# TEMPERLEY-LIEB AND NON-CROSSING PARTITION PLANAR ALGEBRAS

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Abstract. For each complex number $\delta \neq 0$, we consider a planar algebra whose space $NC_n(\delta)$ of '$k$-boxes' has a basis consisting of non-crossing partitions of a set of $2k$ points, (usually thought of as being arrayed on two parallel lines, with $k$ points on each line), and with multiplication and other planar algebra structure being defined almost exactly as in the case of the Temperley-Lieb planar algebra $TL(\delta)$. We show that this planar algebra $NC(\delta)$ is a $C^*$-planar algebra when $\delta > 4$. We do this by showing that $NC(\delta^2)$ is isomorphic to the so-called 2-cabling of $TL(\delta)$.

## 1. Introduction

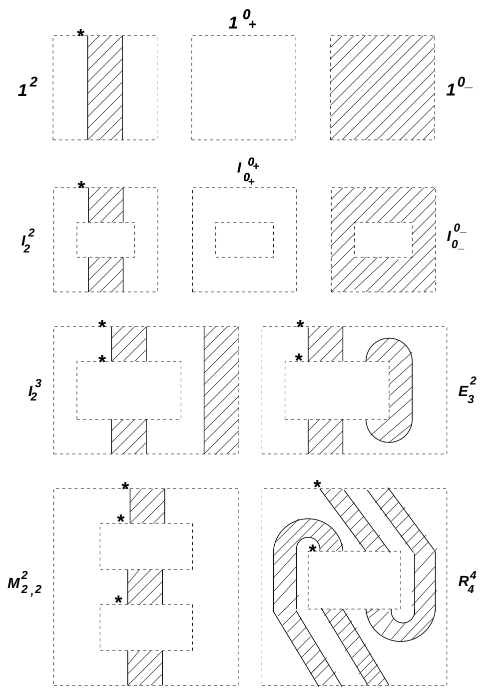
We begin with the basic definitions on planar algebras in §2. (It must be mentioned that, unless explicitly stated otherwise, our planar algebras need not have a $C^*$-structure.) We describe the example of the Temperley-Lieb planar algebra, and introduce the 'non-crossing partition' planar algebra.

§3 is devoted to proving a combinatorial identity relating various features of a planar configuration consisting of a straight line and a system of closed curves each of whose components intersects the line. As can be expected, this seemingly strange identity is a consequence of the Euler characteristic formula; but we need this identity in this form at a couple of instances during the course of proving our main result.

§4 commences with a 'linearisation lemma' of possibly independent combinatorial interest, then proceeds to the statement (and proof) of the main result identifying the planar algebra $NC(\delta^2)$ with the '2-cabling' of $TL(\delta)$, and concludes with some final remarks on values of $\delta$ for which $NC(\delta)$ is a $C^*$-planar algebra.

## 2. The Temperley-Lieb and Non-crossing Partition planar algebras

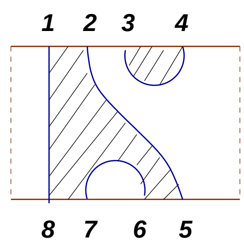
We begin with a very brief summary of planar algebras. For details, the reader is referred to the source [J2] or to [KS]. By $Col$, we denote the set $\{0_+, 0_-, 1, 2, \cdots\}$, whose elements will be referred to as colours. Recall that a planar tangle is an equivalence class, under planar isotopy preserving all relevant data, of subsets of the plane which comprise the following: an external box, denoted $D_0$, and a finite (possibly null) ordered collection of internal boxes denoted $D_1$, $D_2$, $\cdots$. Further, each box has an even number (again, possibly 0) of points marked on its boundary - a box with $2k$ points on its boundary being called a $k$-box or said to be of colour $k$. If a box has at least one point marked on its boundary, one of them is distinguished and marked with a '$*$'. There is also given a collection of disjoint curves each of which is either closed, or joins a marked point on one of the boxes to another such. The whole picture is to be planar and each marked point on a box must be the end-point of one of the curves. Finally, there is given a black-and-white (checkerboard) shading of the regions such that moving away from (resp. towards) the $*$ on one of the internal boxes (resp. the external box) along the curve of which it is the end-point, a black region is to the right. A 0-box is said to be $0_+$ box if the region touching its boundary is white and a $0_-$ box otherwise. A tangle is said to be a $k$-tangle if its external box is of colour $k$. In fact,we will sometimes find it convenient to use the symbol $T^{k_0}_{k_1,\cdots,k_b}$ to denote a $k_0$-tangle with $b$ internal boxes of colours $k_1, \cdots, k_b$. Several examples of tangles are shown in the figure below.




