STRUCTURE OF VON NEUMANN ALGEBRAS OF TYPE III

Masamichi Takesaki

Abstract. In this lecture, we will show that to every von Neumann algebra $\mathcal{M}$ there corresponds unquely a covariant system $\left\{\widetilde{\mathcal{M}}, \tau, \mathbb{R}, \theta\right\}$ on the real line $\mathbb{R}$ in such a way that

$$\mathcal{M} = \widetilde{\mathcal{M}}^{\theta}, \quad \tau\circ\theta_s = e^{-s}\tau, \quad \mathcal{M}' \cap \widetilde{\mathcal{M}} = \mathcal{C},$$

where $\mathcal{C}$ is the center of $\widetilde{\mathcal{M}}$. In the case that $\mathcal{M}$ is a factor, we have the following commutative square of groups which describes the relation of several important groups such as the unitary group $\mathcal{U}(\mathcal{M})$ of $\mathcal{M}$, the normalizer $\widetilde{\mathcal{U}}(\mathcal{M})$ of $\mathcal{M}$ in $\widetilde{\mathcal{M}}$ and the cohomology group of the flow of weights: $\{\mathcal{C}, \mathbb{R}, \theta\}$:

$$\begin{tikzcd}
& 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathbb{T} \arrow[r] \arrow[d] & \mathcal{U}(\mathcal{C}) \arrow[r, "\partial_\theta"] \arrow[d] & \mathrm{B}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathcal{U}(\mathcal{M}) \arrow[r] \arrow[d, "\mathrm{Ad}"'] & \widetilde{\mathcal{U}}(\mathcal{M}) \arrow[r, "\partial_\theta"] \arrow[d, "\widetilde{\mathrm{Ad}}"'] & \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Int}(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{Cnt_r}(\mathcal{M}) \arrow[r, "\dot{\partial}_\theta"] \arrow[d] & \mathrm{H}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \arrow[r] \arrow[d] & 1 \\
& 1 & 1 & 1 &
\end{tikzcd}$$

Contents

Lecture 0. History of Structure Analyis of von Neumann algebras of type III.

The foundation of the theory of von Neumann algebras was laid down by John von Neumann together with a collaborator Francis Joseph Murray in a series of the fundamental papers "Rings of Operators" during the period of 1936 through 1943, although the fundamental bicommutant theorem was proven in 1929. The reason for this 7 year gap was that John von Neumann believed the Wedderbun type theorem for von Neumann algebra: all von Neumann algebras are of type I, unbelievable mistake by the genius. During their active perod, there was no one followed their work, except possibly the work of Gelfand and Neumark in 1943 who succeeded to have given postulates of a C*-algebra in a set of simple axioms with an additional condition which is now known to be unnecessary. The decade after the WW II, there was a tidal wave on the theory of operator algebras notably in the US, the Chicago school, and France lead by Jacque Dixmier. In this period, the knowledge of the topologgical properties of an operator algebra was greatly increased, which allowed the people handles the infinite dimensional non-commutative objects successfully. It is safe to say that the misterious subject introduced by the two pioneers is more or less digested by the specialists by then. The fashion after the WWII was somewhat gone by 1955. However, there were quite a few difficult but important problems were left untouched. The specialists lead by R. Kadison continued to attack the field and obtained deep results although the field remained unpopular, notaly Kadison's transitivity theorem, Glimm's result on type I C*-algebras, Sakai's characterization of von Neumann algebra as a C*-algebra whichi is a dual Banach space. In the 1960's, there was a dramatic change occured: the invasion of theoretical physicists lead by Rudolf Haag, Daniel Kastler, Nico M. Hugenholtz, Hans Borchers, David Ruel, Huzihiro Araki, Derek Robinson and many others who brought in a set of new ideas from the point of view of physics. It changed the scope the field greatly. New ideas and techniques were brought in. A great deal of progress started to follow. The number of specialists increased dramatically too. Then a big brow came when Araki proved in 1964, [Ark], that the most of von Neumann algebras occuring in the theoretical physics were of type III as there was no general method to treat the case of type III. Generally speaking, during the 60's the mathematical physicists lead the field: for example in 1967, [RTPW], R.T. Powers proved the existence of continuously many non-isomorphic factors of type III, which are often called the Powers' factors although they appeared in the work of J. von Neumann in 1938 and 1940 and were studied by L. Pukanszky in 1956, [Pksky]. After the work of Powers, Araki and Woods,[ArWd], worked on infinite tensor product of factors of type I, abbreviated ITPFI, and classified them with invariant called the asymptotic ratio set and denoted $r_\infty(\mathcal{M})$ for such a factor $\mathcal{M}$ toward the end of the 60's. But a von Neumann algebra of type III are still hiden deep in mistery. For instance the commutation theorem $(\mathcal{M}\overline{\otimes}\mathcal{N})' = \mathcal{M}'\overline{\otimes}\mathcal{N}'$ was unkown, while the tensor product $\mathcal{M}\overline{\otimes}\mathcal{N}$ was shown to be of type III if either of the components is of type III, proven by S. Sakai in 1957, [Sk]. So the theory for a von Neumann algebra of type III was badly needed when Tomita proposed his theory at the Baton Rouge Meeting in the spring of 1967. But his preprint was very poorly written and full of poor mistakes: nobady bothers to check the paper. When I wrote to Dixmier in the late spring of 1967 about the validity of Tomita's work, he responded by saying that he was unable to go beyond the third page and mentioned that it was very improtant to decide the validity of Tomita's work. At any rate, Tomita's work was largely ignored by the participants of the Baton Rouge Conference.

For the promiss I made to Hugenholtz and Winnink, I started very seriously in April, 1967, after returning from the US and was able to resque all the major results: not lemmas and small propositions, many of which are either wrong or nonsense. Then I spent the academic year of 1968 through 1969 at Univ. of Pennsylvania, where R.V. Kadison, S. Sakai, J.M. Fell, E. Effros, R.T. Powers, E. Størmer and B. Vowdon were, but non of them believed Tomita's result. So I checked once more and wrote a very detailed notes which was later published as Springer Lecture Notes No.128: simplification was not an issue, but the validity of Tomita's claim. Through writing up the notes, I dicovered that Tomita's work could go much further than his claim: the modular condition, (called the KMS-condition by physicists), and a lot more. I know that the crossed product of a von Neumann algebra by the modular automorphism group is semi-finite which I didn't include in the lecture notes because I thought that the semi-finiteness alone was a half cocked claim. When I mentioned firmly the validity of Tomita's claim, the people at the U. of Pennsylvania decided to run an inspection seminar in which I was allowed to give only the first introductory talk, but not in subsequent seminars, which run the winter of 1969 through the spring and the validity was established at the end. The miracle match of Tomita's work and HHW work produced the theory of the modular automorphism group. I lectured this at UCLA for the academic year of 1969 through 1970 and produced UCLA Lecture Notes which were widely distributed. Among the audience of my lectures were H.A. Dye, R. Herman, M. Walter and Colin Sutherland.

The campus disturbance in Japan prevented me from going home in Sendai. I decided to stay in the US and UCLA offered me a professorship. In the academic year of 1970/71, Tulane University hosted a special year project on Ring Theory and invited me to participate. I participated only the Fall quarter, and lectured on the duality on locally compact groups based on Hopf von Neumann algebra techniques, [Tk4], which were later completed as the theory of Kac algebras, [EnSw], by Enock and Scwartz, students of J. Dixmier. Through these lectures, I recognized the need of unbounded functionals, i.e., the theory of weights, which motivated the joint work with Gert K. Pedersen in the succeeding year, [PdTk].

In the spring of 1971, UCLA hosted a special quarter project on operator algebras which invited three major figures: R. Haag, N.M. Hugenholtz and R.V. Kadson along with Gert K. Pedersen who stayed there throughout the spring quoter of 1971. Short term visitors were J. Glimm, J. Ringrose, B. Johnson and many others. The summer of 1971 was a turning point for the theroy of operator algebras as A. Connes participated in the summer school at Battelle Institue Seattle on the mathematical aspect of statistical mechanics, organized by D. Ruelle, O. Landford and J. Lebowitz.

It was this Battelle Institute Summer School which motivated Connes to work on operator algebras. At the time of the summer school, he wasn't sure what he wanted to work on. But the Summer School gave him enough kick to determine his mind to work on operator algeberas despite discouragement from his supervisor J. Dixmier. At any rate, soon after the summer school, Connes started to produce a series of excellent works on the structure analysis of factors of type III introducing his famous invariant $S(\mathcal{M})$ and $T(\mathcal{M})$ and devided factors of type III into those of type III_λ , $0 \leq \lambda \leq 1$. During this period, I was in close contact with Connes. Maybe it is not an exaggeration to say that the period of 1972 through 1977 recorded historically high speed development in the theory of operator algebras. For my part, for the period of 71 through the spring of 72, I was engaged in the advancement of the theory of weights with Gert K. Pedersen as I recognized the need of the theory of weights in the course of the duality theory in which I was also very much interested. Then in the spring of 1972, W. Arveson came down to UCLA from Berkeley to deliver his new discovery of the constructive proof of a theorem of Borchers, [Brch], which asserts that the energy is a limit of local observables: before him the result rest on the derivation theorem of Sakai which is not constructive as it relies on the compactness of a relevant object. Arveson's theory was so beautiful that I couldn't resist to try on the structure analysis of a factor of type III. Also his theory equally inspiered Connes to develop his theory of Connes spectrum $\Gamma(\alpha)$ of an action α of a locally compact abelian group on a von Neumann algebra. The summer of 1972 was quite dramatic for the structure analysis of a factor of type III on the both sides of Atlantic Oceaan. On the west, Araki was staying at Kingston with E.J. Woods as the host and working on the structure analysis of factors of type III which overlapped heavily with mine and collided at a meeting in Austin, Texas, where the both of us presented almost same results on the structure of a factor of type III. It was E. J. Woods who recognized immediately that the collaboration of myself and Araki would produce a remarkable progress in the structure theory of factors of type III and invited myself to Kingston for the summer of 72. The structure analysis of factors of type III developped in a breath taking pase: on the eastern side Connes was developping his own structure analysis of factor of type III_λ, $0 \leq \lambda < 1$, while Araki and myself are working on a factor of type III which admists a special kind of states.

Observing the rapid progress in Connes' work, J. Dixmier advised him to postpone his Ph D for another year to complete a comprehensive and complete Ph D thesis and Connes followed his advise: thus his historic thesis which does not lose its glory today at all. The fall of 1972, I was able to prove the duality theorm which takes care of all von Neumann algebras of type III. Then I received a phone call from Daniel Kastler inviting me to spend the academic year of 1973/ 74 in Maraseille along with Connes. At that time Connes agreed to apply for the Hedricks Assistant Professorship at UCLA, but Kastler's offer is more attractive. He invited me along with my students: Trond Digernes and Hiroshi Takai. So I accepted the invitation. Guggenheim Foundation also supported my plan to spend a year in Marseille. Then Connes visited me at UCLA in February of 1973, with which we started to work on the flow of weights: of course we didn't think about the title. We both just knew that the duality theorem was not the end of the story, instead it was a beginning of something much deeper theory. The collaboration went very successfully: birth of the theory of flow of weights in which we have proved a technically demanding result: the relative commutanat theorem.

The year 1973 - 74 has gone very quickly too. It took another year to complete the flow of weights paper. After this year, Connes went to Kingston to fulfil his military duty as a French citizen. There he proved his famous results on injective factors and the classification of a single automorphism of an AFD factor, leaving me far behind.

In the succeeding year, I worked on the further completion of the structure theory of a factor of type III, which resulted the canonical construction of the core of a factor of type III and the cocycle conjgacy analysis of amenable discrete countable group actions on an AFD factor, following the work of Vaughan F.R. Jones and Adrian Ocneanu.

Lecture 1. Covariant System and Crossed Product.

§1.1. Topology on Aut($\mathcal{M}$). Let $\mathcal{M}$ be a von Neumann algebra. An automorhism α of $\mathcal{M}$ means a bijective linear map $\alpha : x \in \mathcal{M} \mapsto \alpha(x) \in \mathcal{M}$ such that

$$\begin{matrix} \alpha(xy) = \alpha(x)\alpha(y), \\ \alpha(x^*) = \alpha(x)^*, \end{matrix} \quad x, y \in \mathcal{M}.$$

The set $\mathrm{Aut}(\mathcal{M})$ of all automorphisms of $\mathcal{M}$ forms a group under the composition of maps. If there exists a unitary $u \in \mathcal{M}$ such that

$$\mathrm{Ad}(u)(x) \stackrel{\text{def}}{=} uxu^*,^{1} \quad x \in \mathcal{M},$$

then the automorphism $\mathrm{Ad}(u)$ is called **inner**. The set $\mathrm{Int}(\mathcal{M})$ of all inner automorphims forms a normal subgroup of $\mathrm{Aut}(\mathcal{M})$. We denote by

$$\mathrm{Out}(\mathcal{M}) \stackrel{\text{def}}{=} \mathrm{Aut}(\mathcal{M})/\mathrm{Int}(\mathcal{M})$$

the quotient group of $\mathrm{Aut}(\mathcal{M})$ by $\mathrm{Int}(\mathcal{M})$.

For each $\alpha \in \mathrm{Aut}(\mathcal{M})$, we consider the following;

$$U(\alpha : \omega_1, \cdots, \omega_n) \stackrel{\text{def}}{=} \left\{ \beta \in \mathrm{Aut}(\mathcal{M}) : \begin{matrix} \|\omega_i \circ \alpha - \omega_i \circ \beta\| < 1, \\ \left\|\omega_i \circ \alpha^{-1} - \omega_i \circ \beta^{-1}\right\| < 1, \end{matrix} \quad 1 \le i \le n \right\}.$$

Then the family $\{U(\alpha : \omega_1, \cdots, \omega_n) : \omega_1, \cdots, \omega_n \in \mathcal{M}_*\}$ defines a topology which makes $\mathrm{Aut}(\mathcal{M})$ a topological group.

Remark 1.1. *The group* $\mathrm{Int}(\mathcal{M})$ *is* **not** *necessarily closed.*

Under this neighborhood system, $\mathrm{Aut}(\mathcal{M})$ is a complete topological group. In the case that $\mathcal{M}_*$ is separable[2], $\mathrm{Aut}(\mathcal{M})$ is a Polish group.

§1.2. Locally Compact Group. Now let G be a locally compact group with left invariant Haar measure.[3] Let us fix basic notations concerning locally compact group G. We take the left Haar measure $\mathrm{d}s$ and consider

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[1]The notation $\stackrel{\text{def}}{=}$ defines the left by the right of the equality sign.

[2]We call $\mathcal{M}$ **separable** when the predual $\mathcal{M}_*$ is seprable. A separable von Neumann algebra is not separable in norm. In fact, a seperable von Neumann algebra in norm is finite dimensional.

[3]For those who are not familier with the general theory of locally compact group, one may take the additive group $\mathbb{R}$ of real numbers.

the following Banach spaces relative to the left Haar measure:

$$\int_G f(st)\mathrm{d}t = \int_G f(t)\mathrm{d}t, \quad f \in C_c(G);$$
$$\int_G f(st)\mathrm{d}s = \delta_G(t)\int_G f(s)\mathrm{d}s, \quad f \in C_c(G);$$
$$L^1(G) = \left\{ f : \|f\|_1 = \int_G |f(s)|\mathrm{d}s < +\infty \right\};$$
$$L^2(G) = \left\{ f : \|f\|_2 = \left(\int_G |f(s)|^2 \mathrm{d}s \right)^{\frac{1}{2}} < +\infty \right\};$$
$$L^\infty(G) = \{ f : |\{g \in G : |f(s)| \geq r\}| = 0 \text{ for some } r \in \mathbb{R}_+ \}^4;$$
$$\|f\|_\infty = \inf\{r > 0 : |\{s \in G; |f(s)| > r\}| = 0\}.$$

The Banach space $L^1(G)$ admits an involutive Banach algebra structure:

$$\begin{aligned} (f * g)(t) &= \int_G f(s)g\left(s^{-1}t\right)\mathrm{d}s; \\ f^*(s) &= \delta_G(s)^{-1}\overline{f(s^{-1})}, \end{aligned} \qquad f, g \in L^1(G).$$

We also introduce the notation:

$$f^\vee(s) = f\left(s^{-1}\right)$$

On the Hilbert space $L^2(G)$ of square integrable functions on G relative to the left Haar measure, we define unitary representations of G:

$$\begin{aligned} (\lambda_G(s)\xi)(t) &= \xi\left(s^{-1}t\right); \\ (\rho_G(t)\xi)(s) &= \delta_G(t)^{-\frac{1}{2}}\xi(st), \end{aligned} \qquad \xi \in \mathfrak{H} = L^2(G), s, t \in G.$$

They are called the left and right regular representations of G respectively. Integrating λ_G and ρ_G, we get the $*$-representations of $L^1(G)$:

$$\begin{aligned} (\lambda_G(f)\xi)(t) &= \int_G f(s)(\lambda_G(s)\xi)(t)\mathrm{d}s \\ &= \int_G f(s)\xi\left(s^{-1}t\right)\mathrm{d}s = (f * \xi)(t); \\ (\rho_G(f)\xi)(s) &= \int_G f(t)(\rho_G(t)\xi)(s)\mathrm{d}t = \int_G f(t)\delta_G(t)^{-\frac{1}{2}}\xi(st)\mathrm{d}t \\ &= \int_G f\left(s^{-1}t\right)\delta_G\left(s^{-1}t\right)^{-\frac{1}{2}}\xi(t)\mathrm{d}t = \left(\xi * \left(\delta_G^{-\frac{1}{2}}f\right)^\vee\right)(s). \end{aligned}$$

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[4]The notation $|E|$ for a Borel subset $E \subset G$ means the measure of E relative to a fixed Haar measure

An **action** of G on a von Neumann algebra $\mathcal{M}$ is a continuous map: $g \in G \mapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$ such that

$$\alpha_{gh} = \alpha_g \circ \alpha_h, \quad g, h \in G.$$

We call the system $\{\mathcal{M}, G, \alpha\}$ a **covariant system** on G. In the case that $G = \mathbb{R}$, the action α is called a **one parameter automorphism group**. Indeed, this is most relevant to us for the structure theory of von Neumann algebra of type III.

§**1.3. Perturbaton by Cocyles and Various Equivalence of Actions.**

When two actions α on $\mathcal{M}$ and β on $\mathcal{N}$ of a locally compact group G are considered, we say that α and β are **conjugate** and write $\alpha \cong \beta$ if there exists an isomorphism $\sigma : \mathcal{M} \mapsto \mathcal{N}$ such that

$$\sigma \circ \alpha_g \circ \sigma^{-1} = \beta_g, \quad g \in G.$$

A continuous map: $g \in G \mapsto u(g) \in \mathcal{U}(\mathcal{M})$ which satisfies the following cocycle condition:

$$u(gh) = u(g)\alpha_g(u(u_h)), \quad g, h \in G,$$

is called an α**-cocycle,** or more precisely an α**-one cocycle.** We denote the set of all α-cocycles by $\mathrm{Z}_\alpha(G, \mathcal{U}(\mathcal{M}))$.

