applying item (1) above to $(\frac{N-z}{r})$.

(3) $\mathcal{F}$ is closed under countable monotone limits, and is hence a 'monotone class'.

Reason: If $E_n \in \mathcal{F} \ \forall n$ and if $E_n \uparrow E$ (resp., $E_n \downarrow E$), then $1_{E_n}(N) \stackrel{SOT}{\to} 1_E(N)$ so that $\mathcal{M}(E) = \overline{(\bigcup \mathcal{M}(E_n))}$ (resp., $\mathcal{M}(E) = \bigcap \mathcal{M}(E_n)$) whence also $E \in \mathcal{F}$.

(4) $\mathcal{F}$ contains all (open or closed) discs.

Reason: The assertion regarding closed discs is item (2) above, and open discs are increasing unions of closed discs.

(5) $\mathcal{F}$ contains all (open or closed) half-planes.

Reason: This is because (i) every open half-plane is an increasing union of closed discs (for example, $R_a := \{z \in \mathbb{C} : \mathrm{Re}\, z > a\} = \bigcup_{n=1}^{\infty} \{z \in \mathbb{C} : |z - (a+n)| \leq n\}$); and (ii) every closed half-plane is a decreasing intersection of open half-planes (eg: $\{\mathrm{Re}\, z > a - \frac{1}{n}\} \downarrow \{\mathrm{Re}\, z \geq a\}$.)

However, we will only need this fact for the special half-planes $R_a, L_b = \{z \in \mathbb{C} : \mathrm{Re}\, z \leq b\}, U_c = \{z \in \mathbb{C} : \mathrm{Im}\, z > c\}, D_d = \{z \in \mathbb{C} : \mathrm{Im}\, z \leq d\}$.

(6) $\mathcal{F}$ is closed under finite intersections and countable disjoint unions.

Reason: $1_{\cap_{i=1}^n E_i} = \prod_{i=1}^n 1_{E_i} \Rightarrow \mathcal{M}(\bigcap_{i=1}^n E_i) = \bigcap_{i=1}^n \mathcal{M}(E_i)$ so if $E_1, \ldots E_n \in \mathcal{F}$, and $x \in \mathcal{M}(\bigcap_{i=1}^n E_i)$, then $x \in \mathcal{M}(E_i) \ \forall i$ and $Tx \in \mathcal{M}(\cap_{i=1}^n E_i)$, so $\bigcap_{i=1}^n E_i \in \mathcal{F}$. Similarly $\mathcal{M}(\coprod_{n=1}^{\infty} E_n) = [\bigcup_{n=1}^{\infty} \mathcal{M}(E_n)]$ implies that $\mathcal{F}$ is closed under countable disjoint unions.

(7) $\mathcal{F} = \mathcal{B}_{\mathbb{C}}$.

Reason: It follows from items (5) and (6) above that $\mathcal{F}$ contains $(a,b] \times (c,d] = R_a \cap L_b \cap U_c \cap D_d$ and the collection $\mathcal{A}$ of all finite disjoint unions of such rectangles. Since $\mathcal{A} \cup \{\emptyset, \mathbb{C}\}$ is an algebra of sets which generates $\mathcal{B}_{\mathbb{C}}$ as a σ-algebra, and since $\mathcal{F}$ is a monotone class containing $\mathcal{A} \cup \{\emptyset, \mathbb{C}\}$, the desired conclusion is a consequence of the monotone class theorem.

□

REMARK 2.7.2. Putnam proved – see [Put] – this extension to Fuglede's theorem: if $N_i, i = 1, 2$ is a normal operator on $\mathcal{H}_i$ and if $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ satisfies $TN_1 = N_2 T$, then, we also necessarily have $TN_1^* = N_2^* T$. (A cute 2×2 matrix proof of this – see [Hal2] – applies Fuglede's theorem to the operators on $\mathcal{H}_1 \oplus \mathcal{H}_2$ given by the operator matrices $\begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}$ and $\begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$.)