The basic operation that one can perform on tangles is substitution of one into a box of another: if $T$ is a tangle that has some internal boxes, say, $D_{i_1}, \cdots, D_{i_j}$ of colours $k_{i_1}, \cdots, k_{i_j}$ and if $X_1, \cdots, X_j$ are arbitrary tangles of colours $k_{i_1}, \cdots, k_{i_j}$, then we may substitute $X_t$ into the box $D_{i_t}$ of $T$ for each $t$ - such that the '$*$'s match' - to get a new tangle that will be denoted $T \circ_{(D_{i_1},\cdots,D_{i_j})} (X_1, \cdots, X_j)$.

A planar algebra $P$ (over $\mathbb{C}$) is a collection $\{P_k : k \in Col\}$ of complex vector spaces and maps $Z_T : P_{k_1} \otimes P_{k_2} \otimes \cdots \otimes P_{k_b} \to P_{k_0}$ for each $k_0$-tangle $T$ with internal boxes of colours $k_1, k_2, \cdots, k_b$. The collection of maps is to be 'compatible with substitution of tangles and renumbering of internal boxes' in an obvious manner. Further, planar algebras are required to be non-degenerate in the sense that for each $k \in Col$, the map $Z_{I_k^k}$ is the identity map of $P_k$. A pleasant verification then shows that each $P_k$ has the structure of an associative, unital algebra where the multiplication is given by $Z_{M_{k,k}^k}$ and the unit by $Z_{1^k}(1)$ - which we will denote by $\mathbb{1}_k$. The tangles $I_k^k$, $M_{k,k}^k$ and $1^k$ are the straightforward generalisations to $k$-tangles of the tangles $I_2^2$, $M_{2,2}^2$ and $1^2$ illustrated earlier.

In a sense, the simplest planar algebras are the Temperley-Lieb planar algebras which we shall now describe. Fix a non-zero complex number $\delta$. The planar algebra $P = TL(\delta)$ has $P_{0_\pm} = \mathbb{C}$ and for $k \geq 1$, $P_k =$ complex vector space with basis consisting of the set $\mathcal{K}_k$ of all Kauffman $k$-diagrams. Recall that such a diagram consists of an isotopy class of a planar (i.e., non-crossing) arrangement of $k$ curves in a box with their ends tied to $2k$ marked points on the boundary; an example, with $k = 4$ is illustrated below:




It will be convenient to regard Kauffman diagrams as being endowed with a black and white shading with the leftmost region being white. We will denote $P_k$ by $TL_k(\delta)$.

Next, the action of a tangle $T$ on this collection of vector spaces is defined as follows. Suppose that $T$ has internal boxes $D_1, \cdots, D_b$ of colours $k_1, \cdots, k_b$ and that Kauffman diagrams $S_1, ..., S_b$ are given with $S_i \in \mathcal{K}_{k_i}$. Insert these into the appropriate boxes of $T$ so that the point numbered 1 is aligned with the *-point of that box, and then delete the boundaries of the internal boxes, to get a picture with, say, $l$ loops. Delete these loops and consider the Kauffman diagram, say, $S \in \mathcal{K}_{k_0}$ that is obtained. Define $Z_T(S_1 \otimes \cdots \otimes S_b) = \delta^l S$ and extend by linearity. It is intutively obvious that this prescription indeed defines a planar algebra.

We will now define by analogy a planar algebra associated to non-crossing partitions. Recall that a non-crossing partition on $2k$ points, is a partition of a set of $2k$ marked points on a circle with the property that the convex hulls of any two distinct equivalence classes of the partition are disjoint; the collection of such partitions will be denoted by $NC_k$. The planar algebra $\tilde{P} = NC(\delta)$, defined for fixed non-zero $\delta \in \mathbb{C}$, has $\tilde{P}_{0_\pm} = \mathbb{C}$ and for $k \geq 1$, $\tilde{P}_k =$ complex vector space with basis $NC_k$. For the sake of convenience, however, we shall think of the $2k$ points as being arrayed with $k$ points on each of two parallel lines. The tangle action on $\tilde{P}$ is defined almost exactly as in $P$ except that the count of loops is replaced with that of internal equivalence classes. Again, it should be clear that this does prescribe a planar algebra, and as before, we denote $\tilde{P}_k$ by $NC_k(\delta)$.

## 3. A topological diversion

We shall need the following result in the proof of our main theorem.