For each α-cocycle $u \in \mathrm{Z}_\alpha(G, \mathcal{U}(\mathcal{M}))$, we define the perturbation ${}_u\alpha$ of α by the cocycle u as follows:

$${}_u\alpha_g = \mathrm{Ad}(u(g)) \circ \alpha_g, \quad g \in G,$$

which is a new action of G on $\mathcal{M}$. The cocycle identity guarantees that the perturbed ${}_u\alpha$ is an action as seen via a simple computation. We say that ${}_u\alpha$ the **pertubed action** of G by the cocycle u.

For each $w \in \mathcal{U}(\mathcal{M})$, we set

$$(\partial_w u)(g) = w^* u(g) \alpha_g(w).$$

It is easily seen that

$${}_{(\partial_w u)}\alpha_g = \mathrm{Ad}(w)^{-1} \circ {}_u\alpha_g \circ \mathrm{Ad}(w), \quad g \in G,$$

so that ${}_u\alpha \cong {}_{(\partial_w(u))}\alpha$. More precisely, we write with $v = \partial_w u, b \in \mathrm{Z}_\alpha(G, \mathcal{U}(\mathcal{M}))$,

$${}_u\alpha \cong {}_v\alpha \quad \mathrm{mod} \quad \mathrm{Int}(\mathcal{M}).^5$$

[5]The notation mod $\mathrm{Int}(\mathcal{M})$ indicates the conjugation is actually given by the group $\mathrm{Int}(\mathcal{M})$ of inner automorphisms.

The set of all cocycle peturbations of an action α of G will be denoted by $[\alpha]$ and called the cocycle perturbation class of α. For each $\beta \in [\alpha]$, i.e., $\beta_g = {}_u\alpha_g, g \in G$, for some α-cocycle u, we consider the set $\mathrm{Z}_\beta(G, \mathcal{U}(\mathcal{M}))$ of all β-cocycles, and

$$\mathrm{Z}_{[\alpha]}(G, \mathcal{U}(\mathcal{M})) = \bigcup \{\mathrm{Z}_\beta(G, \mathcal{U}(\mathcal{M})) : \beta = {}_u\alpha \text{ for some } u \in \mathrm{Z}_\alpha(G, \mathcal{U}(\mathcal{M}))\}.$$

Each $u \in \mathcal{U}(\mathcal{M})$ acts on $\mathrm{Z}_{[\alpha]}(G, \mathcal{U}(\mathcal{M}))$ by the following:

$$\partial_u(v)(g) = u^* v(g)\beta_g(u), \quad g \in G,\ v \in \mathrm{Z}_\beta(G,\ \mathcal{U}(\mathcal{M})),\ \beta \in [\alpha].$$

We denote the set of all $\partial_{\mathcal{U}(\mathcal{M})}$-orbits by the following:

$$\mathrm{H}_{[a]}(G, \mathcal{U}(\mathcal{M})) = \mathrm{Z}_{[\alpha]}(G, \mathcal{U}(\mathcal{M}))/\partial(\mathcal{U}(\mathcal{M})).$$

Two actions α and β of G on $\mathcal{M}$ are said to be **cocycle conjugate** if there exists $u \in \mathrm{Z}_\alpha(G, \mathcal{U}(\mathcal{M}))$ such that

$$\beta \cong {}_u\alpha.$$

§**1.4. Crossed Product of a von Neumann algebra $\mathcal{M}$ by an Action of a Locally Compact Group G.** Fix a covariant system $\{\mathcal{M}, G, \alpha\}$. We represent $\mathcal{M}$ on a Hilbert space $\mathfrak{H}$ faithfully. We view the Hilbert space $\mathfrak{K} = \mathfrak{H} \otimes L^2(G)$ as the Hilbert space $L^2(\mathfrak{H}, G)$ of all $\mathfrak{H}$-valued square integrable functions on G relative to the left Haar measure. Set

$$\begin{aligned} (\pi_\alpha(x)\xi)(g) &= \alpha_g^{-1}(x)\xi(g), \\ (u_G(h)\xi)(g) &= \xi\big(h^{-1}g\big), \end{aligned} \quad \xi \in \mathfrak{K} = L^2(\mathfrak{H}, G), x \in \mathcal{M}.$$

It then follows that π_α is a normal faithful representation of $\mathcal{M}$ and u is a unitary representation of G on $\mathfrak{K}$ which are linked in the following way:

$$u(g)\pi_\alpha(x)u(g)^* = \pi_\alpha(\alpha_g(x)), \quad g \in G, x \in \mathcal{M},$$

the relation called the **covariance.** The **crossed product** $\mathcal{M} \rtimes_\alpha G$ is by definition the von Neumann algebra generated by $\pi_\alpha(\mathcal{M})$ and $u(G)$.

Theorem 1.2. *The crossed product $\mathcal{M} \rtimes_\alpha G$ does not depend on the Hilbert space on which $\mathcal{M}$ acts.*

Sketch of Proof. Suppose that $\mathcal{M}$ is represented on Hibert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ faithfully. Consider the direct sum Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and write the actions of $\mathcal{M}$ on $\mathfrak{H}_1$ by π_1 and on $\mathfrak{H}_2$ by π_2 respectively. Then represent $\mathcal{M}$ diagonally on $\mathfrak{H}$, i.e.

$$x \in \mathcal{M} \mapsto \pi(x) = \begin{pmatrix} \pi_1(x) & 0 \\ 0 & \pi_2(x) \end{pmatrix} \in \mathcal{L}(\mathfrak{H}).$$

Let e_1 and e_2 be the projections to the first and second components of $\mathfrak{H}$ respectively. Then we have

$$\{\pi_1(\mathcal{M}), \mathfrak{H}_1\} \cong \pi(\mathcal{M})_{e_1}, \quad \{\pi_2(\mathcal{M}), \mathfrak{H}_2\} \cong \pi(\mathcal{M})_{e_2}.$$

Consider the Hilbert space $\mathfrak{K} = L^2(\mathfrak{H}, G)$ on which $\pi(\mathcal{M}) \rtimes_\alpha G$ is represented. Then the Hilbert space $\mathfrak{K}$ is the direct sum of $\mathfrak{K}_1 = L^2(\mathfrak{H}_1, G)$ and $\mathfrak{K}_2 = L^2(\mathfrak{H}_2, G)$. Now the crossed product $\pi(\mathcal{M}) \rtimes_\alpha G$ is generated by two kinds of operators:

$$\pi_\alpha(x), \quad x \in \pi(\mathcal{M}), \qquad u_G(s), \quad s \in G.$$

Let f_1 and f_2 be the projections of $\mathfrak{K}$ to $\mathfrak{K}_1$ and $\mathfrak{K}_2$ respectively, i.e.,

$$f_1 = e_1 \otimes 1_{L^2(G)}, \quad f_2 = e_2 \otimes 1_{L^2(G)}.$$

Now we observe that

$$e_1,\ e_2 \in \pi(\mathcal{M})';$$
$$\bigvee \{ve_1v^* : v \in \mathcal{U}(\pi(\mathcal{M})')\} = 1 = \bigvee \{we_1w^* : w \in \mathcal{U}(\pi(\mathcal{M})')\}.$$

As $v \otimes 1_{L^2(G)} \in \mathcal{U}\big((\pi(\mathcal{M}) \rtimes_\alpha G)'\big)$ for every $v \in \mathcal{U}(\pi(\mathcal{M})')$. we have

$$\bigvee \big\{vf_iv^* : v \in \mathcal{U}\big((\pi(\mathcal{M}) \rtimes_\alpha G)'\big)\big\} = 1, \quad i = 1, 2.$$

Hence the maps $x \in \pi(\mathcal{M}) \rtimes_\alpha G \mapsto xf_i \in \pi_i(\mathcal{M}) \rtimes_\alpha G, i = 1, 2$, are isomorphisms. Hence we have

$$\pi_1(\mathcal{M}) \rtimes_\alpha G \cong \pi(\mathcal{M}) \rtimes_\alpha G \cong \pi_2(\mathcal{M}) \rtimes_\alpha G.$$

♡

Let $g \in G \mapsto U(g) \in \mathcal{U}(\mathfrak{H})$ be a unitary reprresentation of G on the Hilbert space on which $\mathcal{M}$ acts also. If we have

$$U(g)xU(g)^* = \alpha_g(x), \quad x \in \mathcal{M}, g \in G,$$

then we say that the action $\mathcal{M}$ on $\mathfrak{H}$ and the unitary representation U of G are **covariant** and/or that the covariant system $\{\mathcal{M}, G, \alpha\}$ acts on $\mathfrak{H}$ **covariantly.**

Theorem 1.3. *Suppose that an action of a covariant system $\{\mathcal{M}, G, \alpha\}$ on a Hilbert space $\mathfrak{H}$ is covariant via a unitary representation U of G on $\mathfrak{H}$, then we have the following*:

i) *Let π be the representation of the commutant $\mathcal{M}'$ on $\mathfrak{K} = \mathfrak{H} \otimes L^2(G)$ given by the following*:

$$(\pi(y)\xi)(g) = y\xi(g), \quad y \in \mathcal{M}', \quad g \in G, \quad \xi \in \mathfrak{K}.$$

Then define $v_G : g \in G \mapsto v_G(g) \in \mathcal{U}(\mathfrak{K})$ *by the following*:

$$(v_G(g)\xi)(h) = \delta_G(g)^{-\frac{1}{2}} U(g)\xi(hg), \quad g, h \in G,^{6}$$

where $\delta_G : g \in G \mapsto \delta_G(g) \in \mathbb{R}_+^*$ *be the modular function of* G *which makes the above* v_G *a unitary representation* G *on* $\mathfrak{K}$. *The representation* π *of the commutant* $\mathcal{M}'$ *of* $\mathcal{M}$ *and unitary representation* v *of* G *is covariant in the sense that*

$$U(g)\pi(y)U(g)^* = \pi\big(\alpha'_g(y)\big), \quad y \in \mathcal{M}', \quad g \in G,$$

where α' *is the action of* G *on the commutant* $\mathcal{M}'$ *given by the following*:

$$\alpha'_g(y) = U(g)yU(g)^*, \quad y \in \mathcal{M}', \quad g \in G.$$

ii) *The commutant of the crossed product* $\mathcal{M} \rtimes_\alpha G$ *represented on* $\mathfrak{K}$ *is given by*

$$(\mathcal{M} \rtimes_\alpha G)' = \{\pi(\mathcal{M}') \cup v_G(G)\}''.$$

iii) *The von Neumann algebra* $(\mathcal{M} \rtimes_\alpha G)'$ *is naturally isomorphic to the crossed product* $\mathcal{M}' \rtimes_{\alpha'} G$.

iv) *If* $u \in \mathrm{Z}_\alpha(G, \mathcal{U}(\mathcal{M}))$, *then the unitary* U *defined by the following*:

$$(U\xi)(g) = u\big(g^{-1}\big)\xi(g), \quad g \in G, \xi \in \mathfrak{K},$$

gives a spatial isomorphism:

$$U(\mathcal{M} \rtimes_\alpha G)U^* = \mathcal{M} \rtimes_{{}_u\alpha} G.$$

Thus the isomorphism class of the crossed product is stable under a cocycle perturbation.

Corollary 1.4. *Let* ρ_G *be the right regular representation of* G *on the Hilbert space* $L^2(G)$, *i.e.,*

$$(\rho_G(g)\xi)(h) = \delta_G(g)^{-\frac{1}{2}}\xi(hg), \quad \xi \in L^2(G), \quad g, h \in G.$$

Then the unitary representation ρ_G *gives rise to an action* ρ *of* G *on* $\mathcal{L}\big(L^2(G)\big)$ *as follows*:

$$\rho_g(x) = \rho_G(g)x\rho_G(g)^*, \quad x \in \mathcal{L}\big(L^2(G)\big), \quad g \in G,$$

[6] The unitary representation v_G is the tensor product $U \otimes \rho_G$ of the unitary representation U and the right regular representation of G.

which has the following properties:

$$\mathcal{M} \rtimes_\alpha G \cong \big(\mathcal{M}\overline{\otimes}\mathcal{L}\big(L^2(G)\big)\big)^{\alpha\otimes\rho}$$

where $\mathcal{N}^\beta$ *for a covariant system* $\{\mathcal{N}, G, \beta\}$ *means the fixed point subalgebra of* $\mathcal{N}$ *under any action* β.

Unless G is discrete, there is no way to represent a general element of the crossed product $\mathcal{M} \rtimes_\alpha G$. But fortunately, there are prenty many elements of the crossed product $\mathcal{M} \rtimes_\alpha G$ which are written in the form:

$$x = \int_G \pi_\alpha(x(g))u(g)\mathrm{d}g,$$

where $g \in G \mapsto x(g) \in \mathcal{M}$ is integrable in the sense that

$$\|x\|_1 = \int_G \|x(g)\|\mathrm{d}g < +\infty.$$

With this in mind, we define an involutive Banach algebra structure on $L^1(\mathcal{M}, G)$, the space obtained as the completion of the set $C_c(\mathcal{M}, G)$ of all σ-strongly*continuous $\mathcal{M}$-valued functions on G under the L^1-norm:

$$\begin{aligned}(x * y)(g) &= \int_G x(h)\alpha_h(y(h^{-1}g))\mathrm{d}h;\\ x^*(g) &= \delta_G(g)^{-1}\alpha_g\big(x\big(g^{-1}\big)\big)^*; \qquad x, y \in L^1(\mathcal{M}, G).\\ \|x\|_1 &= \int_G \|x(g)\|\mathrm{d}g,\end{aligned}$$

For each $x \in L^1(\mathcal{M}, G)$, we set

$$\pi(x) = \int_G \pi_\alpha(x(g))u(g)\mathrm{d}g \in \mathcal{L}\big(L^2(\mathfrak{H}, G)\big).$$

Then π is a $*$-representation of $L^1(\mathcal{M}, G)$ and the image $\pi\big(L^1(\mathcal{M}, G)\big)$ is σ-weakly dense in $\mathcal{M} \rtimes_\alpha G$. The space $C_c(\mathcal{M}, G)$ of $L^1(\mathcal{M}, G)$ is a self-adjoint subalgebra and dense. The square $C_c(\mathcal{M}, G) * C_c(\mathcal{M}, G)$, i.e., the set of all linear combinations of the products:

$$C_c(\mathcal{M}, G) * C_c(\mathcal{M}, G) = \left\{\sum_{i=1}^n x_i * y_i : x_i, y_i \in C_c(\mathcal{M}, G), i = 1, \cdots, n\right\}$$

lies in the domain of the operator valued weight:

$$\mathcal{E}(x * y) = (x * y)(e) \in \mathcal{M}, \quad x, y \in C_c(\mathcal{M}, G),$$

where we identify $C_c(\mathcal{M}, G)$ with its image

$$\pi(C_c(\mathcal{M}, G)) \subset \mathcal{M} \rtimes_\alpha G.$$

We denote the domain of $\mathcal{E}$ and the positive part of the domain by the following:

$$\mathcal{D}(\mathcal{M}, G) = C_c(\mathcal{M}, G) * C_c(\mathcal{M}, G);$$

$$\mathcal{D}(\mathcal{M}, G)_+ = \left\{\sum_{i=1}^n x_i^* * x_i : x_1, \cdots, x_n \in C_c(\mathcal{M}, G)\right\}.$$

§**1.5 Dual Weight.** We fix a covariant system $\{\mathcal{M}, G, \alpha\}$. We now investigate a semi-finite normal weight on $\mathcal{M}$ and the corresponding semi-finite normal weight on $\mathcal{M} \rtimes_\alpha G$.

Theorem 1.4. *The map $\mathcal{E} : \mathcal{D}(\mathcal{M}, G)_+ \mapsto \mathcal{M}$ is extended to a semifinite normal faithful operator valued weight from $\mathcal{M} \rtimes_\alpha G$ to the subalgebra $\mathcal{M}$ where $\mathcal{M}$ is identified with its image $\pi_\alpha(\mathcal{M})$ in the crossed product $\mathcal{M} \rtimes_\alpha G$.*

Definition 1.5. For a semi-finite normal weight φ on $\mathcal{M}$, the semi-finite normal weight $\widehat{\varphi}$ on $\mathcal{M} \rtimes_\alpha G$ defined by the following:

$$\widehat{\varphi} = \varphi \circ \mathcal{E}$$

is called the **dual** weight of φ.

Theorem 1.6. *Under the above notations and the set up the dual weight $\widehat{\varphi}$ on $\mathcal{M} \rtimes_\alpha G$ of a faithful semi-finite normal weight φ enjoys the following properties*:

i) *The semi-finite normal weight $\widehat{\varphi}$ is faithful.*
ii) *The modular automorphism group $\sigma^{\widehat{\varphi}}$ acts on the generators of the crossed product $\mathcal{M} \rtimes_\alpha G$ as follows*:

$$\sigma_t^{\widehat{\varphi}}(\pi_\alpha(x)) = \pi_\alpha(\sigma_t^\varphi(x)), \quad x \in \mathcal{M};$$
$$\sigma_t^{\widehat{\varphi}}(u(g)) = \delta_G(g)^{\mathrm{i}t} u(g) \pi_\alpha\big((\mathrm{D}\varphi\circ\alpha_g : \mathrm{D}\varphi)_t\big), \quad t \in \mathbb{R}.$$

iii) *If ψ is another faithful semi-finite normal weight, then we have the Connes cocycle derivative given the following*:

$$\left(\mathrm{D}\widehat{\psi} : \mathrm{D}\widehat{\varphi}\right)_t = \pi_\alpha((\mathrm{D}\psi : \mathrm{D}\varphi)_t), \quad t \in \mathbb{R}.$$

In the theorem, if we take the modular covariant system $\{\mathcal{M}, \mathbb{R}, \sigma^\varphi\}$, then the modular automophism group $\left\{\sigma_t^{\widehat{\varphi}} : t \in \mathbb{R}\right\}$ is given by $\{\mathrm{Ad}(u_\mathbb{R}(t)) : t \in \mathbb{R}\}$, which gives the following result at once.