2.7.2 Functional calculus for several commuting normal operators

This section addresses the analogue of the statement that a family of commuting normal operators on a finite-dimensional Hilbert space can be simultaneously diagonalised, equivalently, that an arbitrary family $\{N_j : j \in I\}$ of pairwise commuting normal operators admits a joint functional calculus – i.e., an appropriate continuous and measurable 'joint functional calculus' identifying (algebraically and topologically) appropriate closures of the *-algebras generated by the family $\{N_j; j \in I\}$.

Suppose $\{X_i : i \in I\}$ is a (possibly infinite, maybe even uncountable) family of self-adjoint operators on $\mathcal{H}$. For each finite set $F \subset I$, let Σ_F be the joint spectrum of $\{X_j : j \in F\}$. Recall that $\mathbf{\Sigma}_F \subset \prod_{i\in F}\sigma(X_i)$. Let $\mathrm{pr}_F : \prod_{i\in I}\sigma(X_i) \to \prod_{i\in F}\sigma(X_i)$ denote the natural projection.

We start with a mild generalisation of Lemma 2.5.2(2).

LEMMA 2.7.3. *If $F \subset E \subset I$ are finite sets, and if $\mathrm{pr}_F^E : \prod_{i\in E}\sigma(X_i) \to \prod_{i\in F}\sigma(X_i)$ is the natural projection, then $\Sigma_F = \mathrm{pr}_F^E(\Sigma_E)$.*

Proof. This assertion is easily seen to follow by induction on $|E \setminus F|$ from the special case of the Lemma when $|E \setminus F| = 1$. (*Reason:* If the result is known for $F_n \subset E, |E \setminus F_n| = n$ and if $|E \setminus F| = n+1$, we can find F_n such that $F \subset F_n \subset E$, $|E \setminus F_k| = k$, and observe that $\pi_F^E = \pi_F^{F_n} \circ \pi_{F_n}^E$, and deduce the truth of the assertion for $n+1$ from that of n and 1, thus: $\Sigma_F = \mathrm{pr}_F^{F_n}(\Sigma_{F_n}) = \mathrm{pr}_F^{F_n}(\mathrm{pr}_{F_n}^E(\Sigma_E))$.)

So suppose $E = \{1, 2, \ldots, k+1\}$ and $F = \{1, 2, \ldots, k\}$. Suppose $(\lambda_1, \ldots, \lambda_k) \in \Sigma_F$. If $\epsilon > 0$, it is seen from Corollary 2.5.4 and Remark 2.3.3(2) that $\mathcal{M}(\epsilon) := \tilde{\pi}(1_{\{(t_1,\ldots,t_k)\in\Sigma_F : |t_i-\lambda_i|<\epsilon\ \forall i\in F\}}) \neq 0$ and is invariant under each $X_i, 1 \leq i \leq k+1$. If $X_{k+1}(\epsilon) = X_{k+1}|_{\mathcal{M}(\epsilon)}$ and $\lambda_{k+1} \in \sigma(X_{k+1}(\epsilon))$, it is seen that $\exists x(\epsilon) \in S(\mathcal{M}(\epsilon))$ such that $\|(X_{k+1} - \lambda_{k+1})x(\epsilon)\| < \epsilon$. Since $\|(X_i - \lambda_i)x\| < \epsilon\ \forall x \in S(\mathcal{M}(\epsilon))$, we see that $\{x(\frac{1}{n})\}$ is a sequence of unit vectors such that $\|(X_i - \lambda_i)x(\frac{1}{n})\| < \frac{1}{n}\ \forall n$, and indeed $(\lambda_1, \ldots \lambda_{k+1}) \in \mathbf{\Sigma}_E$ so $\mathbf{\Sigma}_E \subset \mathrm{pr}_F^E(\mathbf{\Sigma}_E)$. The reverse inclusion is obvious, and the proof is complete. □

For each finite $F \subset I$, let $\mathbf{\Sigma}(F) = \mathrm{pr}_F^{-1}(\mathbf{\Sigma}_F)$ and let $\mathbf{\Sigma} = \bigcap_F \mathbf{\Sigma}(F)$.