PROPOSITION 3.1. *Consider a configuration consisting of:*

- *a system of $C$ disjoint closed curves in the plane*
- *a checkerboard shading of the resulting regions*
- *a line intersecting each of the curves (with the number $P$ of points of intersection being $2m$).*

*Then,*

$$C - 2B = m - B_+ - B_- \ , \tag{3.1}$$

*where $B_+$ (resp., $B_-$) denotes the number of black regions above (resp., below) the line.*

*Proof:* We now wish to 'symmetrise' (between black and white) the desired identity. We shall henceforth view our configuration as being embedded on the surface of a sphere, 'line' will be taken to mean 'great circle', and of the two components of the complement of this great circle in the sphere, one will be arbitrarily declared as 'top' and the other 'bottom'. On interchanging the roles of black and white, we find that the statement (3.1) would imply (and be equivalent to):

$$C - 2W = \frac{1}{2}P - W_+ - W_- \tag{3.2}$$

with the symbols $W, W_\pm$ having their natural meanings.

In order to prove these two identities, it suffices (and is necessary) to prove their 'symmetric' (=sum) and 'antisymmetric' (=difference) versions, namely:

$$2C - 2R = P - R_+ - R_- \tag{3.3}$$

$$2(W - B) = (W_+ - B_+) + (W_- - B_-) \tag{3.4}$$

where we have used the symbols $R, W$ and $B$ to denote, respectively, the # of regions, the # of white regions and the # of black regions. We shall also write $R_\pm$ to have their obvious meanings.

For equation (3.3), consider the 'polygonation' of the sphere obtained by taking (i) the vertices to be the $2m$ points of intersection, (so $V = 2m$) (ii) the edges to be the parts of the closed curves bounded by vertices as well as the parts of the line between two successive vertices, (so $E = 2m + 2m$) and (iii) the faces to be the resulting regions (so $F = R_+ + R_-$), and we find - since $C - R = -1$ - that

$$\begin{aligned}
(2C - 2R) - P + R_+ + R_- &= -2 - 2m + F \\
&= -2 + (V - E) + F \\
&= 0
\end{aligned}$$

since the Euler characteristic of $S^2$ is 2, thereby establishing (3.3).

To complete the proof, we shall establish equation (3.4) by induction on $m$. To set the ball rolling, notice that when $m = 0$, there are no curves, and both sides of the desired equation are seen to be $\pm 2$, according as the sphere is shaded white or black.

Suppose then that the result holds for $m-1$, and that we have a configuration with $2m$ points. It is easy to see that there must exist at least one 'special' pair of adjacent points on the line which are connected by one of the curves. We may assume, without loss of generality, that the region bounded by the line and that curve is shaded white and lies below the line. We consider two cases now:

*Case 1*: The curve meets the line in only these two special points.

In this case, consider the configuration - call it $\mathcal{C}'$ - obtained by simply removing this pargticular curve. Then $\mathcal{C}'$ has $2m-2$ points of intersection, so we know, by induction, that

$$2(W' - B') = (W'_+ - B'_+) + (W'_- - B'_-) \ ,$$

where the primed symbols have the natural interpretation. The assumptions of this case imply that

$$W' = W - 1, W'_+ = W_+, W'_- = W_- - 1$$

while

$$B' = B - 1, B'_+ = B_+ - 1, B'_- = B_-$$

and we may deduce that

$$\begin{aligned} 2(W-B) &= 2(W'-B') \\ &= (W'_+ - B'_+) + (W'_- - B'_-) \\ &= (W_+ - B_+ + 1) + (W_- - 1 - B_-) \\ &= (W_+ - B_+) + (W_- - B_-) \end{aligned}$$

as desired.

*Case 2*: The curve meets the line in more than these two special points.

In this case, consider the configuration - call it $\mathcal{C}'$ as before - obtained by moving the curve up so that these two special points of intersection are eliminated. The assumptions of this case ensure that $\mathcal{C}'$ is an admissibe configuration - i.e., satisfies the hypotheses - and has only $2m-2$ points of intersection, so that, by induction hypothesis,

$$2(W' - B') = (W'_+ - B'_+) + (W'_- - B'_-) \ .$$

The assumptions are easily seen to imply that

$$W' = W - 1, W'_+ = W_+, W'_- = W_- - 1$$

while

$$B' = B - 1, B'_+ = B_+ - 1, B'_- = B_-$$

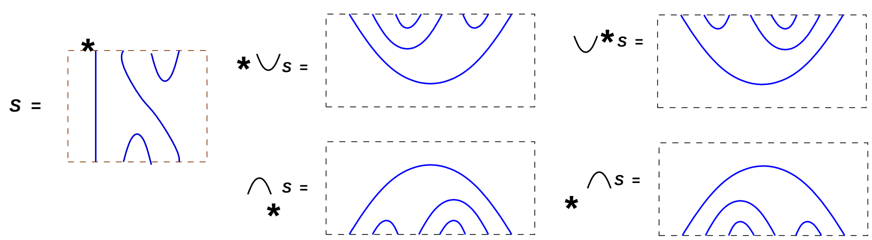
and we may deduce that

$$\begin{aligned} 2(W-B) &= 2(W'-B') \\ &= (W'_+ - B'_+) + (W'_- - B'_-) \\ &= (W_+ - B_+ + 1) + (W_- - 1 - B_-) \\ &= (W_+ - B_+) + (W_- - B_-) \end{aligned}$$

as desired; and the proof of equation (3.4) and hence also of Proposition 3.1, is complete. □