Corollary 1.7. *The crossed product $\widetilde{\mathcal{M}} = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ of the modular covariant system $\{\mathcal{M}, \mathbb{R}, \sigma^\varphi\}$ is semi-finite and its algebraic type is independent of the choice of a faithful semi-finite normal weight φ. The faithful semi-finite normal trace on the crossed product $\widetilde{\mathcal{M}}$ is given by the following formula*:

$$\tau(x) = \widehat{\varphi}\left(h^{-\frac{1}{2}} x h^{-\frac{1}{2}}\right), \quad x \in \widetilde{\mathcal{M}}_+,$$

where the non-singular positive self-adjoint opertator h affiliated with $\widetilde{\mathcal{M}}$ is the logarithmic generator of the one parameter unitary group $\{u_\mathbb{R}(t) : t \in \mathbb{R}\}$, i.e.,

$$u_\mathbb{R}(t) = h^{\mathrm{i}t}, \quad t \in \mathbb{R}.$$

Lecture 2: Duality for the Crossed Product by Abelian Groups.

§**2.1. Fourier Transform and Plancherel Formula.** We now consider a locally compact abelian group G. We denote the group operation of G additively. Let $\widehat{G}$ be the Kampen-Pontrjagin dual of G, i.e., the group of all continous homomorphism from G to the torus $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. We write the duality of $\left\{G, \widehat{G}\right\}$ in the following way:

$$
\begin{aligned}
(g,p) \in G \times \widehat{G} \mapsto \langle g,p\rangle \in \mathbb{T} \quad & \text{jointly continuous bicharacter;} \\
\langle g+h,p\rangle = \langle g,p\rangle\langle h,p\rangle, \quad & \langle g,p+q\rangle = \langle g,p\rangle\langle g,q\rangle, \\
& g,h \in G, p,q \in \widehat{G}
\end{aligned}
$$

We choose the Haar measures dg on G and dp on $\widehat{G}$ in such a way that the Plancherel formula holds, i.e.,

$$
\begin{aligned}
(\mathcal{F}f)(p) &= \int_G \overline{\langle g,p\rangle} f(g)\mathrm{d}g, \quad f \in L^1(G), \\
\left(\widehat{\mathcal{F}}\widehat{f}\right)(g) &= \int_{\widehat{G}} \langle g,p\rangle \widehat{f}(p)\mathrm{d}p, \quad \widehat{f} \in L^1\left(\widehat{G}\right), \\
\left(f \,\middle|\, \widehat{\mathcal{F}}\widehat{f}\right)_{L^2(G)} &= \left(\mathcal{F}f \,\middle|\, \widehat{f}\right)_{L^2\left(\widehat{G}\right)}, \\
& f \in L^2(G) \cap L^1(G), \widehat{f} \in L^2\left(\widehat{G}\right) \cap L^2\left(\widehat{G}\right); \\
\widehat{\mathcal{F}}\circ\mathcal{F}f = f, &\quad \text{if } f \in L^1(G) \text{ and } \mathcal{F}f \in L^1\left(\widehat{G}\right); \\
\mathcal{F}\circ\widehat{\mathcal{F}}\widehat{f} = \widehat{f}, &\quad \text{if } \widehat{f} \in L^1\left(\widehat{G}\right) \text{ and } \widehat{\mathcal{F}}\widehat{f} \in L^1(G); \\
\|f\|_2 = \|\mathcal{F}f\|_2, &\quad f \in L^1(G) \cap L^2(G); \\
\left\|\widehat{\mathcal{F}}\widehat{f}\right\|_2 = \left\|\widehat{f}\right\|_2, &\quad \widehat{f} \in L^1\left(\widehat{G}\right) \cap L^2\left(\widehat{G}\right).
\end{aligned}
$$

If $f \in L^2(G)$, then for each compact subset $K \Subset G$ there exists $\widehat{f}_K \in L^2\left(\widehat{G}\right)$ such that

$$\widehat{f}_K(p) = \int_K \overline{\langle g,p\rangle} f(g)\mathrm{d}g,$$

and the net $\left\{\widehat{f}_K : K \Subset G\right\}$ converges to $\widehat{f} \in L^2\left(\widehat{G}\right)$ in L^2-norm as K increases to G. We call this $\widehat{f}$ **the extended Fourier transform** of f and write $\widehat{f} = \mathcal{F}f$. Hence the extended Fourier transform $\mathcal{F}$ is indeed an isometry from $L^2(G)$ onto $L^2\left(\widehat{G}\right)$.

We use the following notations and name:

$$
\begin{aligned}
\widehat{\mathcal{F}}\left(L^1\left(\widehat{G}\right)\right) &= A(G) = L^2(G) * L^2(G) \subset C_0(G); \\
\mathcal{F}(L^1(G)) &= A\left(\widehat{G}\right) = L^2\left(\widehat{G}\right) * L^2\left(\widehat{G}\right) \subset C_0\left(\widehat{G}\right).
\end{aligned}
$$

The algebras $A(G)$ and $A\left(\widehat{G}\right)$ are respectively called the **Fourier** algebra.

We now discuss the real additive group $\mathbb{R}$. In this case, we have

$$\begin{gathered}
\widehat{\mathbb{R}} = \mathbb{R}; \\
(s,p) \in \mathbb{R} \times \mathbb{R} \mapsto \langle s,p \rangle = e^{\mathrm{i}sp} \in \mathbb{T}; \\
(\mathfrak{F}f)(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\mathrm{i}sp} f(s) \mathrm{d}s, \quad f \in L^1(\mathbb{R}); \\
\left(\widehat{\mathfrak{F}}\widehat{f}\right)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mathrm{i}sp} \widehat{f}(p) \mathrm{d}p, \quad \widehat{f} \in L^1(\mathbb{R}).
\end{gathered}$$

Viewing the Fourier transform $\mathfrak{F}$ as an operator on $L^2(\mathbb{R})$, we have

$$\begin{gathered}
\widehat{\mathfrak{F}} = \mathfrak{F}^*, \quad \mathfrak{F}\mathfrak{F}^* = 1, \quad \mathfrak{F}^*\mathfrak{F} = 1, \quad \text{unitary} \\
\mathfrak{F}^2 f(s) = f(-s), \quad \left(\widehat{\mathfrak{F}}^2 f\right)(p) = f(-p), \quad \mathfrak{F}^4 = 1, \quad \widehat{\mathfrak{F}}^4 = 1.
\end{gathered}$$

Namely the Fourier transform on $L^2(\mathbb{R})$ is a unitary operator with period 4. If $f(s) = e^{-\pi s^2}$, then

$$(\mathfrak{F}f)(p) = f(p), \quad p \in \mathbb{R}.$$

Define the convolution and the involution by the following:

$$\begin{gathered}
(f * g)(s) = \int_{\mathbb{R}} f(t) g(s-t) \mathrm{d}t, \quad f, g \in L^1(\mathbb{R}); \\
f^*(s) = \overline{f(-s)}.
\end{gathered}$$

With this algebraic structure, $L^1(\mathbb{R})$ is an involutive Banach algebra. Furthermore, the Fourier transform is indeed the Gelfand transform of the Banach algebra $L^1(\mathbb{R})$ as seen below:

$$\begin{gathered}
\mathfrak{F}f \in C_0(\mathbb{R}), \quad f \in L^1(\mathbb{R}); \\
\mathfrak{F}(f * g) = (\mathfrak{F}f)(\mathfrak{F}g), \quad f, g \in L^1(\mathbb{R}); \\
(\mathfrak{F}f^*) = \overline{(\mathfrak{F}f)}; \\
\mathfrak{F}(fg) = (\mathfrak{F}f) * (\mathfrak{F}g), \quad f, g \in C_0(\mathbb{R}) \cap L^1(\mathbb{R}); \\
\mathfrak{F}\overline{f} = (\mathfrak{F}f)^*, \quad f \in L^1(\mathbb{R}).
\end{gathered}$$

§2.2. Uniqueness of the Heisenberg Commutation Relation.

Let G be an abelian locally compact group with Kampen - Pontrjagin dual $\widehat{G}$. A pair U, V of unitary representations of G and $\widehat{G}$ on the same Hilbert space $\mathfrak{H}$ is called **covariant** if the following covariance condition holds:

$$U(s)V(p) = \overline{\langle s,p \rangle} V(p) U(s), \quad (s,p) \in G \times \widehat{G}.$$

The map:

$$((s_1, p_1); (s_2, p_2)) \in \left(G \times \widehat{G}\right)^2 \mapsto m_G(g_1, g_2) = \overline{\langle s_2, p_1 \rangle} \in \mathbb{T}$$

where

$$g_i = (s_i, p_i) \in G \times \widehat{G}, \quad i = 1, 2,$$

is a 2-cocycle satisfying the cocycle identity:

$$m_G(g_1, g_2) m_G(g_1 g_2, g_3) = m_G(g_1, g_2 g_3) m_G(g_2, g_3), \quad g_1, g_2, g_3 \in G \times \widehat{G}.$$

We fix the Haar measures[7] on G and $\widehat{G}$ respectively so that the Plancherel formula holds. Consider the Hilbert space $\mathfrak{H} = L^2(G)$ and define:

$$\begin{gathered}
(\rho_G(s)\xi)(t) = \xi(t+s), \quad s, t \in G; \\
(\lambda_G(s)\xi)(t) = \xi(t-s) \\
(\mu_G(p)\xi)(t) = \overline{\langle t, p \rangle} \xi(t), \quad p \in \widehat{G}, \xi \in L^2(G).
\end{gathered}$$

Then the following computation:

$$\begin{gathered}
(\lambda_G(s)\mu_G(p)\xi)(t) = (\mu_G(p)\xi)(t-s) = \overline{\langle t-s, p \rangle} \xi(t-s); \\
(\mu_G(p)\lambda_G(s)\xi)(t) = \overline{\langle t, p \rangle} (\lambda_G(s)\xi)(t) = \overline{\langle t, p \rangle} \xi(t-s); \\
\lambda_G(s)\mu_G(p) = \overline{\langle s, p \rangle} \mu_G(p)\lambda_G(s)
\end{gathered}$$

shows that the pair $\{\lambda_G, \mu_G\}$ is covariant. We then consider the effect of the Fourier transform on them:

$$\begin{aligned}
(\mathcal{F}\lambda_G(s)\mathcal{F}^*\xi)(p) &= \int_G \overline{\langle t, p \rangle} (\lambda_G(s)\mathcal{F}^*\xi)(t) \mathrm{d}t = \int_G \overline{\langle t, p \rangle} (\mathcal{F}^*\xi)(t-s) \mathrm{d}t \\
&= \int_G \overline{\langle t+s, p \rangle} (\mathcal{F}^*\xi)(t) \mathrm{d}t = \overline{\langle s, p \rangle} \int_G \overline{\langle t, p \rangle} (\mathcal{F}^*\xi)(t) \mathrm{d}t \\
&= \overline{\langle s, p \rangle} (\mathcal{F}\mathcal{F}^*\xi)(p) = \overline{\langle s, p \rangle} \xi(p), \quad \xi \in L^2\left(\widehat{G}\right) \cap L^1\left(\widehat{G}\right), \\
(\mathcal{F}\mu_G(p)\mathcal{F}^*\xi)(q) &= \int_G \overline{\langle t, q \rangle} (\mu_G(p)\mathcal{F}^*\xi)(t) \mathrm{d}t = \int_G \overline{\langle t, q \rangle \langle t, p \rangle} (\mathcal{F}^*\xi)(t) \mathrm{d}t \\
&= \int_G \overline{\langle t, p+q \rangle} (\mathcal{F}^*\xi)(t) \mathrm{d}t = (\mathcal{F}\mathcal{F}^*\xi)(p+q) \\
&= \left(\rho_{\widehat{G}}(p)\xi\right)(q)
\end{aligned}$$

for sufficiently many ξ's to conclude

$$\begin{gathered}
\mathcal{F}\lambda_G(s)\mathcal{F}^* = \mu_{\widehat{G}}(s), \quad s \in G; \\
\mathcal{F}\mu_G(p)\mathcal{F}^* = \rho_{\widehat{G}}(p) = \lambda_{\widehat{G}}(p)^*, \quad p \in G.
\end{gathered}$$

[7]For those of you who are not familier with the theory of locally compact abelian groups, you can assume $G = \widehat{G} = \mathbb{R}$ and $\sqrt{1/2\pi}$ times the Lebesgue measure as the Haar measures on G and $\widehat{G}$ with which Plancherel formula holds.

Theorem 2.1. i) *The covariant representations* $\{\lambda_G, \mu_G\}$ *is irreducible in the sense that there is no non-trivial closed subspace* $\mathfrak{M} \subset L^2(G)$ *jointly invariant under* ρ_G *and* μ_G. *Equivalently, we have*

$$\Big(\lambda_G(G) \cup \mu_G\Big(\widehat{G}\Big)\Big)'' = \mathfrak{L}\big(L^2(G)\big).$$

ii) *The covariant pair* $\{\lambda_G, \mu_G\}$ *of unitary representations is a unique covariant representation in the sense that if* $\{U, V\}$ *is another covariant pair of unitary representations of* G *and* $\widehat{G}$ *on the same Hilbert space* $\mathfrak{H}$, *then there exists a Hilbert space* $\mathfrak{K}$ *such that*

$$\big\{\lambda_G \otimes 1_{\mathfrak{K}}, \mu_G \otimes 1_{\mathfrak{K}}, L^2(G) \otimes \mathfrak{K}\big\} \cong \{U, V, \mathfrak{H}\} \quad \textit{Unitarily Equivalent.}$$

The algebra $L^\infty(G)$ of all essentially bounded functions on G relative to the Haar measure acts on $L^2(G)$ by multiplication:

$$(\pi(f)\xi)(s) = f(s)\xi(s), \quad f \in L^\infty(G), \xi \in L^2(G).$$

Then $\pi(L^\infty(G)) \subset \mathfrak{L}\big(L^2(G)\big)$ is a maximal abelian subalgebra of $\mathfrak{L}\big(L^2(G)\big)$. For each $\widehat{f} \in L^1\Big(\widehat{G}\Big)$, the multiplication operator by the Fourier transform $f = \widehat{\mathfrak{F}}\widehat{f} \in C_0(G)$ of $\widehat{f}$ is given by the following;

$$\begin{aligned}
(\pi(f)\xi)(s) &= \left(\int_{\widehat{G}} \langle s, p\rangle \widehat{f}(p)\mathrm{d}p\right)\xi(s) = \int_{\widehat{G}} \widehat{f}(p)\langle s, p\rangle \xi(s)\mathrm{d}p \\
&= \left(\left(\int_{\widehat{G}} \widehat{f}(p)\mu_G(p)\mathrm{d}p\right)\xi\right)(s) \\
&= \Big(\mu_G\Big(\widehat{f}\Big)\xi\Big)(s),
\end{aligned}$$

where

$$\mu_G\Big(\widehat{f}\Big) = \int_{\widehat{G}} \widehat{f}(p)\mu_G(p)\mathrm{d}p \in \mathfrak{L}\big(L^2(G)\big).$$

Consequently,

$$\pi(f) = \mu_G\Big(\widehat{f}\Big), \quad f = \mathfrak{F}^*\widehat{f}, \quad \widehat{f} \in L^1\Big(\widehat{G}\Big).$$

Since $A(G) = \widehat{\mathfrak{F}}\Big(L^1\Big(\widehat{G}\Big)\Big)$ is norm dense in $C_0(G)$, we have

$$\mu_G\Big(\widehat{G}\Big)'' = \pi(L^\infty(G)).$$

For each $f \in L^1\Big(G \times \widehat{G}\Big)$, set

$$\pi(f) = \iint\limits_{G\times\widehat{G}} f(s,p)\rho_G(s)\mu_G(p)\mathrm{d}s\mathrm{d}p;$$

$$\begin{aligned}
\pi(f)\pi(g) &= \left(\iint\limits_{G\times\widehat{G}} f(s,p)\rho_G(s)\mu_G(p)\mathrm{d}s\mathrm{d}p\right)\left(\iint\limits_{G\times\widehat{G}} g(t,q)\rho_G(t)\mu_G(q)\mathrm{d}t\mathrm{d}q\right)\\
&= \iiiint\limits_{(G\times\widehat{G})^2} \overline{\langle t,p\rangle} f(s,p)g(t,q)\rho_G(s+t)\mu_G(p+q)\mathrm{d}s\mathrm{d}p\mathrm{d}t\mathrm{d}q\\
&= \iiiint\limits_{(G\times\widehat{G})^2} \overline{\langle t,p\rangle} f(s,p)g(t-s,q-p)\rho_G(t)\mu_G(q)\mathrm{d}s\mathrm{d}p\mathrm{d}t\mathrm{d}q\\
&= \pi(f*g)
\end{aligned}$$

where

$$(f*g)(t,q) = \iint\limits_{G\times\widehat{G}} \overline{\langle t,p\rangle} f(s,p)g(t-s,q-p)\mathrm{d}s\mathrm{d}p$$

Then we have

$$\begin{aligned}
(\pi(f)\xi)(t) &= \iint\limits_{G\times\widehat{G}} \langle t,p\rangle f(s,p)\xi(s+t)\mathrm{d}s\mathrm{d}p\\
&= \iint\limits_{G\times\widehat{G}} \langle t,p\rangle f(s-t,p)\xi(s)\mathrm{d}s\mathrm{d}p.
\end{aligned}$$

Let $\{\mathcal{M}, G, \alpha\}$ be a covariant system on a locally compact abelian group G. Assume that the von Neumann algebra $\mathcal{M}$ acts on a Hilbert space $\mathfrak{H}$. Then the crossed product:

$$\mathcal{N} = \mathcal{M} \rtimes_\alpha G,$$

is generated by the two kind of operators:

$$\pi_\alpha(x), \quad x \in \mathcal{M}, \qquad u(s), \quad s \in G,$$

and operates on the Hilber space

$$\mathfrak{K} = \mathfrak{H} \otimes L^2(G) = L^2(G, \mathfrak{K}).$$

Now the unitary representation $v : p \in \widehat{G} \mapsto v(p) \in \mathcal{U}(\mathfrak{K})$ defined by the formula:

$$(v(p)\xi)(s) = \overline{\langle s,p\rangle}\xi(s), \quad \xi \in \mathfrak{K}, \quad s \in G, \quad p \in \widehat{G},$$

behaves in the following way:

$$v(p)\pi_\alpha(x)v(p)^* = \pi_\alpha(x), \quad x \in \mathcal{M}, p \in \widehat{G};$$
$$v(p)u(s)v(p)^* = \overline{\langle s, p\rangle}u(s), \quad s \in G.$$

Thus the unitary representation v of the dual group $\widehat{G}$ gives rise to an action $\widehat{\alpha}$ of $\widehat{G}$:

$$\widehat{\alpha}_p(x) = v(p)xv(p)^*, \quad x \in \mathcal{M} \rtimes_\alpha G, \quad p \in \widehat{G}.$$

Defition 2.2. The action $\widehat{\alpha}$ of $\widehat{G}$ is said to be **dual to** the original action α or the dual action of α. The covariant system $\left\{\mathcal{M} \rtimes_\alpha G, \widehat{G}, \widehat{\alpha}\right\}$ is called the **dual covariant system** or **the covariant system dual to** the original system $\{\mathcal{M}, G, \alpha\}$.