THEOREM 2.7.4. *With the foregoing notation, we have:*

(1) $\mathbf{\Sigma}$ is a non-empty compact set, which we shall refer to as the joint spectrum of $\{X_j : j \in I\}$.

(2) There exists a unique isomorphism $\pi : C(\Sigma) \to C^*(\{X_j : j \in I\})$ such that $\pi(\mathrm{pr}_{\{j\}}) = X_j\ \forall j \in I$.

*(3) There exists a probability measure μ defined on $\mathcal{B}_\Sigma$, unique up to mutual absolute continuity, such that the continuous functional calculus π above 'extends' to an isometric *-algebra monomorphism $\tilde{\pi}$ of $L^\infty(\Sigma, \mathcal{B}_\Sigma, \mu) \to B(\mathcal{H})$ with the property that a norm-bounded sequence $\{f_n : n \in \mathbb{N}\} \subset$*

$L^\infty(\Sigma, \mathcal{B}_\Sigma, \mu)$ converges in (μ) measure if and only if the image of this sequence under this 'joint measurable functional calculus' is SOT-convergent.

Proof. (1) It is clear that $\mathbf{\Sigma}$ is the closed subset of $\mathbb{R}^I$ consisting of those tuples $((\lambda_i))_{i\in I}$ such that for any finite $F \subset I$, it is possible to find a sequence of unit vectors $x_n^F, n \in \mathbb{N}$ such that $\|(X_i - \lambda_i)x_n^F\| \to 0 \; \forall i \in F$ so that, in particular $\mathbf{\Sigma}$ is a closed subset of $\prod_{i\in I}\sigma(X_i)$ and hence compact. It is not hard to see (from Lemma 2.7.3 and Lemma 2.5.2) that $\{\mathbf{\Sigma}(F) : F$ a finite subset of $I\}$ is a family of non-empty compact sets with the finite intersection property, and that hence, their intersection, i.e., $\mathbf{\Sigma}$, is also non-empty and compact.

(2) On the one hand, the family $\{\mathrm{pr}_F : F$ a finite subset of $I\}$ linearly spans a self-adjoint subalgebra of functions which separates points of $\mathbf{\Sigma}$, which is dense in $C(\mathbf{\Sigma})$. It then follows from Proposition 2.5.3 (2) that there is a unique isometric *-algebra isomorphism $\pi : C(\mathbf{\Sigma}) \to C^*(\{X_i : i \in I\})$ such that $\pi(\mathrm{pr}_{\{j\}}) = X_j$.

(3) This follows immediately from Remark 2.3.3.

□

Suppose now that $N_j = A_j + iB_j$ (resp., $\lambda_j = \alpha_j + i\beta_j$) is the Cartesian decomposition of N_j as in the last paragraph (resp., $\lambda_j \in \sigma(N_j)$), and denote their **joint spectrum** by the set $\Sigma = \{\lambda = ((\lambda_j))_{j\in I} \in \mathbb{C}^I$ – or alternatively $\{((\alpha_j, \beta_j))_{j\in I} \in (\mathbb{R}^2)^I\}$ – of those tuples for which it is possible to find a sequence $\{x_n : n \in \mathbb{N}\}$ of unit vectors such that

$$\lim_{n\to\infty} \|(N_j - \lambda_j)x_n\|^2 = \lim_{n\to\infty} \left(\|(A_j - \alpha_j)x_n\|^2 + \|(B_j - \beta_j)x_n\|^2\right) = 0 \; \forall j \in I.$$

In view of Fuglede's theorem, we see that commutativity of the family $\{N_j : j \in I\}$ of normal operators is equivalent to that of the family $\{A_j, B_j : j \in I\}$ of self-adjoint operators. It must be clear that $\{((\alpha_j + i\beta_j)) \in \mathbb{C}^I : (((\alpha_j, \beta_j))) \in \mathbf{\Sigma}(\{A_j, B_j : j \in I\}$ may be defined as the joint spectrum of the family $\{N_j : j \in I\}$ of normal operators, and the exact counterpart of Theorem 2.7.4 (with mild modifications, usually involving changing $\mathbb{R}$ to $\mathbb{C}$ and self-adjoint to normal) for a family of commuting normal operators is valid.