## 4. Relation between the planar algebras

For $S \in \mathcal{K}_{2n}$, define its near relatives - which we shall denote by ${}^* \cup S$ and $\cap_*$ respectively - by 'combing $S$' so that all end-points of strings are 'at the top (resp., bottom)', with the *-point at the extreme left ( resp., extreme right). We shall also have occasion to use ${}_* \cap S$ and $\cup^*$ for the result of horizontally reflecting ${}^* \cup S$ and $\cap_*$ respectively. We illustrate the general case with an example:

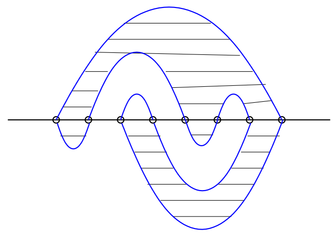



Some notation will help before setting up the crucial linearisation lemma below. To start with, we shall regard elements of $\mathcal{K}_m$ as partitions of the set $[2m] = \{1, 2, \cdots, 2m\}$ into doubleton sets. Given partitions $P, Q$ of the set $[2m]$, we shall write $|P|$ for the number of parts of $P$, and $P \bigvee Q$ for the coarsest partition which is finer than both $P$ and $Q$. Finally, given a partition $P$ of $[2m]$, we shall write $\widetilde{P} = P \bigvee P_0$, where $P_0 = \{\{1,2\}, \cdots, \{2m-1, 2m\}\}$.

Lemma 4.1. *(*Linearisation Lemma*)*

$$|S_1 \bigvee S_2| - 2|\widetilde{S_1} \bigvee \widetilde{S_2}| = m - |\widetilde{S_1}| - |\widetilde{S_2}| \ \forall S_1, S_2 \in \mathcal{K}_m.$$

*Proof:* We begin by re-stating the problem thus: first consider the configuration obtained by 'glueing ${}_* \cap S_1$ and ${}^* \cup S_2$ along the $2m$ common points. Thus, if $S_1$ and $S_2$ are the elements of $\mathcal{K}_4$ shown in the figure displayed in the discussion following equation (4.10), what we have is:



In terms of this configuration, the terms in the statement of the lemma are reinterpreted thus:

- $|S_1 \bigvee S_2|$ is the number $C$ of closed curves
- $|\widetilde{S_1} \bigvee \widetilde{S_2}|$ is the number $B$ of regions shaded black
- $m$ is half the number $P$ of points of intersection of the curves and the line
- $|\widetilde{S_1}|$ is the number $B_+$ of black regions above the line
- $|\widetilde{S_2}|$ is the number $B_-$ of black regions below the line

Thus, in the context of our reformulation, the lemma is a consequence of Proposition 3.1. $\square$

In order to motivate our main result, we begin by recalling that both $\mathcal{K}_{2k}$ and $NC_k$ have the same cardinality given by the Catalan number $\frac{1}{2k+1}\binom{4k}{2k}$. In fact, there is a natural pictorial bijection

$$\mathcal{K}_{2k} \ni S \leftrightarrow \widetilde{S} \in NC_k \ ;$$

this bijection is partially illustrated in Figure 1 for the case $k = 2$ (partial because the sets in question have 14 elements each).