Now we are ready to state the duality theorem on the crossed product of a covariant system on a locally compact abelian group.

Theorem 2.3. (Duality Theorem). *Suppose that $\{\mathcal{M}, G, \alpha\}$ is a covariant system on a locally compact abelian group G. Then the covariant system $\left\{\mathcal{M} \rtimes_\alpha G, \widehat{G}, \widehat{\alpha}\right\}$ dual to $\{\mathcal{M}, G, \alpha\}$ has the following properties*:

i) *The fixed point algebra $(\mathcal{M} \rtimes_\alpha G)^{\widehat{\alpha}}$ under the dual action $\widehat{\alpha}$ is precisely $\pi_\alpha(\mathcal{M})$;*
ii) *A faithful semi-finite normal weight ω on the $\mathcal{M} \rtimes_\alpha G$ is dual to a some faithful semi-finite normal weight φ on $\mathcal{M}$, i.e., $\omega = \widehat{\varphi}$ if and only if ω is invariant under the dual action $\widehat{\alpha}$;*
iii) *Concerning the second crossed product, we have*

$$(\mathcal{M} \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \cong \mathcal{M} \overline{\otimes} \mathcal{L}\big(L^2(G)\big)$$

under the isomorphism Φ determined uniquely by the following properties:

$$\begin{aligned} &(\Phi(\pi_\alpha(x))\xi)(s) = \alpha_s^{-1}(x)\xi(s), \quad x \in \mathcal{M}, s \in G; \\ &(\Phi(\pi_{\widehat{\alpha}}(u(s)))\xi)(t) = \xi(t-s), \quad s, t \in G; \qquad \xi \in L^2(\mathfrak{H}, G) \\ &\big(\Phi\big(u_{\widehat{G}}(p)\big)\xi\big)(t) = \overline{\langle t, p\rangle}\xi(t), \quad p \in \widehat{G}; \end{aligned}$$

where $u_{\widehat{G}}(\cdot)$ means the unitary representation of the dual group $\widehat{G}$ corresponding to the second crossed product;
iv) *The isomorphism Φ conjugates the second dual action $\widehat{\widehat{\alpha}}$ to the action $\alpha \otimes \rho$ where ρ means the inner automorphism action of G on $\mathcal{L}\big(L^2(G)\big)$ given by the right regular representation ρ_G defined by the following*:

$$(\rho_G(s)\xi)(t) = \xi(s+t), \quad \xi \in L^2(G), s, t \in G;$$

v) *For each faithful semi-finite normal weight φ, the second dual semi-finite normal weight $\hat{\hat{\varphi}}$ is related to the original weight φ in the following way:*

$$\left(\left(\mathrm{D}\left(\hat{\hat{\varphi}}\circ\Phi^{-1}\right) : \mathrm{D}(\varphi\otimes\mathrm{Tr})\right)_t \xi\right)(s) = (\mathrm{D}(\varphi\circ\alpha_s) : \mathrm{D}\varphi)_t\xi(s)$$

for each $\xi \in L^2(\mathfrak{H}, G), s \in G$ and $t \in \mathbb{R}$.

Brief Sketch of the Proof of Theorem 2.3.(iii). The second crossed product $\mathcal{N} = (\mathcal{M}\rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$ acts on the Hilbert space $\mathfrak{H}_\varphi \otimes L^2(G) \otimes L^2\left(\widehat{G}\right)$, where $\mathfrak{H}_\varphi$ is the semi-cyclic Hilbert space constructed by a faithful semi-finite normal weight φ on $\mathcal{M}$ on which the original von Neumann algebra $\mathcal{M}$ acts. Making use of the inverse Fourier transform $1 \otimes 1 \otimes \widehat{\mathcal{F}}$, we represent the second crossed product von Neumann algebra $\mathcal{N}$ on $\mathfrak{K} = L^2(\mathfrak{H}_\varphi, G \times G,)$ which is then generated by the following three types of operators:

$$\begin{gathered} U(s) = 1 \otimes \lambda_G(s) \otimes \lambda_G(s), \quad s \in G; \\ V(p) = 1 \otimes 1 \otimes \mu_G(p), \quad p \in \widehat{G}; \\ \pi_{\widehat{\alpha}}\circ\pi_\alpha(x) = \pi_\alpha(x) \otimes 1, \quad x \in \mathcal{M}. \end{gathered}$$

Consider the following unitary operator which is called the **fundamental unitary of** G:

$$(W\xi)(r,s) = \xi(r+s,s), \quad r,s \in G,\ \xi \in \mathfrak{K}.$$

Observe

$$\begin{gathered} W(1\otimes 1\otimes \mu_G(p))W^* = 1\otimes 1\otimes \mu_G(p), \quad p \in \widehat{G}; \\ W(1\otimes\lambda_G(s)\otimes\lambda_G(s))W^* = 1\otimes 1\otimes\lambda_G(s), \quad s \in G; \\ W(\pi_{\widehat{\alpha}}\circ\pi_\alpha(x))W^* = \pi(x), \quad x \in \mathcal{M}, \end{gathered}$$

where

$$(\pi(x)\xi)(r,s) = \alpha^{-1}_{(r+s)}(x)\xi(r,s), \quad x \in \mathcal{M},\ \xi \in \mathfrak{K}.$$

We then prove

$$\pi(\mathcal{M}) \vee (\mathbb{C}\otimes\mathbb{C}\otimes L^\infty(G)) = \pi_\alpha(\mathcal{M})\overline{\otimes}L^\infty(G), \tag{*}$$

which will implies the following:

$$\begin{aligned} W\mathcal{N}W^* &= (\mathbb{C}\otimes\mathbb{C}\otimes L^\infty(G)) \vee \left(\mathbb{C}\otimes\mathbb{C}\otimes\left(\lambda_G(G)''\right)\right) \vee \pi(\mathcal{M}) \\ &= (\pi_\alpha(\mathcal{M})\overline{\otimes}L^\infty(G)) \vee (\mathbb{C}\otimes\mathbb{C}\otimes\mathcal{R}(G)) \\ &= \pi_\alpha(\mathcal{M})\overline{\otimes}\mathcal{L}\big(L^2(G)\big). \end{aligned}$$

This completes the proof.

Proof of $(*)$ *for a finite abelian group* G. Observe

$$\pi(x) = \int_G^{\oplus} \pi_\alpha(\alpha_s(x))\mathrm{d}s \in \pi_\alpha(\mathcal{M})\overline{\otimes} L^\infty(G),$$

so that

$$\pi(\mathcal{M}) \subset \pi_\alpha(\mathcal{M})\overline{\otimes} L^\infty(G).$$

Hence we have

$$W(\pi_\alpha(\mathcal{M}) \otimes \mathbb{C})W^* = \pi(\mathcal{M}) \subset \pi_\alpha(\mathcal{M})\overline{\otimes} L^\infty(G).$$

On the other hand, W commutes with $\mathbb{C} \otimes \mathbb{C} \otimes L^\infty(G)$, which yields the following:

$$W(\pi_\alpha(\mathcal{M})\overline{\otimes} L^\infty(G))W^* \subset \pi_\alpha(\mathcal{M})\overline{\otimes} L^\infty(G).$$

Simlarly, we have

$$W^*(\pi_\alpha(\mathcal{M})\overline{\otimes} L^\infty(G))W \subset \pi_\alpha(\mathcal{M})\overline{\otimes} L^\infty(G).$$

Hence, we conclude that

$$W(\pi_\alpha(\mathcal{M})\overline{\otimes} L^\infty(G))W^* = \pi_\alpha(\mathcal{M})\overline{\otimes} L^\infty(G).$$

This completes the proof of $(*)$.

♡

Lecture 3: Second Cohomology of Locally Compact Abelian Group.

§3.1. Two cocycles of a separable locally compact abelian group. Fix a separable locally compact abelian group G, where we write the group operation multiplicatively to shorten formulae.

A Borel function $\mathrm{m} : G \times G \mapsto \mathbb{T}$ is called a **2-cocycle** if the following cocycle identity holds:

$$\mathrm{m}(g_1, g_2)\mathrm{m}(g_1g_2, g_3) = \mathrm{m}(g_1, g_2g_3)\mathrm{m}(g_2, g_3), \quad g_1, g_2, g_3 \in G.$$

If there exists a Borel function $\mathrm{b} : G \mapsto \mathbb{T}$ such that

$$\mathrm{m}(g_1, g_2) = \mathrm{b}(g_1)\mathrm{b}(g_2)\overline{\mathrm{b}(g_1g_2)}, \quad g_1, g_2 \in G,$$

then m is called a **coboundary** and written

$$\mathrm{m} = \partial \mathrm{b}.$$

We also say that the function b **cobounds** m. The set of all cocycles is written $\mathrm{Z}^2(G, \mathbb{T})$ and becomes an abelian group under the pointwise multiplication, i.e.,

$$(\mathrm{mn})(s_1, s_2) = \mathrm{m}(s_1, s_2)n(s_1, s_2), \quad s_1, s_2 \in G, \quad \mathrm{m}, \mathrm{n} \in \mathrm{Z}^2(G, \mathbb{T}).$$

The set $\mathrm{B}^2(G, \mathbb{T})$ of all coboundaries is a subgroup of $\mathrm{Z}^2(G, \mathbb{T})$. The quotient group:

$$\mathrm{H}^2(G, \mathbb{T}) = \mathrm{Z}^2(G, \mathbb{T})/\mathrm{B}^2(G, \mathbb{T})$$

is called the **second cohomology group** of G.

Fixing a cocycle $\mathrm{m} \in \mathrm{Z}^2(G, \mathbb{T})$, we consider

$$G_{\mathrm{m}} = \mathbb{T} \times G,$$

and define a map:

$$((\lambda_1, s_1), (\lambda_2, s_2)) \in G_{\mathrm{m}} \times G_{\mathrm{m}} \mapsto (\lambda_1\lambda_2\mathrm{m}(s_1, s_2), s_1s_2) \in G_{\mathrm{m}}.$$

Observe that G_{m} is a group. We consider the Borel structure in G_{m} generated by product sets $\{E \times F : E \subset \mathbb{T}, F \subset G\}$ of Borel subsets $E \subset \mathbb{T}$ and $F \subset G$ and consider the integration:

$$\int_{G_{\mathrm{m}}} f(\lambda, s)\mathrm{d}\lambda\mathrm{d}s, \quad \text{for Borel function } f \geq 0,$$

where $\mathrm{d}\lambda$ means the normalized Haar measure on the torus $\mathbb{T}$. This integration gives an invariant standard measure on G_{m}, which makes the group G_{m} a locally compact group. Hence each $\mathrm{m} \in \mathrm{Z}^2(G, \mathbb{T})$ gives rise to an exact sequence of locally compact groups equipped with associated cross-section, (i.e., a right inverse of the map π_{m}):

$$1 \longrightarrow \mathbb{T} \xrightarrow{i} G_{\mathrm{m}} \underset{\overleftarrow{u_{\mathrm{m}}}}{\xrightarrow{\pi_{\mathrm{m}}}} G \longrightarrow 1$$

such that $i(\mathbb{T})$ is contained in the center of G_{m} and

$$u_{\mathrm{m}}(r)u_{\mathrm{m}}(s) = i(\mathrm{m}(r,s))u_{\mathrm{m}}(rs), \quad r, s \in G.$$

If we have another cocycle $\mathrm{n} \in \mathrm{Z}^2(G, T)$, then we have another exact sequence:

$$1 \longrightarrow \mathbb{T} \xrightarrow{i} G_{\mathrm{n}} \underset{\overleftarrow{u_{\mathrm{m}}}}{\xrightarrow{\pi_{\mathrm{n}}}} G \longrightarrow 1.$$

These exact sequences are in the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{T} \arrow[r, "i"] \arrow[d, equal] & G_{\mathrm{m}} \arrow[r, "\pi_{\mathrm{m}}"] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{T} \arrow[r, "i"] & G_{\mathrm{n}} \arrow[r, "\pi_{\mathrm{n}}"] & G \arrow[r] & 1
\end{tikzcd}$$

if and only

$$[\mathrm{m}] = [\mathrm{n}] \in \mathrm{H}^2(G, \mathbb{T}).$$

In particular, the above exact sequence split in the sense that the quotient map π_{m} admits a left inverse u if and only if $\mathrm{m} \in \mathrm{B}^2(G, \mathbb{T})$.

For every closed subgroup $H \subset G$ and the subgroup of the dual group $\widehat{G}$:

$$H^{\perp} = \left\{ p \in \widehat{G} : \langle s, p \rangle = 1, s \in H \right\},$$

the Kampen-Pontrjagin Duality Theorem for locally compact abelian groups asserts that

$$\widehat{H} \cong \widehat{G}/H^{\perp}, \quad \widehat{G/H} \cong H^{\perp},$$

under the natural correspondence. This means that the above exact sequence splits in the sense that the quotiend map π_{m} admits a left inverse if and only if G_{m} is commutative.

For each $\mathrm{m} \in \mathrm{Z}^2(G, \mathbb{T})$ consider the following **asymmetrization**:

$$(\mathrm{ASm})(s_1, s_2) \stackrel{\mathrm{def}}{=} \mathrm{m}(s_1, s_2)\overline{\mathrm{m}(s_2, s_1)}, \quad s_1, s_2 \in G.$$

Theorem 3.1. i) *If* m *is a 2-cocycle on a locally compact abelian group G, then the assymetrization* ASm *is a skew symmetric bicharacter in the following sense*:

$$\begin{aligned}(\mathrm{ASm})(s_1s_2, r) &= (\mathrm{ASm})(s_1, r)(\mathrm{ASm})(s_2, r);\\ (\mathrm{ASm})(r, s) &= \overline{(\mathrm{ASm})(s, r)},\end{aligned} \qquad r, s_1, s_2, s \in G,$$

Furthermore, the cocycle m *is a coboundary if and only* ASm $= 1$*, in the sense that* $\mathrm{ASm}(r, s) = 1, r, s \in G$*. Consequently, the second cohomology group* $\mathrm{H}^2(G, \mathbb{T})$ *is isomorlphic to the group* $X^2(G, \mathbb{T})$ *of all continuous* $\mathbb{T}$*-valued squew symmetric bicharacters on* G*.*

ii) *If G is closed under the squar root, i.e. if every element of G is square of another element of G, then every 2-cocycle is cohomologous to a skew symmetric bicharacter.*

We are now returning to the additive notation on the group G. A skew symmetric bicharacter $\chi \in X^2(G, \mathbb{T})$ is said to be **symplectic** if for every non-zero $s \in G$ there exists some $t \in G$ such that

$$\chi(s, t) \neq 1.$$

Example 3.2. i) *Let E be an even dimensional real vector space. Let σ be a symplectic bilinear form on E, i.e.,*

$$\begin{gathered}\sigma(r, s) = -\sigma(s, r);\\ \sigma(r, s) = 0 \quad \textit{for all } s \in E \quad \Rightarrow \quad r = 0,\end{gathered}$$

Then the bicharacter

$$\chi(r, s) = \exp(\mathrm{i}\sigma(r, s)), \quad r, s \in E.$$

is a symplectic bicharacter of E.

ii) *Let $E = R^n \times R^n, n \in \mathbb{N}$ and write element $r \in E$ by*

$$\begin{aligned}r &= (x_1, \cdots, x_n; p_1, \cdots, p_n);\\ s &= (y_1, \cdots, y_n; q_1, \cdots, q_n),\end{aligned} \quad x_i, y_i, p_j, q_j \in \mathbb{R}, 1 \leq i, j \leq n.$$

Then set

$$\sigma(r, s) = \sum_{i=1}^{n}(x_iq_i - p_iy_i)$$

Then σ is a symplectic bilinear form on E.

iii) *Every symplectic bilinear form on an even dimensional real vector space E is of the above form after choosing an appropriate coordinate system.*

Theorem 3.3. *Suppose that E is an even dimensional real vector space equipped with a symplectic bilinear form σ and set*

$$\chi_\sigma(s,t) = \exp(\mathrm{i}(\sigma(s,t))), \quad s,t \in E.$$

i) *The Hilbert space $L^2(G)$ of square integrable functions on E relative to the fixed Haar measure is a unimodular left Hilbert algebra under the twisted convolution and the involution defined by the following:*

$$\begin{aligned} (x*y)(s) &= \int_G \chi(s,t)x(s)y(t-s)\mathrm{d}s; \\ x^*(s) &= \overline{x(-s)}, \end{aligned} \qquad x,y \in L^2(G), s \in G.$$

ii) *Choose a basis $\{e_1,\cdots,e_n;f_1,\cdots,f_n\}$ of E so that for a pair of vectors:*

$$\begin{aligned} u &= \sum_{i=1}^n x_i e_i + \sum_{j=1}^n p_j f_j; \\ v &= \sum_{i=1}^n y_i e_i + \sum_{j=1}^n q_j f_j, \end{aligned}$$

the symplectic form σ takes the following form:

$$\sigma(u,v) = \sum_{i=1}^n (x_i q_i - y_i p_i).$$

iii) *Set $F = \langle e_1,\cdots,e_n\rangle \subset E$ and consider the Hilbett space $L^2(F)$ of square integrable functions on F relative to the Lebesgue measure. Then the twisted convolution:*

$$f*\xi(x) = \iint_E \exp(-\mathrm{i}\langle p,y\rangle)f(x,p)\xi(y-x)\mathrm{d}x\mathrm{d}p.$$

for every $f \in L^2(E)$ and $\xi \in L^2(F)$ converges and gives a bounded operator:

$$\xi \in L^2(F) \mapsto \pi(f)\xi = f*\xi \in L^2(F).$$

The map $\pi : L^2(E) \mapsto \pi(f) \in \mathcal{L}\big(L^2(F)\big)$ gives rise to an isomorphism between $L^2(E)$ and the algebra $\mathcal{L}^2\big(L^2(F)\big)$ of Hilbert-Schmidt class operators on $L^2(F)$ and

$$\mathrm{Tr}(\pi(g)^*\pi(f)) = (f \mid g), \quad f,g \in L^2(F).$$

Lecture 4. Arveson Spectral of an Action of a Locally Compact Abelian Group G on a von Neumann algebra $\mathcal{M}$.