EXERCISE 2.7.5. *(1) Formulate and prove the precise statement of the 'normal version' of Theorem 2.7.4.*

(2) Also state and prove a formulation of the 'joint spectral theorem' for a family of commuting normal operators in terms of projection-valued measures.

2.8 Typical uses of the spectral theorem

We now list some simple consequences of the spectral theorem (i.e., the functional calculi) for a normal operator.

PROPOSITION 2.8.1. *1. Let $T \in B(\mathcal{H})$ be a normal operator. Then*

(a) T is self-adjoint if and only if $\sigma(T) \subset \mathbb{R}$.

(b) T is a projection if and only if $\sigma(T) \subset \{0,1\}$.

(c) T is unitary if and only if $\sigma(T) \subset \{z \in \mathbb{C} : |z| = 1\}$.

2. The following conditions on an operator $A \in B(\mathcal{H})$ are equivalent:

(a) There exists some Hilbert space $\mathcal{K}$ and an operator $T \in B(\mathcal{H}, \mathcal{K})$ such that $A = T^*T$.

(b) $\langle Ax, x \rangle \geq 0 \ \forall x \in \mathcal{H}$.

(c) A is self-adjoint and $\sigma(A) \subset [0, \infty)$.

(d) A is normal and $\sigma(A) \subset [0, \infty)$.

(e) There exists a self-adjoint operator $B \in B(\mathcal{H})$ such that $A = B^2$.

Such an operator A is said to be **positive**, *and we write $A \geq 0$, and more generally, we shall write $A \geq C$ if and only if A, C are self-adjoint operators satisfying $A - C \geq 0$.*

3. If $A \geq 0$, there exists a unique $B \geq 0$ such that $A = B^2$, and we denote this unique positive square root of A by $A^{\frac{1}{2}}$.

4. Let $U \in B(\mathcal{H})$ be a unitary operator. Then there exists a self-adjoint operator $A \in B(\mathcal{H})$ such that $U = e^{iA}$, where the right hand side is interpreted as the result of the continuous functional calculus for A; further, given any $a \in \mathbb{R}$, we may choose A to satisfy $\sigma(A) \subset [a, a + 2\pi]$.

5. If $T \in B(\mathcal{H})$ is a normal operator, and if $n \in \mathbb{N}$, then there exists a normal operator $A \in B(\mathcal{H})$ such that $T = A^n$.

6. Any self-adjoint operator T admits a unique decomposition $T = T_+ - T_-$, where $T_\pm \geq 0$ and $T_+T_- = 0 = T_-T_+$

7. Any self-adjoint contraction (i.e., an operator T satisfying $T = T^*$ and $\|T\| \leq 1$ is expressible as the average of at most two unitary operators, and hence any operator is expressible as a linear combination of at most four unitary operators.

Proof. (1) A normal operator T is self-adjoint (resp., a projection, resp., unitary precisely when it satisfies $T = T^*$, or $T = T^* = T^2$, or $T^*T = 1$ respectively. while the function $f_0 \in C(\Sigma)$, for $\Sigma \subset \mathbb{C}$, defined by $f_0(z) = z$ satisfies $f_0 = \overline{f_0}$ (resp., $f_0 = \overline{f_0} = f_0^2$, resp., $f_0\overline{f_0} = 1$) precisely when $\Sigma \subset \mathbb{R}$ (resp., $\Sigma \subset \{0,1\}$, resp., $\Sigma \subset \{z : |z| = 1\}$).