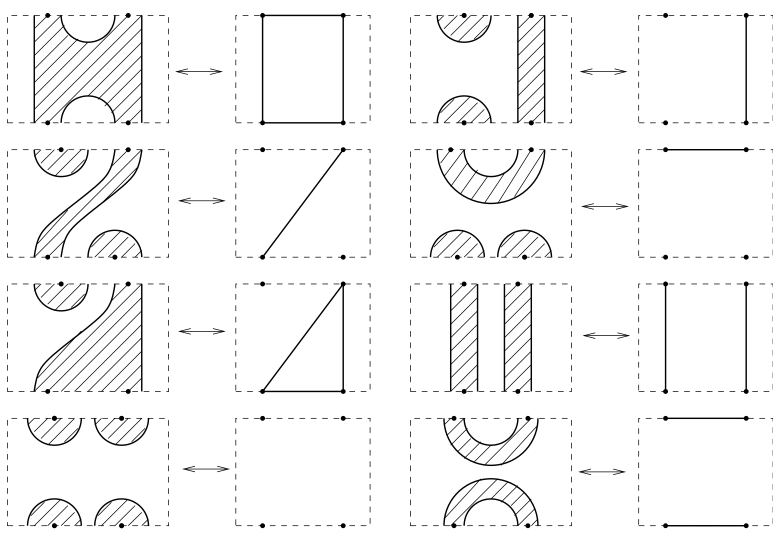


FIGURE 1. Bijection between $\mathcal{K}_4$ and $NC_2$

The $2k$ points of the element $\widetilde{S}$ of $NC_k$ which corresponds to an element $\tilde{S} \in \mathcal{K}_{2k}$ may be chosen as points on the boundary which are midway between an odd point and the next even point, and the equivalence relation is defined as 'belonging to the same black region of $S$'. (It should be observed that if $P \in \mathcal{K}_m$, then the two different notions of $\widetilde{P}$ - the one given here and the one given before the linearisation lemma - are c onsistent when suitably interpreted.)

To state the main result, we recall the notion of a $c$-cabling of a planar algebra $P$ for $c \in \mathbb{N}$ which is denoted by ${}^{(c)}P$ and defined as follows. The spaces ${}^{(c)}P_k$ and the tangle actions are defined by

$$
{}^{(c)}P_k = \begin{cases} P_{ck} & \text{if } k \in \mathbb{N} \\ P_{0_+} & \text{if } k = 0_+ \\ P_{0_+} & \text{if } k = 0_- \text{ and } c \text{ is odd} \\ P_{0_-} & \text{if } k = 0_- \text{ and } c \text{ is even} \end{cases}
$$

and

$$
Z_T^{{}^{(c)}P} = Z_{T^{(c)}}^P
$$

where $T^{(c)}$ denotes the *$c$-cabling of $T$* which is, by definition, the tangle obtained by replacing each of its strings by a parallel cable of $c$ strings.

Recall also that an isomorphism of planar algebras from $P$ to $Q$ is a collection of vector space isomorphisms $\{\phi_k : P_k \to Q_k : k \in Col\}$ which are equivariant with respect to the tangle actions.

Given the bijection between $\mathcal{K}_{2k}$ and $NC_k$, the following result seems intuitively reasonable/plausible, but neither the asserted isomorphism nor the proof of the theorem is so intuitively obvious!

THEOREM 4.2. *For any non-zero $\delta \in \mathbb{C}$, the planar algebras $P = {}^{(2)}TL(\delta)$ and $\tilde{P} = NC(\delta^2)$ are isomorphic.*

*Proof:* Note that both $P$ and $\tilde{P}$ are connected planar algebras (i.e., their $0_\pm$ spaces are 1-dimensional); hence, for any $k \in Col\backslash\{0_-\}$, the trace tangle $tr_k^{0_+}$ may be regarded as taking values in $\mathbb{C}$ and thereby specifying a trace denoted by $\tau = \tau_k$ on $P_k$ and $\tilde{\tau} = \tilde{\tau}_k$ on $\tilde{P}_k$. Note that these are not normalised traces; in fact, $\tau(\mathbb{1}_k) = \delta^{2k} = \tilde{\tau}(\mathbb{1}_k)$.

We shall show that the desired isomorphism is implemented by the maps $\phi_k : P_k \to \tilde{P}_k$ defined as the linear extensions of the maps defined on the bases $\mathcal{K}_{2k}$ of $P_k$ by:

$$
S \mapsto \frac{\tau(S)}{\tilde{\tau}(\tilde{S})}\tilde{S},
$$

for $k \geq 1$ and $id_{\mathbb{C}}$ for $k = 0_\pm$.

Before proceeding with the proof, we wish to observe the following important consequence of the definitions of the tangle actions on $P$ and $\tilde{P}$. Suppose that $T$ is a $k_0$-tangle with internal boxes of colours $k_1, \cdots, k_b$ and $S_i \in \mathcal{K}_{2k_i}$ (where, of course, $2.0_\pm = 0_+$ - see the definition of cabling). We have, by definition,

$$
Z_T^P(S_1 \otimes \cdots \otimes S_b) = Z_{T^{(2)}}^{TL(\delta)}(S_1 \otimes \cdots \otimes S_b) = \delta^l.S,
$$

for some $S \in \mathcal{K}_{2k_0}$ and some $l \in \mathbb{N} \cup \{0\}$. What must be noticed is that then,

$$Z_T^{\tilde{P}}(\tilde{S}_1 \otimes \cdots \otimes \tilde{S}_b) = \delta^{2c}.\tilde{S},$$

for some $c \in \mathbb{N} \cup \{0\}$. The numbers $l$ and $c$ are the number of loops and internal equivalence classes respectively when $S_i$'s are inserted into the boxes of $T^{(2)}$; when necessary, we will indicate their dependence on the tangle $T$ and the Kauffman diagrams $S_1, \cdots, S_b$ by writing $l(T; S_1, \cdots, S_b)$ or $c(T; S_1, \cdots, S_b)$. The point of this observation is that $S \mapsto \tilde{S}$, is almost a planar algebra map except for the tangle dependent multiplicative factors determined by $l$ and $c$, and the content of Theorem 4.2 is that with the correct normalisations, all these factors are 1.