We discuss further the Arveson spctrum of actions of a separable locally compact abelian group G on a von Neumann algebra $\mathcal{M}$. [Arv]. Let G be a locally compact abelian group and X the dual Banach space of another Banach space X_* on which G acts in such a way that the map $s \in G \mapsto \langle \alpha_s(x), \varphi \rangle \in \mathbb{C}$ is continuous for every $\varphi \in X_*$. An **action** G on X means that a homomorphism $\alpha : s \in G \mapsto \mathrm{GL}_w(X)$, where $\mathrm{GL}_w(X)$ means the set of all $\sigma(X, X_*)$-continous invertible operators on X, such that

$$\lim_{s\to 0} \|\varphi \circ \alpha_s - \varphi\| = 0, \quad \varphi \in X_*$$
$$\sup\{\|\alpha_s\| : s \in G\} < +\infty.$$

Let α be an action of G on X which will be fixed for a while. Recall that the Fourier algebra $A\left(\widehat{G}\right)$ is the Fourier transform of the convolution algebra $L^1(G)$, equivalently the set of all convolution of two square integrable functions on $\widehat{G}$ relative to the Haar measue. For each $f \in L^1(G)$, we write and set

$$\widehat{f}(p) = \left(\widehat{\mathcal{F}}f\right)(p) = \int_G \langle s, p\rangle f(s)\mathrm{d}s, \quad p \in \widehat{G};$$
$$\alpha_{\widehat{f}}(x) = \int_G f(s)\alpha_s(x)\mathrm{d}s, \quad f \in A\left(\widehat{G}\right), \quad x \in X.$$

Then $\alpha_{\widehat{f}} \in \mathcal{L}(X)$ and map $\alpha : \widehat{f} \in A\left(\widehat{G}\right) \mapsto \alpha_{\widehat{f}} \in \mathcal{L}(X)$ is a representation of a Banach algebra $A\left(\widehat{G}\right)$ on the Banach space X, i.e.,

$$\alpha_{\widehat{f}\widehat{g}} = \alpha_{\widehat{f}} \circ \alpha_{\widehat{g}}, \quad \widehat{f},\ \widehat{g} \in A\left(\widehat{G}\right).$$

For a fixed element $x \in X$, the set

$$I_\alpha(x) = \left\{\widehat{f} \in A\left(\widehat{G}\right) : \alpha_{\widehat{f}}(x) = 0\right\}$$

is a closed[8] ideal of $A\left(\widehat{G}\right)$. Hence we get a closed subset

$$\mathrm{Sp}_\alpha(x) = \left\{p \in \widehat{G} : \widehat{f}(p) = 0, \widehat{f} \in I_\alpha(x)\right\} \subset \widehat{G}.$$

and call it the α**-spectrum** of $x \in X$.

With

$$I(\alpha) = \left\{\widehat{f} \in A\left(\widehat{G}\right) : \alpha_{\widehat{f}} = 0\right\} = \bigcap_{x \in X} I_\alpha(x),$$

[8]The norm topology in $A\left(\widehat{G}\right)$ is inherited from $L^1(G)$.

the zero point set of the ideal $I(\alpha)$ of $A\Big(\widehat{G}\Big)$ is called the **Arveson spectrum** of α and written $\mathrm{Sp}(\alpha)$.

The ***α*-spectrum $\mathbf{Sp}_{\boldsymbol{\alpha}}\boldsymbol{(x)}$** of $x \in X$ is the oscillation mode of the map: $s \in G \mapsto \alpha_s(x) \in X$ as seen below. Then the following properties of the α-spectrum of $x \in \mathcal{M}$ is easily shown:

$$\mathrm{Sp}_\alpha(\alpha_{\widehat{f}}(x)) \subset \mathrm{supp}(\widehat{f}) \cap \mathrm{Sp}_\alpha(x).$$

DEFINITION 4.1. For a subset $E \subset \widehat{G}$, we set

$$X^\alpha(E) = \{x \in X : \mathrm{Sp}_\alpha(x) \subset E\}.$$

Example 4.2. *Suppose that a map $x(\cdot)$ is a X-valued continuous function on $\widehat{G}$ such that*

$$\alpha_s(x(p)) = \langle s, p\rangle x(p), \quad p \in \widehat{G}.$$

Set

$$x = \int_{\widehat{G}} x(p)\mathrm{d}p \in X.$$

For each $f \in L^1(G)$, let

$$\widehat{f}(p) = \Big(\widehat{\mathcal{F}}f\Big)(p) = \int_G \langle s, p\rangle f(s)\mathrm{d}s,$$

and compute

$$\begin{aligned}
\alpha_{\widehat{f}}(x) &= \int_G f(s)\alpha_s\left(\int_{\widehat{G}} x(p)\mathrm{d}p\right)\mathrm{d}s \\
&= \iint\limits_{G\times\widehat{G}} f(s)\alpha_s(x(p))\mathrm{d}s\mathrm{d}p = \iint\limits_{G\times\widehat{G}} f(s)\langle s, p\rangle(x(p))\mathrm{d}s\mathrm{d}p \\
&= \int_{\widehat{G}} \widehat{f}(p)x(p)\mathrm{d}p.
\end{aligned}$$

Then conclude that

$$\widehat{f} \in I_\alpha(x) \quad \Leftrightarrow \quad \mathrm{supp}(f) \cap \mathrm{supp}(x(\cdot)) = \emptyset.$$

Thus we get in this case

$$\mathrm{Sp}_\alpha(x) = \mathrm{supp}(x(\cdot)).$$

Theorem 4.3. (Arveson's Theorem). *Suppose that β be another action of G on X such that $\beta_s \in \mathrm{GL}_w(X)$ and*

$$\lim_{s\to 0}\|\varphi\circ\beta_s - \varphi\| = 0, \quad \varphi \in X_*.$$

Then the following statements are equivalent:

i) $\alpha_s = \beta_s, s \in G$;
ii) $X^\alpha(E) \subset X^\beta(E)$ *for all compact subset* $E \subset \widehat{G}$;
iii) $X^\beta(E) \subset X^\alpha(E)$ *for all compact subset* $E \subset \widehat{G}$;
iv) $X^\beta(E) = X^\alpha(E)$ *for all compact subset* $E \subset \widehat{G}$.

The proof is quite technical. So we omite the proof.

The following result indicates the property of the Arveson spectrum.

Proposition 4.4. *If $\{\mathcal{M}, G, \alpha\}$ is a covariant system on locally compact abelian group G, then the Arveson spectrum of each $x \in \mathcal{M}$ enjoys the following properties*:

i) $\mathrm{Sp}_\alpha(x^*) = -\mathrm{Sp}_\alpha(x), x \in \mathcal{M}$, i′) $\mathcal{M}^\alpha(E)^* = \mathcal{M}^\alpha(-E)$, $E \subset \widehat{G}$,
ii) $\mathrm{Sp}_\alpha(xy) \subset \overline{\mathrm{Sp}_\alpha(x) + \mathrm{Sp}_\alpha(y)}$, ii′) $\mathcal{M}^\alpha(E)\mathcal{M}^\alpha(F) \subset \mathcal{M}^\alpha\big(\overline{E+F}\big)$.

We continue to work on the covariant system $\{\mathcal{M}, G, \alpha\}$ with G a locally compact abelian group, and set

$$p^\alpha(E) = \sup\{s_\ell(x) : x \in \mathcal{M}^\alpha(E)\};$$
$$q^\alpha(E) = \sup\{s_r(x) : x \in \mathcal{M}^\alpha(E)\},$$

for each closed subset $E \subset \widehat{G}$. where $s_\ell(x)$ and $s_r(x), x \in \mathcal{M}$ mean respectively the left and right support of x, i.e.,

$$s_\ell(x) = \sigma\text{-strong}^* \lim_{n\to\infty} (xx^*)^{\frac{1}{n}}, \quad s_r(x) = \sigma\text{-strong}^* \lim_{n\to\infty} (x^*x)^{\frac{1}{n}}, \quad x \in \mathcal{M}.$$

Theorem 4.5. *Under the above notation, both $p^\alpha(E)$ and $q^\alpha(E)$ for a closed subset $E \subset \widehat{G}$ are projections in the center $\mathcal{C}_\alpha$ of the fixed point subalgebra $\mathcal{M}^\alpha$.*

Making use of the above discusion, the following refined theorem of Borchers can be shown.

Theorem 4.6 (Borchers-Arveson). *For a one parameter automorphism group $\{\mathcal{M}, \mathbb{R}, \alpha\}$ of a von Neumann algebra $\mathcal{M}$, the following two conditions are equivalent*:

i) *There exists a one parameter unitary group $\{u(t) : t \in \mathbb{R}\}$ in the unitary group $\mathcal{U}(\mathcal{M})$ such that*

$$\alpha_t(x) = u(t)xu(t)^*, \quad x \in \mathcal{M}, \quad t \in \mathbb{R};$$

and the one parameter unitary group $\{u(t) : t \in \mathbb{R}\}$ *has a positive spectrum in the sense that the spectral integration*:

$$u(t) = \int_0^\infty e^{\mathrm{i}\lambda t} \mathrm{d}e(\lambda)$$

gives the spectral decompostion of $\{u(t)\}$ *over the positive real line* $[0, \infty)$;

ii)

$$\inf\{q_\alpha([t, +\infty)) : t \in \mathbb{R}\} = 0.$$

If the condition (ii) *holds, then the spectral measure* $e(\cdot)$ *of* $\{u(t)\}$ *is given explicitly in the following form*:

$$e([t, +\infty)) = \inf\{q^\alpha([s, +\infty)) : s < t\}.$$

Returning to the original setting $\{X, X_*, \alpha\}$ on locally compact abelian group G, we have the following

Theorem 4.7. i) *Let* $\mathfrak{A}$ *be the Banach subalgebra of* $\mathcal{L}(X)$ *generated by* $\alpha_{\widehat{f}}, f \in A\left(\widehat{G}\right)$. *Then the spectrum* $\mathrm{Sp}(\mathfrak{A})$ *of the commutative Banach algebra* $\mathfrak{A}$ *is naturally identified with the Arveson spectrum* $\mathrm{Sp}(\alpha)$ *of the action* α.

ii) *The following two conditions on* $\{X, X_*, \alpha\}$ *are equivalent*:

a) *The map*: $s \in G \mapsto \alpha_s \in \mathcal{L}(X)$ *is contiuous in norm*;

b) *The Arveson spectrum* $\mathrm{Sp}(\alpha)$ *is compact.*

Definition 4.8. Let A be a C*-algebra and $\delta : x \in A \mapsto \delta(x)$ is called a **derivation** if δ is a linear map such that

$$\delta(xy) = \delta(x)y + x\delta(y), \quad x, y \in A.$$

When we have

$$\delta(x^*) = \delta(x)^*, x \in A,$$

it is called a **$*$-derivation**. Setting

$$\delta^*(x) = \delta(x^*)^*, \quad x \in A,$$

we obtain another derivation δ^* and we decompose the derivation into the linear combination of $*$-derivations:

$$\delta = \frac{1}{2}(\delta + \delta^*) + \mathrm{i}\frac{1}{2\mathrm{i}}(\delta - \delta^*).$$

Theoem 4.9. (Sakai's Theorem). *A derivation* δ *on a C***-algebra* A *is always bounded and generates a norm continuous one parameter automorphism group of* A *in the sense*

$$\alpha_t(x) = \sum_{n \in \mathbb{Z}_+} \frac{t^n}{n!}\delta^n(x), \quad t \in \mathbb{R}.$$

Each automorphism α_t *is a* $*$*-automorphism if and only if* δ *is a* $*$*-derivation.*

Theorem 4.10. (Sakai - Kadison). *A derivation on a von Neumann algebra* $\mathcal{M}$ *is inner in the sense that there exists* $a \in \mathcal{M}$ *such that*

$$\delta(x) = [a, x] = ax - xa, \quad x \in \mathcal{M}.$$

Lecture 5: Connes Spectrum.

We return to the study of a covariant system $\{\mathcal{M}, G, \alpha\}$ over a locally compact abelian group G. We denote the fixed point subalgera of $\mathcal{M}$ under the action α by $\mathcal{M}^\alpha$, i.e.,

$$\mathcal{M}^\alpha = \{x \in \mathcal{M} : \alpha_s(x) = x, \quad s \in G\},$$

or in the previous lecture's notation $\mathcal{M}^\alpha = \mathcal{M}^\alpha(0)$.

Definition 5.1. In the above setting, if $e \in \mathrm{Proj}(\mathcal{M}^\alpha)$, then the action α leaves the reduced algebra $\mathcal{M}_e$ globally invariant, so that the restriction $\alpha_s^e,\ s \in G$, of α_s to the reduced algebra $\mathcal{M}_e = e\mathcal{M}e$ makes sense. We call this restriction $\{\mathcal{M}_e, G, \alpha^e\}$ a **reduced** covariant system and α^e the **reduced** action of α by the projection $e \in \mathcal{M}^\alpha$. The intersection:

$$\Gamma(\alpha) = \bigcap \{\mathrm{Sp}(\alpha^e) : e \in \mathrm{Proj}(\mathcal{M}^\alpha), e \neq 0\}$$

is called the **Connes spectrum** of α. Let us state the main theorem concerning the Connes spectrum of a covariant system $\{\mathcal{M}, G, \alpha\}$ with G a locally compact abelian group.

Theorem 5.1 (Connes - Takesaki). *Suppose that $\{\mathcal{M}, G, \alpha\}$ with G is a covariant system over a locally compact abelian group G. Then the Connes spectrum $\Gamma(\alpha)$ has the following properties*:

i) *The Connes spectrum $\Gamma(\alpha)$ is a closed subgroup of $\widehat{G}$*;
ii) $\Gamma(\alpha) + \mathrm{Sp}(\alpha) = \mathrm{Sp}(\alpha)$;
iii) *If β is another action of G on $\mathcal{M}$ which is cocycle conjugate to α, then* $\Gamma(\alpha) = \Gamma(\beta)$;
iv) *The Connes spectrum is precisely the kernel of the restriction*

$$\left\{\mathcal{C}_{\mathcal{N}}, \widehat{G}, \widehat{\alpha}\right\}$$

of the dual system $\widehat{\alpha}$ to the center $\mathcal{C}_{\mathcal{N}}$ of the crossed product $\mathcal{N} = \mathcal{M} \rtimes_\alpha G$.

Corollary 5.2. *If $\mathcal{M}$ is a factor of type* III*, then $\Gamma(\sigma^\varphi)$ is a closed subgroup of the real additive group $\mathbb{R}$ and independent of the choice of a faithful semi-finite normal weight φ on $\mathcal{M}$, hence it is an algebraic invariant of the factor $\mathcal{M}$.*

Definition 5.3. A factors of type III is called of **type** III_λ with $0 < \lambda < 1$ if

$$\Gamma(\sigma^\varphi) = -(\log\, \lambda)\mathbb{Z} \quad \text{for any faithful semi-finite normal weight } \varphi;$$

of **type** III_1 if

$$\Gamma(\sigma^\varphi) = \mathbb{R} \quad \text{for any faithful semi-finite normal weight } \varphi;$$

and of **type** III_0 in the remaining case.

Lecture 6: Examples.

We discuss examples. Fix $0 < \lambda < 1$ and let

$$M_n = \mathrm{M}(2, \mathbb{C}), \quad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, define a state ω_n^λ on M_n by the following:

$$\omega_n^\lambda \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \frac{\lambda^{\frac{1}{2}}}{\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}} x_{11} + \frac{\lambda^{-\frac{1}{2}}}{\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}} x_{22}, \quad \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in M_n.$$

Consider the algebraic infinite tensor product:

$$M_0 = \prod_{n \in \mathbb{N}}{}^{\otimes} M_n,$$

which is the set of linear combinations of the elements of the from:

$$x_1 \otimes \cdots x_n \otimes 1 \otimes 1 \otimes \cdots, \quad x_i \in M_i, \quad i \in \mathbb{N}.$$

Then M_0 is naturally an involutive algebra on $\mathbb{C}$, which admits a unique C*-norm such that

$$\|x_1 \otimes \cdots x_n \otimes 1 \otimes 1 \otimes \cdots\| = \prod_{i=1}^{n} \|x_i\|.$$

The completion M of M_0 under this norm is called the infinite C*-tensor product. We will write this C*-algebra as follows:

$$M = \prod_{n=1}^{\infty}{}^{\hat{\otimes}} M_n.$$

On this C*-algebra M we define a state ω^λ in the following way:

$$\omega^\lambda(x_1 \otimes \cdots x_n \otimes 1 \otimes 1 \otimes \cdots) = \prod_{i=1}^{\infty} \omega_i(x_i)$$

where $x_{n+1} = 1, x_{n+2} = 1, \cdots$, which gives rise to a representation $\{\pi_\lambda, \mathfrak{H}_\lambda\}$ of M via the GNS construction and obtain the von Neumann algebra equipped with a faithful normal state ω^λ:

$$\{\mathcal{R}_\lambda, \omega^\lambda\} = \prod_{n=1}^{\infty}{}^{\otimes} \{M_n, \omega_n^\lambda\}.$$

To define an infinite tensor product of von Neumann algebras we need to specify state on each component von Neumann algebra. It is easy to see that each ω_n^λ is a faithful state on M_n and its modular automorphism group σ_n^λ is given by the following:

$$\sigma_{t,n}^\lambda = \mathrm{Ad}\begin{pmatrix} \lambda^{\frac{it}{2}} & 0 \\ 0 & \lambda^{\frac{-it}{2}} \end{pmatrix}.$$

From the modular conditon, it is easy to see that the modular automorhism group of ω^λ is the tensor product automorphism group:

$$\sigma_t^{\omega^\lambda} = \prod_{n=1}^{\infty}{}^{\otimes}\sigma_{t,n}^\lambda, \quad t \in \mathbb{R}.$$

We write σ_t^λ for this heavy notation $\sigma_t^{\omega^\lambda}$ instead.