(2) The implications $(e) \Rightarrow (a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$ are obvious. As for $(d) \Rightarrow (e)$, note that (d) implies that A is self-adjoint by $1(a)$. If the function defined on $[0, \infty)$, by $f(t) = t^{\frac{1}{2}}$, denotes the positive square-root, then the condition (c) implies that $f \in C(\sigma(A))$, and we see that $B = f(A)$ works.

(Notice that $B \in C^*(A)$ by construction). As for $(b) \Rightarrow (c)$, the self-adjointness of A follows from Corollary 1.5.3 (2), and the positivity of elements of $\sigma(A)$ follows then from Theorem 1.6.2(2).

(3) Suppose B_1 is another prospective positive square root of A. Since $B \in C^*(A) \subset C^*(B_1) \cong C(\sigma(B_1))$, there must be a non-negative $g \in C(\sigma(B_1))$ such that $B = g(B_1)$. As $B^2 = A = B_1^2$, we must have $g(t)^2 = t^2 \ \forall t \in \sigma(B_1)$, and we must have $g(t) = t$ so $B = B_1$.

(4) Let $\phi : \mathbb{C} \setminus \{0\} \to \{z \in \mathbb{C} : \text{Im } z \in [a, a + 2\pi)\}$ be any (measurable) branch of the logarithm - for instance, we might set $\phi(z) = log|z| + i\theta$, if $z = |z|e^{i\theta}$, $a \leq \theta < a + 2\pi$. Setting $A = \phi(U)$, we find - since $e^{\phi(z)} = z$ - that $U = e^{iA}$.

(5) This is proved like 4 above, by taking some measurable branch of the logarithm defined everywhere in $\mathbb{C}\setminus\{0\}$ and choosing the $z^{\frac{1}{n}}$ as the exponential of $\frac{1}{n}$ times this choice of logarithm.

(6) Define $T_\pm = f_\pm(T)$ where $f_\pm$ are the obviously continuous functions $f_\pm : \mathbb{R} \to \mathbb{R}$ defined by $f_\pm = (|f_0| \pm f_)/2$. Then indeed

$$f_0 = f_+ - f_-, f_\pm \geq 0 \quad \text{and} \quad f_+f_-(= f_-f_+) = 0$$

and hence

$$T_0 = T_+ - T_-, T_\pm \geq 0 \quad \text{and} \quad T_+T_- = 0 = (T_+T_-)^* = T_-T_+.$$

As for uniqueness, if $T = A_+ - A_-$ with $A_\pm \geq 0, A_+A_- = 0$, note first that

$$A_+A_- = 0 \Rightarrow A_-A_+ = (A_+A_-)^* = 0$$

and hence that

$$(A_+ + A_-)^2 = A_+^2 + A_-^2 = (A_+ - A_-)^2 = T^2 = |T|^2$$

where $|T|$ represents the image, under the functional calculus for T, of the function $f(t) = |t|$; and we may deduce from the uniqueness of the positive square root of a positive operator that $(A_+ + A_-) = |T|$ and hence we must have

$$A_\pm = \tfrac{1}{2}(|T| \pm T) = T_\pm \ ,$$

as desired.

(7) Consider $v_\pm \in C([-1, 1])$ defined by $v_\pm(t) = t \pm i\sqrt{1 - t^2}$. Note that $t = \frac{1}{2}(v_+(t) + v_-(t))$ and $|v_\pm(t)| = 1$ for $t \in [-1, 1]$. Define $U_\pm = v_\pm(T)$.

As $U_\pm$ are unitary with average T, it follows, by scaling, that every self-adjoint operator is a linear combination of at most[2] two unitary operators, and the Cartesian decomposition completes the proof of the proposition. □

[2]The reason for the 'at most' is that T might have already been self-adjoint and unitary (i.e., satisfying $T^2 = 1$)