So we see that if $T^{k_0}_{k_1,\cdots,k_b}$ is a $k_0$-tangle with $b$ internal boxes of colours $k_0, \cdots, k_b$, and if $S_i \in \mathcal{K}_{2k_i}, 0 \le i \le b$, then

$$Z_T^P(\otimes_{i=1}^b S_i) = \delta^{l(T;S_1,\cdots,S_b)} S_0 \Rightarrow Z_T^{\tilde{P}}(\otimes_{i=1}^b \widetilde{S_i}) = \delta^{2c(T;S_1,\cdots,S_b)} \widetilde{S_0}. \tag{4.5}$$

We need to verify that for every tangle $T$ as above, we have

$$\phi_{k_0}(Z_T^P(\otimes_{i=1}^b S_i)) = Z_T^{\tilde{P}}(\otimes_{i=1}^b \phi_{k_i}(S_i)) \ \forall\ S_i \in \mathcal{K}_{2k_i}, 1 \le i \le b\ .$$

On the one hand,

$$\begin{aligned} \phi_{k_0}(Z_T^P(\otimes_{i=1}^b S_i)) &= \delta^{l(T;S_1,\cdots,S_b)} \phi_{k_0}(S_0) \\ &= \delta^{l(T;S_1,\cdots,S_b)} \frac{\tau(S_0)}{\tilde{\tau}(\widetilde{S_0})}\ \widetilde{S_0}\ , \end{aligned}$$

while on the other, we have

$$\begin{aligned} Z_T^{\tilde{P}}(\otimes_{i=1}^b \phi_{k_i}(S_i)) &= \left(\prod_{i=1}^b \frac{\tau(S_i)}{\tilde{\tau}(\widetilde{S_i})}\right) Z_T^{\tilde{P}}(\otimes_{i=1}^b \widetilde{S_i}) \\ &= \left(\prod_{i=1}^b \frac{\tau(S_i)}{\tilde{\tau}(\widetilde{S_i})}\right) \delta^{2c(T;S_1,\cdots,S_b)} \widetilde{S_0}\ . \end{aligned}$$

Thus, we need to verify that

$$\delta^{l(T;S_1,\cdots,S_b)} \frac{\tau(S_0)}{\tilde{\tau}(\widetilde{S_0})} = \left(\prod_{i=1}^b \frac{\tau(S_i)}{\tilde{\tau}(\widetilde{S_i})}\right) \delta^{2c(T;S_1,\cdots,S_b)}$$

or equivalently that

$$\frac{\delta^{l(T;S_1,\cdots,S_b)} \tau(S_0)}{\prod_{i=1}^b \tau(S_i)} = \frac{\delta^{2c(T;S_1,\cdots,S_b)} \tilde{\tau}(\widetilde{S_0})}{\prod_{i=1}^b \tilde{\tau}(\widetilde{S_i})}$$

i.e., that

$$\frac{\tau(Z_T^P(\otimes_{i=1}^b S_i))}{\prod_{i=1}^b \tau(S_i)} = \frac{\tilde{\tau}(Z_T^{\tilde{P}}(\otimes_{i=1}^b \widetilde{S_i}))}{\prod_{i=1}^b \tilde{\tau}(\widetilde{S_i})} \tag{4.6}$$

Our proof will be based on the fact - see [KS], Theorem 3.5 - that the collection of all planar tangles is generated (with respect to composition) by the following set of tangles:

$$\mathcal{T} = \{1^{0_+}, 1^{0_-}\} \cup \{R_k^k : k \geq 2\} \cup \{E_{k+1}^k, M_{k,k}^k, I_k^{k+1} : k \in \; Col\} \tag{4.7}$$

This result implies that in order to complete the proof of our theorem, we only need to establish equation (4.6) for each $T \in \mathcal{T}$.

The verification of equivariance for the actions of the tangles $1^{0_\pm}$ is a combination of vacuousness and tautology.