With

$$T = -\frac{2\pi}{\log\lambda} > 0,$$

we have

$$\sigma_{T,n}^\lambda = \mathrm{id}, \quad n \in \mathbb{N},$$

so that each component modular automorphism group σ_n^λ has common period T and consequently

$$\sigma_T^\lambda = \mathrm{id}.$$

Let $\mathfrak{S}$ be the finite permutation group of $\mathbb{N}$, i.e. each element $\sigma \in \mathfrak{S}$ permute only finitely many members of $\mathbb{N}$, which then acts on $\mathcal{M}_0$ in the following way:

$$\sigma(x_1 \otimes \cdots x_n \otimes 1 \otimes 1 \otimes \cdots) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)} \otimes \cdots$$

if $\sigma(n+k) = n+k, k \in \mathbb{N}$, which is then extended by linearity to M_0 and leaves the state ω^λ invariant. Furthermore, the automorphism σ acts only on the first n-components and leaves the rest invariant and the first n-tensor product is isomorphic to the matrix algebra $M(2^n, \mathbb{C})$ of order 2^n. Hence it is given by a unitary, say

$$U(\sigma) \in \prod_{k=1}^{n}{}^{\otimes} M_k.$$

As σ leaves ω^λ invariant, $U(\sigma)$ belongs to the centerlizer:

$$\left(\prod_{k=1}^{n}{}^{\otimes} M_k\right)^{\omega^\lambda}.$$

Now for each $n \in \mathbb{N}$, consider the permutation σ_n which permutes the first n terms with the n terms starting from $2^n + 1$ and leaves the rest unchanged. Then it is easy to see that

$$\lim_{n\to\infty} \omega^\lambda(\sigma_n(x)y) = \omega^\lambda(x)\omega^\lambda(y), \quad x, y \in M_0,$$

which is called the **mixing** property of ω^λ. This property entails the following σ-weak convergence:

$$\lim_{n\to\infty} \sigma_n(x) = \omega^\lambda(x) \quad \sigma\text{-weakly for every } x \in \mathcal{R}_\lambda.$$

Now each σ_n is given by $\mathrm{Ad}(U(\sigma_n))$ and $U(\sigma_n) \in (\mathcal{R}_\lambda)_{\omega^\lambda}$. Consequently

$$(\mathcal{R}_\lambda)'_{\omega^\lambda} \cap \mathcal{R}_\lambda = \mathbb{C}.$$

In particular, the centralizer $\mathcal{R}_{\lambda,\omega^\lambda}$ of ω^λ is a factor. Hence $\Gamma\left(\sigma^{\omega^\lambda}\right) = -(\log\lambda)\mathbb{Z} \subset \mathbb{R}$. Thus the factor $\mathcal{R}_\lambda$ is of type III_λ. We now come to the result of Powers in 1967:

Theorem 6.1 (R.T. Powers). *If* $0 < \lambda \neq \mu < 1$, *then*

$$\mathcal{R}_\lambda \not\cong \mathcal{R}_\mu.$$

Now we fix $0 < \lambda \neq \mu < 1$ such that

$$\frac{\log \lambda}{\log \mu} \notin \mathbb{Q},$$

which guarantees the following:

$$\lambda^{\mathrm{i}t}\mu^{-\mathrm{i}t} \neq 1 \quad \text{except for } t = 0.$$

To each $n \in \mathbb{N}$, we asign the 3×3 matrix algebra:

$$M_n = \mathrm{M}(3, \mathbb{C}),$$

and consider the state given by the following:

$$\omega_n \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{31} & x_{33} \end{pmatrix} = \frac{1}{1+\lambda+\mu}x_{11} + \frac{\lambda}{1+\lambda+\mu}x_{22} + \frac{\mu}{1+\lambda+\mu}x_{33}$$

The modular automorphism group $\{\sigma_t^n\}$ of ω_n is give by the following one parameter unitary group

$$u_n(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^{\mathrm{i}t} & 0 \\ 0 & 0 & \mu^{\mathrm{i}t} \end{pmatrix}, \quad t \in \mathbb{R}.$$

The choice of λ and μ entails that

$$u_n(t) \notin \mathbb{T} \quad \text{for all non-zero } t \in \mathbb{R}.$$

Consequently, the modular automorphism group $\{\sigma_t^n\}$ is not periodic. Let M be the infinite C*-tensor product:

$$M = \prod_{n=1}^{\infty} {}^{\hat{\otimes}} M_n,$$

and set

$$\{\mathcal{R}, \omega\} = \prod_{n=1}^{\infty} {}^{\otimes} \{M_n, \omega_n\},$$

i.e.,

$$\omega = \prod_{n=1}^{\infty} {}^{\otimes} \omega_n, \quad \mathcal{R} = \pi_\omega(M)''.$$

By the similar arguments as in the previous case, we can conclude that

$$\mathcal{R}'_\omega \cap \mathcal{R} = \mathbb{C}.$$

Thus we get

$$\Gamma(\sigma^\omega) = \mathbb{R}.$$

Thus the factor $\mathcal{R}$ is of type III_1. This factor is not isomorphic to any of $\mathcal{R}_\lambda, 0 < \lambda < 1$, and does not admit a faithful semi-finite normal weight with periodic modular automorphism group.

Lecture 7: Crossed Product Construction of a Factor.

§7.1. Transformation Groups of a Standard Measure Space. Let $\{X, \mu\}$ be a standard measure space with σ-finite measure μ which gives rise to an abelian von Neumann algebra $\mathcal{A} = L^\infty(X, \mu)$ of all essentially bounded complex valued measurable functions on X which acts on the Hilbert space $\mathfrak{H}_0 = L^2(X, \mu)$ of square integrable functions. Let G be a separable locally compact group. By an action G on $\{X, \mu\}$ we mean a Borel map $T : (g, x) \in G \times X \mapsto T_g x \in X$ such that

i) for each $s \in G$ the map: $x \in X \mapsto T_s x \in X$ is a non-singular transformation of X, i.e.,

$$\mu(T_s(N)) = 0 \quad \Leftrightarrow \quad \mu(N) = 0, \quad N \subset X;$$

ii)

$$T_s \circ T_t = T_{st}, \quad s, t \in G;$$

iii)

$$T_e x = x, \quad x \in X, \quad \text{for the identity } e \in G.$$

We naturally assume that the action is **effective** in the sense described below:

$$T_s x = x \quad \text{for almost every } x \in X \quad \Rightarrow \quad s = e.$$

We call the system $\{G, X, \mu, T\}$ **a G-measure space.** This is equivalent to an action α of G on the abelian von Neumann algebra $\mathcal{A} = L^\infty(X, \mu)$ and the relation between the action α and the transformation T is described in the following:

Let α be an action of G on the abelian von Neumann algebra $\mathcal{A}$ which entails a non-singular transformation group $\{T_g : g \in G\}$ of the measure space $\{X, \mu\}$:

$$\alpha_s(a)(x) = a\big(T_s^{-1} x\big), \quad a \in \mathcal{A} = L^\infty(X, \mu),\ s \in G,\ x \in X.$$

In this setting, we have the Radon-Nikodym derivative:

$$\rho(s, x) = \frac{\mathrm{d}\mu \circ T_s}{\mathrm{d}\mu}(x), \quad s \in G, x \in X.$$

The ρ-function satisfies the following cocycle identity:

$$\rho(st, x) = \rho(s, T_t x)\rho(t, x), \quad s, t \in G, x \in X,$$

and

$$\int_X f\big(T_s^{-1} x\big) \mathrm{d}\mu(x) = \int_X f(x)\rho(s, x) \mathrm{d}\mu(x), \quad f \in L^1(X, \mu).$$

This tells us that the transformation rules on $L^p(X,\mu)$ and $L^q(X,\mu), 1 \le p \ne q < +\infty$, are different:

$$\begin{aligned}
\alpha_s(f)(x) &= \rho\big(s^{-1},x\big)^{\frac{1}{q}} f\big(T_s^{-1}x\big), \quad f \in L^p(X,\mu), \quad s \in G;\\
\alpha_s(f)(x) &= \rho\big(s^{-1},x\big)^{\frac{1}{q}} f\big(T_s^{-1}x\big), \quad f \in L^q(X,\mu), \quad s \in G;\\
\|\alpha_s(f)\|_p &= \left(\int_X \Big(\rho\big(s^{-1},x\big)^{\frac{1}{p}} \big|f\big(T_s^{-1}x\big)\big|\Big)^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&= \left(\int_X \rho\big(s^{-1},x\big)\big|f\big(T_s^{-1}x\big)\big|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&= \left(\int_X \big|f\big(T_s^{-1}\circ T_s x\big)\big|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} = \left(\int_X |f(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&= \|f\|_p.
\end{aligned}$$

For $p = 2$, we write $U(s)\xi, \xi \in L^2(X,\mu)$, and obtain a unitary representation $U(\cdot)$ of G on $L^2(X,\mu)$ which implements the action α in the following sense:

$$\alpha_s(a) = U(s)aU(s)^*, \quad s \in G, \quad a \in \mathcal{A} = L^\infty(X,\mu),$$

where we regards $\mathcal{A}$ as a von Neumann algebra acting on $L^2(X,\mu)$ by multiplication. The measurability of the map:

$$s \in G \mapsto \omega_{\xi,\eta}(U(s)) = (U(s)\xi \mid \eta), \quad \xi,\eta \in L^2(X,\mu)$$

entails the strong continunity of $s \in G \mapsto U(s) \in \mathcal{L}\big(L^2(X,\mu)\big)$, and hence it is a unitary representation of G and $\{\mathcal{A}, G, \alpha\}$ is a covariant system. Associated with this is the crossed product von Neumann algebra

$$\mathcal{R}(X,\mu,G,\alpha) = \mathcal{A} \rtimes_\alpha G.$$

The measure space theory always involves almost everywhere arguments, which is quite sticky in the case that G is a continuous group as it is not countable. For this reason, if we can replace $\{X,\mu,G,T\}$ by a topological transformation group, it is much easier to handle. So we consider the C*-covariant system.

§7.2. C*-Covariant System for a Commutative Covariant System. We begin by defenition:

Definition 7.1. Let A be a C*-algebra and G a locally compact group . An action α of G on A is a map $\alpha : s \in G \mapsto \alpha_s \in \mathrm{Aut}(A)$ is a homomorphism such that

$$\lim_{s\to t}\|\alpha_s(x) - \alpha_t(x)\| = 0, \quad x \in A.$$

Proposition 7.2. *If $\{\mathcal{M}, G, \alpha\}$ is a covariant system, then the set*

$$A_c^\alpha = \left\{x \in \mathcal{M} : \lim_{s \to e} \|\alpha_s(x) - x\| = 0\right\}$$

is a unital σ-weakly dense C-subalgebra of $\mathcal{M}$. If $\mathcal{M}_*$ is separable, then we can choose a separable σ-weakly dense unital C*-subalgebra $A \subset A_c$ which is globally invariant under α and the restriction $\{A, G, \alpha\}$ is a* C**-covariant system.*

§**7.3. Continuity as a Result of Measurability.** As $\lim_{s \to e} \|\lambda_s(f) - f\|_1 = 0, f \in L^1(G)$, we can consider the subset of $\mathcal{M}$:

$$\left\{\alpha_f(x) : x \in \mathcal{M}, f \in L^1(G)\right\}$$

which is a subset of A_c and σ-weakly dense in $\mathcal{M}$, because

$$\alpha_s(\alpha_f(x)) = \alpha_{\lambda_s(f)}(x) \quad \text{and} \quad \|\alpha_f(x)\| \le \|f\|_1 \|x\|, \quad x \in \mathcal{M}, f \in L^1(G).$$

So if $\{X, \mu, G, T\}$ is an action of separable locally compact group G on a standard measure space, then we obtain a covariant system $\{\mathcal{A}, G, \alpha\}$ as before and here we take a seprable C*-covairant system $\{A, \alpha\}$ to obain a compact space $Y = \mathrm{Sp}(A)$ on which G acts as a topological transformation group leaving a probability Radon measure ν on Y quasi-invariant corresponding to the original measure μ on X after replacing μ by an equivalent probability measure. So we will consider only the case that $\{X, \mu\}$ is a compact space and μ is a probability Radon measure and $\{X, G, T\}$ is a toplogical transformation group of X which leaves the measure μ quasi-invariant.

§**7.4. Free and Ergodic Covariant System.** With $\{X, G, \mu, T\}$ as above, we then consider the crossed product

$$\mathcal{R}(X, G, \mu, T) = \mathcal{A} \rtimes_\alpha G, \quad \mathcal{A} = L^\infty(X, \mu).$$

We often omit the letter T and write $\mathcal{R}(X, G, \mu)$.

Definition 7.3. If the fixed point reduces to the scalar, $\mathcal{A}^\alpha = \mathbb{C}$, then we say that the system $\{X, G, \mu, T\}$ is **ergodic**. The action T is called **free** if for any compact subset $K \subset G$ such that $e \notin K$ and for any Borel subset $E \subset X$ with $\mu(E) > 0$ then there exists a Borel subset $F \subset E$ such that

$$T_s(F) \cap F = \emptyset, \quad s \in K, \quad \text{and} \quad \mu(F) > 0.$$

In terms of von Neumann algebra lungage, the freeness is phrased as in the following: for any compact subset $K \subset G$ such that $e \notin K$ and any non-zero projection $e \in \mathrm{Proj}(\mathcal{A})$ there exists $f \in \mathrm{Proj}(\mathcal{A})$ such that

$$f \perp \alpha_s(f), \quad s \in K, \quad \text{and} \quad f \ne 0.$$

Theorem 7.4. *If $\{\mathcal{A}, G, \alpha\}$ is a covariant system with $\mathcal{A}$ abelian, then the following two conditions on the covariant system are equivalent*:

i) *The action T of G on X is free*;
ii) *The original abelian von Neumann algebra $\mathcal{A}$ in the crossed product $\mathcal{A} \rtimes_\alpha G$ is maximal abelian.*

For the proof on a general locally compact group G, it involves the duality theory of unitary representations of G. For an abelian group G, we can prove as an application of the Arveson - Connes spectral theory for actions.

Corollary 7.5. *Under the assumption that $\{\mathcal{A}, G, \alpha\}$ is free, the crossed product $\mathcal{A} \rtimes_\alpha G$ is a factor if and only if the system is ergodic.*

Theorem 7.6. *Let $\{\mathcal{A}, G, \alpha\}$ is an ergodic and free covariant system with $\mathcal{A}$ abelian, then we have the following properties on the crossed product $\mathcal{R} = \mathcal{A} \rtimes_\alpha G$*:

i) *$\mathcal{R}$ is of type I if and only if the measue is concentrated on a single orbit, equivalently the action is transitive.*
ii) *$\mathcal{R}$ is of type II_1 if and only if G is discrete and there exists a finite invariant measure ν equivalent to the original measure μ.*
iii) *$\mathcal{R}$ is of type II_∞ if and only if*
   a) *The action is not transitive*;
   b) *There exists a Borel measure ν equivalent to μ such that*

$$\delta_G(s)\frac{\mathrm{d}\nu\circ T_s}{\mathrm{d}\nu}(x) = 1 \quad \textit{for every } s \in G \textit{ and } x \in X.$$

   where $\delta_G(\cdot)$ is the modular function of G, and
   c) *if G is discrete, then the measure ν is infinite.*
iv) *$\mathcal{R}$ is of type* III *if and only if there is no nontrivial Borel measure equivalent to μ which satisfies the condition* (iii-b).

In the sequel, we consider only ergodic case as the genral case is decomposed into ergodic systems.

S7.5. Concrete Example of Each Type. Now we are going to discuss examples which realizes the above criteria. We continue to use the above notations unless we say contrary.

Example 7.7. *If we take $\{X, \mu\}$ to be the group G itself with left Haar measure μ, then the left translation action of G on X is obviously transitive.*

i) *If G is infinite, then $\mathcal{R}$ is a factor of type I_∞ .*
ii) *If G is of oder n, then $\mathcal{R}$ is a factor of type I_n.*

Example 5.8. *Let $X = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the one dimensional torus. Fix an irrational number θ and set*

$$Tx = \exp(2\pi \mathrm{i}\theta)x, \quad x \in X.$$

The Lebesgue measure μ is invariant under T. Viewing T as an action of the additive integer group $G = \mathbb{Z}$, then the action is free and ergodic with finite invariant measure. As each orbit is countable, the action can not be transitive. Hence the resulted factor $\mathcal{R}$ is of type II_1.

Example 7.9. *Take $G = \mathbb{Q}$ to be the additive group of rational numbers and let it act on $X = \mathbb{R}$ by translation*:

$$T_s x = x + s, \quad x \in X = \mathbb{R},\ s \in \mathbb{Q}.$$

Take the Lebesgue measure $\mathrm{d}x$ *on X. Then the action is free and ergodic. The measure is invariant under the translation action T of $G = \mathbb{Q}$ and infinite. Hence the resulted factor $\mathcal{R}$ is a factor of type* II_∞ .

Example 7.10. *Let G be the $ax + b$ group, i.e., it is given by the following*:

$$G = \left\{ g = g(a, b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Q}_+^*, b \in \mathbb{Q} \right\}.$$

Let G act on $X = \mathbb{R}$ in the following way:

$$T_{g(a,b)} x = ax + b$$

The subgroup

$$H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Q} \right\}$$

acts on $X = \mathbb{R}$ ergodically by translation. So any other invariant measure on X equivalent to the Lebesgue measure is propotional to the Lebesgue measure. But each element $g(a, 0), a \in \mathbb{Q}_+^*, a \neq 1$, transforms the Lebesgue measure μ to $a\mu$. Hence it does not leave μ invariant. Consequently, there is no invariant measure equivalent to the Lebesgue measure. So the resulted factor is of type III.

Example 7.11. *Let $X = \mathbb{R}$ and*

$$G = \left\{ \begin{pmatrix} \lambda^n & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}(\lambda) \right\}$$

for a fixed $0 < \lambda < 1$, where $\mathbb{Z}(\lambda)$ means the additive subgroup of $\mathbb{R}$ generated by the powers $\lambda^n, n \in \mathbb{Z}$, of λ:

$$\mathbb{Z}(\lambda) = \left\{ \sum_{k=-n}^{n} \frac{a_{-n}}{\lambda^{-n}} + \cdots + \frac{a_{-1}}{\lambda} + a_0 + a_1\lambda \cdots + a_n\lambda^n : a_i \in \mathbb{Z}, n \in \mathbb{N} \right\},$$

then the resulted factor $\mathcal{R}$ is of type III_λ.

Example 7.12. *Fix an irrational positive number θ. On $L^2(\mathbb{R})$ we define two unitaries*:

$$(U\xi)(x) = \xi(x+1) \quad (V\xi)(x) = \exp(2\pi\mathrm{i}\theta x)\xi(x), \quad \xi \in L^2(\mathbb{R}).$$

The von Neumann algebra $\mathcal{R}$ generated by $\{U, V\}$ is of type II_1 and the relative dimension $\dim_{\mathcal{R}}\big(L^2(\mathbb{R})\big)$ of $L^2(\mathbb{R})$ over $\mathcal{R}$ is θ.

§**7.6 Flow of Weights for Ergodic Free Transformation Group.** First we begin by definition:

Definition 7.13 For a factor $\mathcal{M}$, the restriction $\{\mathcal{C}, \mathbb{R}, \theta\}$ of the trace scaling covariant system $\{\mathcal{N}, \mathbb{R}, \tau\}$ with

$$\mathcal{M} = \mathcal{N} \rtimes_\theta \mathbb{R}$$

to the center $\mathcal{C} = \mathcal{C}_{\mathcal{N}}$ of $\mathcal{N}$ is called the **flow of weights on $\mathcal{M}$**.

We now dicuss the flow of weights on the factor $\mathcal{R} = \mathcal{R}(X, \mu, G)$ constructed from a free erogodic system $\{X, G, \mu\}$ in the case that G is discrete. The continuous case can be handled similarly. So suppose that the covariant system $\{\mathcal{A}, G, \alpha\}$ is free and ergodic with the ρ-function $\rho(\cdot,\cdot)$ on $G \times X$. Recall the cocycle identity:

$$\rho(g, s) = \frac{\mathrm{d}\mu\circ g}{\mathrm{d}\mu}(x), \quad x \in X, g \in G;$$
$$\rho(gh, x) = \rho(g, hx)\rho(h, x), \quad g, h \in G, x \in X.$$

The cocycle identity allows us to make the product space $\widetilde{X} = \mathbb{R}_+^* \times X$ an $R \times G$-space as follows:

$$\widetilde{T}_{s,g}(\lambda, x) = \big(e^{-s}\rho(g, x)\big)\lambda, gx), \quad (s, g) \in \mathbb{R} \times G,\ (\lambda, x) \in \widetilde{X};$$
$$\int_{\widetilde{X}} f(\lambda, x)\mathrm{d}\tilde{\mu}(\lambda, x) = \iint_{\mathbb{R}_+^* \times X} f(\lambda, x)\mathrm{d}\lambda\mathrm{d}\mu(x), \quad f \geq 0.$$

Since

$$\int_0^\infty f(a\lambda)\mathrm{d}\lambda = \frac{1}{a}\int_0^\infty f(\lambda), \mathrm{d}\lambda, \quad a > 0,$$

we have for a positive measurbale function f on $\widetilde{X}$,

$$\begin{aligned}\iint_{\widetilde{X}} f\circ\widetilde{T}_{s,g}(\lambda, x)\mathrm{d}\tilde{\mu}(\lambda, x) &= \int_X \int_0^\infty f\big(e^{-s}\rho(g, x)\lambda, gx\big)\mathrm{d}\lambda\mathrm{d}\mu(x)\\ &= \int_X e^s\rho(g, x)^{-1}\Big(\int_0^\infty f(\lambda, gx)\mathrm{d}\lambda\Big)\mathrm{d}\mu(x)\\ &= e^s \iint_{\mathbb{R}_+^* \times X} f(\lambda, x)\mathrm{d}\lambda\mathrm{d}\mu(x) = e^s \iint_{\widetilde{X}} f(\lambda, x)\mathrm{d}\tilde{\mu}(\lambda, x).\end{aligned}$$

Theorem 7.14 (Connes - Takesaki). *The flow of weights for the factor* $\mathcal{R} = \mathcal{R}(X, \mu, G)$ *associated with the ergodic free* G*-measure space* $\{X, \mu, G\}$*, the core* $\{\mathcal{N}, \mathbb{R}, \tau, \theta\}$ *is given by the following:*

i) *The core von Neumann algebra* $\mathcal{N}$ *is the group measure space von Neumann algebra* $\mathcal{N} = \mathcal{R}(\widetilde{X}, \tilde{\mu}, G)$ *of the system:*

$$\begin{aligned} \mathcal{N} &= L^\infty\Big(\widetilde{X}, \tilde{\mu}\Big) \rtimes_{\tilde{\alpha}} G; \\ (\tilde{\alpha}_g(a))(\lambda, x) &= a\Big(\widetilde{T}_{0,g}^{-1}(\lambda, x)\Big) \\ &= a\big(\rho(g^{-1}, x)\lambda, g^{-1}x\big), \ a \in L^\infty\Big(\widetilde{X}, \tilde{\mu}\Big); \end{aligned}$$

ii) *The one parameter automorphism group* θ *which scales down the trace is the lifting of the flow:*

$$\widetilde{T}_{s,e} : (\lambda, x) \in \widetilde{X} \mapsto \big(e^{-s}\lambda, x\big) \in \widetilde{X}$$

to the crossed product $\mathcal{N} = L^\infty\Big(\widetilde{X}, \tilde{\mu}\Big) \rtimes_{\tilde{\alpha}} G$.

The flow $\{\mathcal{C}, \mathbb{R}, \theta\}$ *of weights on* $\mathcal{R}$ *is then given by the flow on the fixed point subalgebra:*

$$\mathcal{C} = L^\infty\Big(\widetilde{X}, \tilde{\mu}\Big)^G.$$

The flow is the restriction of the flow $\{T_{s,e} : s \in \mathbb{R}\}$ *to the fixed point subalgebra* $\mathcal{C} = L^\infty\Big(\widetilde{X}, \tilde{\mu}\Big)^G$.

Lecture 8: Structure of a Factor of Type III. Trace Scaling One Parameter Automorphism Group

§8.1. Trace Scaling One Parameter Automorphism Group. Suppose that $\{\mathcal{N}, \mathbb{R}, \theta, \tau\}$ is a covariant system over the real line $\mathbb{R}$ which scales a faithful semifinite norma trace τ down, i.e.,

$$\tau\circ\theta_s = e^{-s}\tau, \quad s \in \mathbb{R}.$$

Set $\mathcal{M} = \mathcal{N}^\theta$. Fix $T > 0$. Then for any non-zero projection $e \in \mathcal{M}$, there exists a projection $e_0 \leq e$ such that

$$e_0 \perp \theta_T(e_0).$$

Such a projection is called a **wondering projection of $\boldsymbol{\theta_T}$.** Set

$$s_T(e_0) = \sum_{n\in\mathbb{Z}} \theta_T^n(e_0) \in \mathcal{N}^{\theta_T}.$$

Choose a maximal family of wondering projections $\{e_i : i \in I\}$ for θ_T such that

$$\theta_T(e_i) \perp e_i, \quad i \in I;$$
$$s_T(e_i) \perp s_T(e_j) \quad \text{for} \quad i \neq j, i, j \in I.$$

The maximality of the family $\{e_i : i \in I\}$ implies that

$$\theta_T(e) \perp e, \quad \sum_{n\in\mathbb{Z}} \theta_T^n(e) = 1, \quad \text{with} \quad e = \sum_{i\in I} e_i.$$

Set

$$f_n = \sum_{|k|\leq n} \theta_T^k(e), \quad n \in \mathbb{N}.$$

Then we have

$$f_n \nearrow 1$$

$$\begin{aligned}
I_\theta(f_n) &= \int_{\mathbb{R}} \theta_s\left(\sum_{|k|\leq n} \theta_T^k(e)\right) \mathrm{d}s = \sum_{|k|\leq n} \theta_T^k\left(\int_{\mathbb{R}} \theta_s(e)\mathrm{d}s\right) \\
&= \sum_{|k|\leq n} \theta_T^k\left(\sum_{m\in\mathbb{Z}} \int_{mT}^{(m+1)T} \theta_s(e)\mathrm{d}s\right) \\
&= \sum_{|k|\leq n} \theta_T^k\left(\sum_{m\in\mathbb{Z}} \theta_T^m\left(\int_0^T \theta_s(e)\mathrm{d}s\right)\right) \\
&= \int_0^T \left(\sum_{|k|\leq n} \theta_T^k\left(\sum_{m\in\mathbb{Z}} \theta_T^m(e)\right)\right) \mathrm{d}s = (2n+1)T < +\infty.
\end{aligned}$$

$$\mathfrak{m}_{I_\theta} \supset f_n \mathcal{N} f_n, \quad n \in \mathbb{N}.$$

Hence we conclude the following:

Lemma 8.1. *In the above setting, the action θ is integrable in the sense that*

$$\overline{\mathfrak{m}_{I_\theta}}^{\sigma\text{-weak closure}} = \mathcal{N}.$$

Now choose a faithful semi-finite normal weight $\varphi \in \mathfrak{W}_0(\mathcal{M})$ and set

$$\widehat{\varphi} = \varphi{\circ}I_\theta \in \mathfrak{W}_0(\mathcal{N}).$$

Then the faithful semi-finite normal weight $\widehat{\varphi}$ is θ-invariant and therefore we get

$$\begin{aligned}\theta_s((\mathrm{D}\widehat{\varphi} : \mathrm{D}\tau)_t) &= \left(\mathrm{D}\widehat{\varphi}{\circ}\theta_s^{-1} : \mathrm{D}\tau{\circ}\theta_s^{-1}\right)_t = (\mathrm{D}\widehat{\varphi} : \mathrm{D}(e^s\tau))_t \\ &= e^{-\mathrm{i}st}(\mathrm{D}\widehat{\varphi} : \mathrm{D}\tau)_t\end{aligned}$$

for every $s, t \in \mathbb{R}$. Hence we obtain a one parameter unitary group $\left\{h_\varphi^{\mathrm{i}t} : t \in \mathbb{R}\right\}$ such that

$$\theta_s\left(h_\varphi^{\mathrm{i}t}\right) = e^{-\mathrm{i}st}h_\varphi^{\mathrm{i}t}$$

such that

$$\begin{aligned}\sigma_t^\varphi &= \mathrm{Ad}\left(h_\varphi^{\mathrm{i}t}\right)|_{\mathcal{M}}, \quad t \in \mathbb{R}; \\ \mathcal{N} &= \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \quad \theta = \widehat{\sigma^\varphi}.\end{aligned}$$

Theorem 8.2.(Connes-Takesaki). *If $\{\mathcal{N}, \mathbb{R}, \tau, \theta\}$ is a trace scaling covariant system, then it is necessarily the dual system such that with $\mathcal{M} = \mathcal{N}^\theta$ the covariant system is decribed in the following way:*

$$\begin{aligned}\sigma_t^\varphi &= \mathrm{Ad}\left(h_\varphi^{\mathrm{i}t}\right)|_{\mathcal{M}}, \quad t \in \mathbb{R}; \\ \mathcal{N} &= \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \quad \theta = \widehat{\sigma^\varphi}\end{aligned}$$

for any faithful semi-finite normal weight $\varphi \in \mathfrak{W}_0(\mathcal{M})$.

Theorem 8.3. (Connes - Takesaki). *If $\{\mathcal{N}, \mathbb{R}, \tau, \theta\}$ is a trace scaling covariant system, then the relative commutant $\mathcal{M}' \cap \mathcal{N}$ of the fixed point $\mathcal{M} = \mathcal{N}^\theta$ is the center $\mathcal{C}$ of $\mathcal{N}$.*

Definition 8.4 The normalizer

$$\widetilde{\mathcal{U}}(\mathcal{M}) = \{u \in \mathcal{U}(\mathcal{N}) : u\mathcal{M}u^* = \mathcal{M}\}$$

of $\mathcal{M}$ in $\mathcal{N}$ is called the **extended unitary group of $\mathcal{M}$.**

Proposition 8.5. *For $u \in \mathcal{U}(\mathcal{N})$ to be a member of the extended unitary group $\widetilde{\mathcal{U}}(\mathcal{M})$ it is necessary and sufficient that*

$$(\partial_\theta u)_t = u^*\theta_t(u) \in \mathcal{U}(\mathcal{C}), \quad t \in \mathbb{R}.$$

The map $c : t \in \mathbb{R} \mapsto c(t) = (\partial_\theta u)_t$ is necessarily a cocycle and hence a member of $\mathrm{Z}_\theta^1(\mathbb{R}, \mathcal{U}(\mathcal{C}))$. Recall that the one parameter automorphism group θ is stable in the sense that every θ-cocycle is a coboundary. Hence the coboundary map $\partial_\theta : u \in \widetilde{\mathcal{U}}(\mathcal{M}) \mapsto \partial_\theta(u) \in \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C}))$ is a surjection. So we have the following exact sequence:

$$
1 \longrightarrow \mathcal{U}(\mathcal{M}) \longrightarrow \widetilde{\mathcal{U}}(\mathcal{M}) \xrightarrow{\partial_\theta} \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \longrightarrow 1
$$

Restricting the above exact sequence to the $\mathcal{U}(\mathcal{C})$, we get the following exact sequences:

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathbb{T} \arrow[r] \arrow[d] & \mathcal{U}(\mathcal{C}) \arrow[r, "\partial_\theta"] \arrow[d] & \mathrm{B}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathcal{U}(\mathcal{M}) \arrow[r] & \widetilde{\mathcal{U}}(\mathcal{M}) \arrow[r, "\partial_\theta"] & \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \arrow[r] & 1
\end{tikzcd}
$$

Adding the last low, we obtain the following characteristic square of $\mathrm{Aut}(\mathcal{M}) \times \mathbb{R}$-equivariant exact sequences:

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathbb{T} \arrow[r, "i"] \arrow[d, "i"'] & \mathcal{U}(\mathcal{C}) \arrow[r, "\partial_\theta"] \arrow[d, "i"'] & \mathrm{B}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \arrow[r] \arrow[d, "i"'] & 1 \\
1 \arrow[r] & \mathcal{U}(\mathcal{M}) \arrow[r, "i"] \arrow[d, "\mathrm{Ad}"'] & \widetilde{\mathcal{U}}(\mathcal{M}) \arrow[r, "\partial_\theta"] \arrow[d, "\widetilde{\mathrm{Ad}}"'] & \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Int}(\mathcal{M}) \arrow[r, "i"] \arrow[d] & \mathrm{Cnt_r}(\mathcal{M}) \arrow[r, "\dot{\partial}_\theta"] \arrow[d] & \mathrm{H}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd} \quad (*)
$$

Summerizing, we get the following:

Theorem 8.6. **(Katayama-Sutherland-Takesaki).** *Associated with a factor $\mathcal{M}$ is the above $\mathrm{Aut}(\mathcal{M}) \times \mathbb{R}$ equivariant characteristic square $(*)$.*

§**8.3. Characteristic Invariant.** Looking at the middle vertical column of the characteristic square, we recognize the following equivariant exact sequence:[9]

$$
\begin{gathered}
N = \mathrm{Cnt_r}(\mathcal{M}), \quad H = \widetilde{\mathcal{U}}(\mathcal{M}), \quad A = \mathcal{U}(\mathcal{C}), \quad G = \mathrm{Aut}(\mathcal{M}) \times \mathbb{R}; \\
E : 1 \longrightarrow A \xrightarrow{\ i\ } H \underset{\overleftarrow{\ \mathfrak{s}\ }}{\xrightarrow{\ j\ }} N \longrightarrow 1 \\
\mathfrak{s}(m)\mathfrak{s}(n)\mathfrak{s}(mn)^{-1} = \mu(m,n) \in A, \quad m,n \in N; \\
\alpha_g\big(\mathfrak{s}(g^{-1}mg)\big)\mathfrak{s}(m)^{-1} = \lambda(m,g) \in A.
\end{gathered}
$$

If we change the cross-section $\mathfrak{s}$ to another one $\mathfrak{s}' : m \in N \mapsto \mathfrak{s}'(m) \in H$, then the difference is in A, i.e.,

$$
\mathfrak{s}'(m)\mathfrak{s}(m)^{-1} = f(m) \in A, \quad m \in N.
$$

We then have

$$
\begin{aligned}
\mathfrak{s}'(m)\mathfrak{s}'(n) &= f(m)\mathfrak{s}(m)f(n)\mathfrak{s}(n) = f(m)f(n)\mathfrak{s}(m)\mathfrak{s}(n) \\
&= f(m)f(n)\mu(m,n)\mathfrak{s}(mn) \\
&= f(m)f(n)f(mn)^{-1}\mu(m,n)\mathfrak{s}'(m,n) \\
&= \mu'(m,n)\mathfrak{s}'(mn); \\
\mu'(m,n) &= f(m)f(n)f(mn)^{-1} = (\partial_1 f)(m,n)\mu(m,n); \\
&\qquad \mu' = (\partial_1 f)\mu; \\
\alpha_g\big(\mathfrak{s}'\big(g^{-1}mg\big)\big) &= \lambda'(m,g)\mathfrak{s}'(m) = \alpha_g\big(f\big(g^{-1}mg\big)\mathfrak{s}\big(g^{-1}mg\big)\big) \\
&= \alpha_g\big(f\big(g^{-1}mg\big)\big)\alpha_g\big(\mathfrak{s}\big(g^{-1}mg\big)\big) \\
&= \lambda(m,g)\alpha_g\big(f\big(g^{-1}mg\big)\big)\mathfrak{s}(m) \\
&= \lambda(m,g)\alpha_g\big(f\big(g^{-1}mg\big)\big)f(m)^{-1}\mathfrak{s}'(m); \\
\lambda'(m,g) &= \lambda(m,g)\alpha_g\big(f\big(g^{-1}mg\big)\big)f(m)^{-1} \\
&= \lambda(m,g)(\partial_2 f)(m,g)
\end{aligned}
$$

The group $\mathrm{Z}(G,N,A)$ of pairs $\{(\lambda,\mu)\} \in A^{N\times G} \times \mathrm{Z}^2(N,A)$ satisfying the natural cocycle identity is called the **characteristic cocycle group** and each element of the group

$$
\mathrm{B}(G,N,A) = \big\{\partial f = (\partial_1 f, \partial_2 f) : f \in A^N\big\}
$$

is called a **coboundary**. Then the quotient group:

$$
\Lambda(G,N,A) = \mathrm{Z}(G,N,A)/\mathrm{B}(G,N,A)
$$

[9] The subgroup $i(A) \lhd H$ is a subgroup of the center of H and G acts on H through α, but N does not act on A.

is then called the **characteristic cohomology group**. If $\chi = [\lambda, \mu]$ comes from the above equivariant short exact sequence E, then it is called the **characteristic invariant of** E and writtend $\chi(E) = [\lambda, \mu] \in \Lambda(G, N, A)$.

Coming back to the $\mathrm{Aut}(\mathcal{M}) \times \mathbb{R}$-equivariant exact sequence:

$$E : 1 \longrightarrow \mathcal{U}(\mathcal{C}) \longrightarrow \widetilde{\mathcal{U}}(\mathcal{M}) \longrightarrow \mathrm{Cnt_r}(\mathcal{M}) \longrightarrow 1,$$

we obtain the characteristic invariant:

$$\chi_{\mathcal{M}} = \chi(E) \in \Lambda_{\mathrm{mod}\times\theta}(\mathrm{Aut}(\mathcal{M}) \times \mathbb{R}, \mathrm{Cnt_r}(\mathcal{M}), \mathcal{U}(\mathcal{C})),$$

which is called the **intrinsic invariant of $\mathcal{M}$.**

Now we state the major result on the cocycle conjugacy classification of amenable group actions on an AFD factor due to A. Connes, V.F.R. Jones, A. Ocneanu, C. Sutherland, Y. Kawahigashi, K. Katayama and M. Takesaki.

Theorem 8.7. i) *If α is an action of a locally compact group G on a factor $\mathcal{M}$, then the pull back of the intrinsic invariant*:

$$\chi_\alpha = \alpha^*(\chi_{\mathcal{M}}) \in \Lambda_{\alpha\times\theta}(G \times \mathbb{R}, N, \mathcal{U}(\mathcal{C}))$$

is a cocycle conjugacy invariant where $N = \alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M})) \triangleleft G$, need not be a closed subgroup but a Borel subgroup.

ii) *If $\mathcal{M}$ is* AFD *and G is an amenable countable discrete group, then the combination*:

$$\begin{aligned}&\big\{\mathrm{mod}, N = \alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M})), \chi_\alpha\big\}\\ &\qquad\in \mathrm{Hom}(G, \mathrm{Aut}_\theta(\mathcal{C})) \times \mathfrak{N} \times \Lambda_{\alpha\times\theta}(G \times \mathbb{R}, N, \mathcal{U}(\mathcal{C}))\end{aligned}$$

is a complete invariant of the cocycle conjugacy class of the action α, where $\mathfrak{N}$ is the set of all normal subgroups of G. Also every combination of invariants with natural restriction occurs as the invariant of an action of G.