*The case of* $T = R_k^k$: We need to verify that for any $S \in \mathcal{K}_{2k}$, we have

$$\frac{\tau(Z_{R_k^k}^P(S))}{\tau(S)} = \frac{\tilde{\tau}(Z_{R_k^k}^{\tilde{P}}(\tilde{S})}{\tilde{\tau}(\tilde{S})} ,$$

or equivalently, that

$$\frac{\tau(S)}{\tilde{\tau}(\tilde{S})} = \frac{\tau(Z_{R_k^k}^P(S))}{\tilde{\tau}(Z_{R_k^k}^{\tilde{P}}(\tilde{S}))} . \tag{4.8}$$

Consider the configuration obtained by glueing $\cap I_{2k}$ and $\cup S$ along their $(4k)$ maked points, along with the line through these points - where we write $I_{2k}$ for the element of $\mathcal{K}_{2k}$ which is the multiplicative identity of $TL_{2k}$. This is a system of closed curves satisfying the hypotheses of Proposition 3.1, where

- the number $C$ of closed curves satisfies $\delta^C = \tau(S)$,
- the number $B$ of black regions satisfies $\delta^{2B} = \tilde{\tau}(\tilde{S})$,
- the number $P$ of points of intersection satisfies $P = 4k$, and
- the numbers $B_\pm$ of black regions above satisfy $B_+ = k$ and $B_- = bl(S)$ is the number of black regions in $S$ ;

and hence, by Proposition 3.1, we have

$$\begin{aligned} \frac{\tau(S)}{\tilde{\tau}(\tilde{S})} &= \delta^{C-2B} \\ &= \delta^{2k-k-B_-} \\ &= \delta^{k-bl(S)} . \end{aligned}$$

On the other hand, a moments' thought reveals that $\widetilde{Z_{R_k^k}^P(S)} = Z_{R_k^k}^{\tilde{P}}(\tilde{S})$ and that $bl(S) = bl(Z_{R_k^k}^P(S))$, so we find, as desired, that

$$\begin{aligned} \frac{\tau(Z_{R_k^k}^P(S))}{\tilde{\tau}(Z_{R_k^k}^{\tilde{P}}(\tilde{S}))} &= \frac{\tau(Z_{R_k^k}^P(S))}{\tilde{\tau}(\widetilde{Z_{R_k^k}^P(S)})} \\ &= \delta^{k-bl(Z_{R_k^k}^P(S)))} \\ &= \delta^{k-bl(S)} . \end{aligned}$$

*The case of* $T = E^k_{k+1}$: We need to verify that for any $S \in \mathcal{K}_{2(k+1)}$, we have

$$\frac{\tau(Z^P_{E^k_{k+1}}(S))}{\tau(S)} = \frac{\tilde{\tau}(Z^{\tilde{P}}_{E^k_{k+1}}(\tilde{S})}{\tilde{\tau}(\tilde{S})} \ . \tag{4.9}$$

It should be observed that we have a 'tangle equation'

$$tr^{0+}_k \circ E^k_{k+1} = tr^{0+}_{k+1}$$

which implies that both sides of equation (4.9) are equal to one.

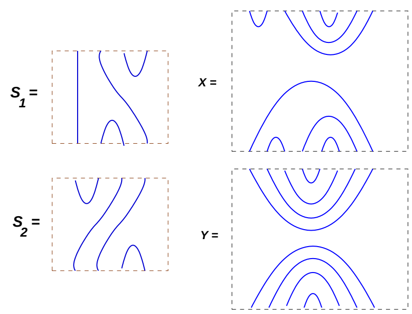
*The case of* $T = M^n_{n,n}$: Equation (4.6) translates, in this case, to

$$\frac{\tau(S_1S_2)}{\tau(S_1)\tau(S_2)} = \frac{\tilde{\tau}(\widetilde{S_1}\widetilde{S_2})}{\tilde{\tau}(\widetilde{S_1})\tilde{\tau}(\widetilde{S_2})} \ \forall \ S_1, S_2 \in \mathcal{K}_{2n}.$$

We shall prove that

$$\frac{\tau(S_1)}{\widetilde{\tau}(\widetilde{S_1})}\frac{\tau(S_2)}{\widetilde{\tau}(\widetilde{S_2})} = \frac{\tau(S_1S_2)}{\widetilde{\tau}(\widetilde{S_1S_2})} \ \forall \ S_1, S_2 \in \mathcal{K}_{2n}. \tag{4.10}$$

Suppose $S_1, S_2 \in \mathcal{K}_{2n}$. Consider the element $X \in \mathcal{K}_{4n}$ (resp., $Y \in \mathcal{K}_{4n}$) defined by requiring that (i) $X$ (resp., $Y$) has no through string, (ii) the 'top half' of $X$ (resp., $Y$) is ${}^*\cup S_2$ (resp., ${}^*\cup I_{2n}$) and (iii) the 'bottom half' of $X$ (resp., $Y$) is $\cap_* S_1$ (resp., $\cap_* I_{2n}$). (See example below.)




Notice that both $X$ and $Y$ have no *through strings* - this phrase having an obvious meaning. (In the above example, the element $S \in \mathcal{K}_4$ has two through strings.)

It is easy to see that

$$\begin{aligned} \tau(XY) &= \tau(S_1)\tau(S_2) \\ \tau(X) &= \tau(S_1S_2) \\ \tau(Y) &= \delta^{2n} \end{aligned}$$

and similarly that

$$\begin{aligned} \widetilde{\tau}(\widetilde{X}\widetilde{Y}) &= \widetilde{\tau}(\widetilde{S_1})\widetilde{\tau}(\widetilde{S_2}) \\ \widetilde{\tau}(\widetilde{X}) &= \widetilde{\tau}(\widetilde{S_1S_2}) \\ \widetilde{\tau}(\widetilde{Y}) &= \delta^{2n} \end{aligned}$$

It follows that it is sufficient to prove equation (4.10) in the special case when neither $S_1$ nor $S_2$ has any through strings.

Suppose then that $S_1, S_2 \in \mathcal{K}_{2n}$ have no through strings, and that their 'top' and 'bottom' halves are ${}^*\cup S_{1+}, {}_*\cap S_{1-}$ and ${}^*\cup S_{2+}, {}_*\cap S_{2-}$ respectively, with $S_{1\pm}, S_{2\pm} \in \mathcal{K}_n$. Deduce from the linearisation lemma (4.1) that

$$\begin{aligned} log_\delta \frac{\tau(S_1)}{\widetilde{\tau}(\widetilde{S_1})} &= |S_{1+} \bigvee S_{1-}| - 2|\widetilde{S_{1+}} \bigvee \widetilde{S_{1-}}| \\ &= n - |\widetilde{S_{1+}}| - |\widetilde{S_{1-}}| \end{aligned}$$

and similarly

$$log_\delta \frac{\tau(S_2)}{\widetilde{\tau}(\widetilde{S_2})} = n - |\widetilde{S_{2+}}| - |\widetilde{S_{2-}}|$$

while

$$\begin{aligned} log_\delta \frac{\tau(S_1S_2)}{\widetilde{\tau}(\widetilde{S_1S_2})} &= |S_{1+} \bigvee S_{2-}| + |S_{1-} \bigvee S_{2+}| - 2|\widetilde{S_{1+}} \bigvee \widetilde{S_{2-}}| - 2|\widetilde{S_{1-}} \bigvee \widetilde{S_{2+}}| \\ &= n - |\widetilde{S_{1+}}| - |\widetilde{S_{2-}}| + n - |\widetilde{S_{1-}}| - |\widetilde{S_{2+}}| \\ &= log_\delta \frac{\tau(S_1)}{\widetilde{\tau}(\widetilde{S_1})} + log_\delta \frac{\tau(S_2)}{\widetilde{\tau}(\widetilde{S_2})} \end{aligned}$$

as desired.

*The case of* $T = I_k^{k+1}$: It is easy to see from the definitions that

$$\begin{aligned} \frac{\tau(Z^P_{I_k^{k+1}}(S))}{\tau(S)} &= \delta^2 \\ &= \frac{\widetilde{\tau}(Z^{\tilde{P}}_{I_k^{k+1}}(\tilde{S}))}{\widetilde{\tau}(\tilde{S})} \end{aligned}$$

for every $S \in \mathcal{K}_{2k}$, and the proof of Theorem 4.2 is finally complete. □

Note that for any non-zero $\delta \in \mathbb{C}$, the planar algebra $TL(\delta)$ has a natural $*$-planar algebra structure where the (conjugate-linear) involution $*$ is defined on the basis elements of $TL_k(\delta)$ by flipping the Kauffman diagrams about the horizontal axis. Further it is known - see [J1] and [GHJ] - that if $\delta \geq 2$, then $TL(\delta)$ is a $C^*$-planar algebra and that if $\delta = 2\ cos(\frac{\pi}{n})$ for $n \geq 3, n \in \mathbb{N}$ then $TL(\delta)$ has a quotient that is a $C^*$-planar algebra. Each of these $C^*$-planar algebras is the standard invariant of the corresponding Jones subfactor of index $\delta^2$.

Similarly, there is a natural $*$-planar algebra structure on $NC(\delta)$ defined on the basis elements of $NC_k(\delta)$ by 'horizontal flip'. Note that with this definition, each $\phi_k$ is a $*$-isomorphism from ${}^{(2)}TL_k(\delta)$ to $NC_k(\delta^2)$ and it follows that for $\delta \geq 2$, $NC(\delta^2)$ is a $C^*$-planar algebra while if $\delta = 2\ cos(\frac{\pi}{n})$ for $n \geq 3, n \in \mathbb{N}$, then $NC(\delta^2)$ admits a quotient that is a $C^*$-planar algebra. Each of these planar algebras is the standard invariant of the one-step basic construction subfactor $N \subseteq M_1$ for the corresponding Jones subfactor $N \subseteq M$ of index $\delta^2$.

## References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