§8.3. Modular Fiber Space and Cross-Section Algebra. Let $\mathcal{M}$ be a factor of type III. We have seen that the crossed product $\mathcal{N} = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ is semi-finite for each faithful semi-finite normal weight φ on $\mathcal{M}$ and that

$$\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R} \cong \mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R}$$

for any other faithful semi-finite normal weight ψ on $\mathcal{M}$. Furthermore, if θ is the action on $\mathcal{N}$ dual to σ^φ, then we have

$$\mathcal{N} \rtimes_\theta \mathbb{R} \cong \mathcal{M}.$$

We want to find a construction of the semi-finite von Neumann algebra $\mathcal{N}$ without fixing a faithful semi-finite normal weight φ on $\mathcal{M}$. So let $\mathfrak{W}_0 =$

$\mathfrak{W}_0(\mathcal{M})$ be the set of all faithful semi-finite normal weights on $\mathcal{M}$. Fix $t \in \mathbb{R}$ and consider a relation $\sim_t$ on the set $\mathcal{M} \times \mathfrak{W}_0$ by the following:

$$(x, \varphi) \sim_t (y, \psi) \quad \Leftrightarrow \quad x(\mathrm{D}\varphi : \mathrm{D}\psi)_t = y$$

The chain rule of the Connes derivatives yields that the relation $\sim_t$ is an equivalence relation on $\mathcal{M} \times \mathfrak{W}_0$. We denote the equivalence class $[x, \varphi]$ by $x\varphi^{\mathrm{i}t}$ and set

$$\mathcal{M}(t) = (\mathcal{M} \times \mathfrak{W}_0)/\sim_t = \left\{x\varphi^{\mathrm{i}t} : x \in \mathcal{M}, \varphi \in \mathfrak{W}_0\right\}.$$

It is easily seen that the map: $x \in \mathcal{M} \mapsto x\varphi^{\mathrm{i}t} \in \mathcal{M}(t)$ is a bijection so that we can define the vector space structure on $\mathcal{M}(t)$, transplanting that of $\mathcal{M}$ and also set

$$\left\|x\varphi^{\mathrm{i}t}\right\| = \|x\|, \quad x \in \mathcal{M}, \varphi \in \mathfrak{W}_0.$$

Then the bijection $x \in \mathcal{M} \mapsto x\varphi^{\mathrm{i}t} \in \mathcal{M}(t)$ gives the dual Banach space structure on $\mathcal{M}(t)$. We then define a binary operation from $\mathcal{M}(s) \times \mathcal{M}(t)$ to $\mathcal{M}(s+t)$ as follows:

$$x\varphi^{\mathrm{i}t} y\varphi^{\mathrm{i}s} = x\sigma_t^{\varphi}(y)\varphi^{\mathrm{i}(s+t)}, \quad s, t \in \mathbb{R},\ x, y \in \mathcal{M}.$$

Then set

$$\mathcal{F} = \dot{\bigcup}\{\mathcal{M}(t) : t \in \mathbb{R}\} : \quad \text{disjoint union.}$$

We call this fibre space $\mathcal{F}$ the **modular fibre space** of $\mathcal{M}$ and write $\mathcal{F}(\mathcal{M})$ when we need to indicate $\mathcal{F}$ coming from $\mathcal{M}$. We then define the involution on $\mathcal{F}$:

$$\left(x\varphi^{\mathrm{i}t}\right)^* = \sigma_{-t}^{\varphi}(x^*)\varphi^{-\mathrm{i}t}, \quad x\varphi^{\mathrm{i}t} \in \mathcal{M}(t).$$

Then we get

$$\begin{aligned} (\lambda x + \mu y)^* &= \bar{\lambda} x^* + \bar{\mu} y^*; \\ (xy)^* &= y^* x^*; \qquad && x, y \in \mathcal{F}, \quad \lambda, \mu \in \mathbb{C}, \\ x^{**} &= x, && \varphi, \psi \in \mathfrak{W}_0, \quad t \in \mathbb{R}, \\ \varphi^{\mathrm{i}t}\psi^{-\mathrm{i}t} &= (\mathrm{D}\varphi : \mathrm{D}\psi)_t, \end{aligned}$$

whenever the sum and the product are possible.

Now for each $\varphi \in \mathfrak{W}_0$ define a map $\Phi_\varphi : \mathcal{M} \times \mathbb{R} \mapsto \mathcal{F}$ as follows:

$$\Phi_\varphi(a, t) = a\varphi^{\mathrm{i}t} \in \mathcal{F}, \quad (a, t) \in \mathcal{M} \times \mathbb{R}.$$

Theorem 8.8. *The maps $\{\Phi_\varphi : \varphi \in \mathfrak{W}_0\}$ have the following properties*:

i) *On a bounded subset $\mathcal{B} \subset \mathcal{M}$, the map $\Phi_\psi^{-1}\circ\Phi_\varphi$ is a homeomorphism on $\mathcal{B} \times \mathbb{R}$ relative to any operator topology on $\mathcal{B}$ except the norm topology and the usual topology on $\mathbb{R}$*;

ii) *On the entire space $\mathcal{M} \times \mathbb{R}$ the map $\Phi_\psi^{-1}\circ\Phi_\varphi$ is a homeomorphism relative to the Arens-Mackey topology $\tau(\mathcal{M}, \mathcal{M}_*)$-topology on $\mathcal{M}$ and the usual topology on $\mathbb{R}$*;

iii) *The Connes derivative takes the following form*:

$$(\mathrm{D}\varphi : \mathrm{D}\psi)_t = \Phi_\psi^{-1}\circ\Phi_\varphi(1, t), \quad t \in \mathbb{R}.$$

In fact, we have, for each $x \in \mathcal{M}$ and $t \in \mathbb{R}$,

$$\Phi_\psi^{-1}\circ\Phi_\varphi(x, t) = (x(\mathrm{D}\varphi : \mathrm{D}\psi)_t, t) \in \mathcal{M} \times \mathbb{R}.$$

This theorem allows us to introduce a topology on the fiber space $\mathcal{F}$ by transplanting the product topology of $\tau(\mathcal{M}, \mathcal{M}_*)$-topology on $\mathcal{M}$ and the usual real line $\mathbb{R}$ through the map Φ_φ which does not depend on the choice of faithful semi-finite normal weight φ.

Remark 8.9. The norm topology on $\mathcal{M}$ cannot be tranplanted to $\mathcal{F}$ independent of the choice of $\varphi \in \mathfrak{W}_0$, because the Connes derivative

$$\{(\mathrm{D}\varphi : \mathrm{D}\psi)_t : t \in \mathbb{R}\}$$

is not norm continuous.

Definition 8.10. *The* **τ-topology** *on $\mathcal{F}$ means the topology introduced by the last theorem based on the Arens-Mackey topology $\tau(\mathcal{M}, \mathcal{M}_*)$-topology on $\mathcal{M}$.*

The above theorem also allows us to introduce a Borel structure on the fiber space $\mathcal{F}$ from the topology introduced above which is indepent of the choice of $\varphi \in \mathfrak{W}_0$. Consequently, we can consider a measurable cross-section as well as an integrable cross-sections of $\mathcal{F}$, i.e., a Borel map $s \in \mathbb{R} \mapsto x(s) \in \mathcal{F}$ such that

$$x(s) \in \mathcal{M}(s), \quad s \in \mathbb{R};$$

$$\|x\|_1 = \int_{\mathbb{R}} \|x(s)\| \mathrm{d}s,$$

recalling that each $\mathcal{M}(s)$ is a Banach space. Let $\Gamma^1(\mathcal{F})$ denote the set of all integrable cross-sections. It then follows that $\Gamma^1(\mathcal{F})$ is an involutive Banach algebra under the following structure:

$$(\lambda x + \mu y)(t) = \lambda x(t) + \mu y(t);$$

$$(xy)(t) = \int_{\mathbb{R}} x(s) y(t - s) \mathrm{d}s;$$

$$x^*(t) = x(-t)^*;$$

$$\|x\|_1 = \int_{\mathbb{R}} \|x(s)\| \mathrm{d}s.$$

We call $\Gamma^1(\mathcal{F})$ the **modular cross-section algebra** for $\mathcal{M}$ and write $\Gamma^1(\mathcal{F}(\mathcal{M}))$ when it is neccessary to indicate where it is from.

To have a von Neumann algebra structure based on the bundle algebra, we need to construct a HIlbert space on which the cross-section algebra acts naturally. So we consider a von Neumann algebra $\mathcal{M}$ represented on a Hilbert space $\mathfrak{H}$. Let $\mathcal{N}$ be the opposit algebra $(\mathcal{M}')^{\circ}$ of the commutant which makes the Hilbert space $\mathfrak{H}$ an $\mathcal{M}-\mathcal{N}$ bimodule. We are going to construct a bundle of Hilbert spaces on which $\mathcal{F}(\mathcal{M})$ acts from the left and $\mathcal{F}(\mathcal{N})$ acts from the right, where $\mathcal{F}(\mathcal{M})$ is the fiber space constructed above based on $\mathcal{M}$ and $\mathcal{F}(\mathcal{N})$ is the one based on $\mathcal{N}$.

Lecture 9: Hilbert Space Bundle.

§**9.1. Spatial Derivative Recall,** [Cnn8, Tk2, Chapter IX, §3]**.** Let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra $\mathcal{M}$ represented on a Hilbert space $\mathfrak{H}$. Consider the opposit von Neumann algebra $\mathcal{N} = (\mathcal{M}')^\circ$ of the commutant $\mathcal{M}'$. Let ψ' be a faithful semi-finite normal weight on $\mathcal{M}'$ and ψ be the corresponding semi-finite normal weight $\psi = (\psi')^\circ$ on $\mathcal{N}$. Let $L^2(\mathcal{N})$ be the standard Hilbert space of $\mathcal{N}$. We then consider the $\mathcal{N}$-right module $\widetilde{\mathfrak{H}}_\mathcal{N} = L^2(\mathcal{N}) \oplus \mathfrak{H}_\mathcal{N}$. We then consider the von Neumann algebra $\mathcal{R} \stackrel{\text{def}}{=} \mathcal{L}\left(\widetilde{\mathfrak{H}}_\mathcal{N}\right)$, i.e., the von Neumann algebra of all bounded operators commuting with the right action of $\mathcal{N}$. Each elelment x of $\mathcal{R}$ has a 2×2 matrix representation:

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

with components from the following spaces:

$$\begin{aligned} x_{11} &\in \mathcal{N}, & x_{12} &\in \mathcal{L}\big(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N}\big)^{10}, \\ x_{21} &\in \mathcal{L}\big(\mathfrak{H}_\mathcal{N}, L^2(\mathcal{N})_\mathcal{N}\big), & x_{22} &\in \mathcal{L}(\mathfrak{H}_\mathcal{N}) = \mathcal{M}. \end{aligned}$$

On $\mathcal{R}$ we define the balanced faithful semi-finite normal weight ρ out of $\varphi \in \mathfrak{W}_0(\mathcal{M})$ of $\psi \in \mathfrak{W}_0(\mathcal{N})$ which is given by the following:

$$\rho \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \psi(x_{11}) + \varphi(x_{22}).$$

Set $e \stackrel{\text{def}}{=} e_{11}$ and $f \stackrel{\text{def}}{=} e_{22}$, i.e., the projection of $\widetilde{\mathfrak{H}}$ to $L^2(\mathcal{N})$ and to $\mathfrak{H}$ respectively which are of course projections in $\mathcal{R}$ such that

$$e\mathcal{R}e = \mathcal{N}, \quad e\mathcal{R}f = \mathcal{L}\big(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N}\big), \quad f\mathcal{R}f = \mathcal{M}.$$

We then set

$$\begin{aligned} \mathfrak{n}_\psi(\mathfrak{H}) &= f\mathfrak{n}_\rho e = \big\{x \in \mathcal{L}\big(\mathfrak{H}_\mathcal{N}, L^2(\mathcal{N})_\mathcal{N}\big) : \psi(x^*x) < +\infty\big\}; \\ \mathfrak{D}(\mathfrak{H}, \psi) &\stackrel{\text{def}}{=} = \Big\{\xi \in \mathfrak{H} : \|\xi x\| \le C_\xi \psi'(xx^*)^{\frac{1}{2}}, x^* \in \mathfrak{n}_\psi \text{ for some } C_\xi \ge 0\Big\}. \end{aligned}$$

For each $x \in \mathfrak{n}_\psi(\mathfrak{H})$, we have $x^*x \in \mathcal{N}$ and with polar decomposition $x = uh, h = (x^*x)^{\frac{1}{2}} \in \mathcal{N}$, we set

$$\eta_\psi(x) = u\eta_\psi(h) \in \mathfrak{H}.$$

Similarly, for each $\varphi \in \mathfrak{W}_0(\mathcal{M})$, we set

$$\mathfrak{n}_\varphi(\mathfrak{H}) = \big\{x \in \mathcal{L}\big(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N}\big) : \varphi(x^*x) < +\infty\big\}.$$

[10] The notation $\mathcal{L}\big(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N}\big)$ means the set of all bounded operators from $\mathfrak{H}$ to $L^2(\mathcal{N})$ which commutes with the right action of $\mathcal{N}$.

Then set

$$S_{\varphi,\psi}\eta_\psi(x) = \eta_\varphi(x^*), \quad x \in \mathfrak{n}_\psi(\mathfrak{H}) \cap \mathfrak{n}_\varphi(\mathfrak{H})^*;$$

$$\Delta_{\varphi,\psi} = \left(S^*_{\varphi,\psi}S_{\varphi,\psi}\right)^{\frac{1}{2}} = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi'},$$

which is a non-singular self-adjoint positive operator on $\mathfrak{H}$ such that

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)^{\mathrm{i}t} x \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)^{-\mathrm{i}t} = \sigma_t^\varphi(x), \quad x \in \mathcal{M};$$

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)^{-\mathrm{i}t} y \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)^{\mathrm{i}t} = \sigma_t^{\psi'}(y), \quad y \in \mathcal{M}',$$

where $\psi' \in \mathfrak{W}_0(\mathcal{M}')$. We call the operator $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}$ the **spatial derivative** of φ by ψ'. The spatial derivative enjoys the following properties:

$$\left(\frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi'}\right)^{\mathrm{i}t} = (\mathrm{D}\varphi_1 : \mathrm{D}\varphi_2)_t \left(\frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi'}\right)^{\mathrm{i}t};$$

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right) = \left(\frac{\mathrm{d}\psi'}{\mathrm{d}\varphi}\right)^{-1}.$$

§**9.2. Hilbert Space Bundle.** We fix a von Neumann algebra $\{\mathcal{M}, \mathfrak{H}\}$ and set $\mathcal{N} = (\mathcal{M}')^\circ$, the opposit von Neumann algebra of the commutant $\mathcal{M}'$. Set

$$X = \mathbb{R} \times \mathfrak{W}_0(\mathcal{M}) \times \mathfrak{H} \times \mathfrak{W}_0(\mathcal{N}).$$

For each $t \in \mathbb{R}$ we set an equivalence relation $\sim_t$ by the following relation:

$$(r_1, \varphi_1, \xi_1, \psi_1) \sim_t (r_2, \varphi_2, \xi_2, \psi_2)$$

whenever

$$\left(\frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi_1^\circ}\right)^{\mathrm{i}r_1} \xi_1 = \left(\frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi_2^\circ}\right)^{\mathrm{i}r_2} \xi_2 (\mathrm{D}\psi_2 : \mathrm{D}\psi_1)_t$$

This is an equivalence relation in X. We then set

$$\mathfrak{H}(t) \stackrel{\mathrm{def}}{=} X/\sim_t$$

and denote the equivalence class $[r, \varphi, \xi, \psi]_t \in \mathfrak{H}(t)$ under this equivalence relation by the following:

$$\varphi^{\mathrm{i}r} \xi \psi^{\mathrm{i}(t-r)} \stackrel{\mathrm{def}}{=} [r, \varphi, \xi, \psi]_t \in \mathfrak{H}(t).$$

Thus in $\mathfrak{H}(t)$ we have

$$\varphi^{\mathrm{i}t}\xi = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi^\circ}\right)^{\mathrm{i}t} \xi \psi^{\mathrm{i}t}, \quad \varphi \in \mathfrak{W}_0(\mathcal{M}), \xi \in \mathfrak{H}, \psi \in \mathfrak{W}_0(\mathcal{N});$$

$$\varphi^{\mathrm{i}t}\xi\psi^{-\mathrm{i}t} = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi^\circ}\right)^{\mathrm{i}t} \xi.$$

Observe the equivalence relation $\sim_t$ on X is generated by the subrelations:

$$\begin{aligned}\varphi_1^{\mathrm{i}t}\varphi_2^{-\mathrm{i}t} &\sim (\mathrm{D}\varphi_1 : \mathrm{D}\varphi_2)_t, \quad \varphi_1, \varphi_2 \in \mathfrak{W}_0(\mathcal{M});\\ \psi_1^{\mathrm{i}t}\psi_2^{-\mathrm{i}t} &\sim (\mathrm{D}\psi_1 : \mathrm{D}\psi_2)_t, \quad \psi_1, \psi_2 \in \mathfrak{W}_0(\mathcal{N});\\ \varphi^{\mathrm{i}t}\xi\psi^{-\mathrm{i}t} &\sim \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi^\circ}\right)^{\mathrm{i}t}\xi.\end{aligned}$$

In the set $\mathfrak{H}(t)$, we define the following vector space structure and inner product:

$$\begin{gathered}\lambda\Big(\varphi^{\mathrm{i}r}\xi\psi^{\mathrm{i}(t-r)}\Big) + \mu\Big(\varphi^{\mathrm{i}r}\eta\psi^{\mathrm{i}(t-r)}\Big) = \varphi^{\mathrm{i}r}(\lambda\xi + \mu\eta)\psi^{\mathrm{i}(t-r)};\\ \Big(\varphi^{\mathrm{i}r}\xi\psi^{\mathrm{i}(t-r)}\Big|\,\varphi^{\mathrm{i}t}\eta\psi^{\mathrm{i}(t-r)}\Big) = (\xi \mid \eta),\end{gathered}$$

which makes $\mathfrak{H}(t)$ a Hilbert space and does not depend on the choice of $\varphi \in \mathfrak{W}_0(\mathcal{M})$ and $\psi \in \mathfrak{W}_0(\mathcal{N})$ nor on $r \in \mathbb{R}$. Each pair $\varphi \in \mathfrak{W}_0(\mathcal{M})$ and $\psi \in \mathfrak{W}_0(\mathcal{N})$ give rise to unitaries from $\mathfrak{H}$ onto $\mathfrak{H}(t)$:

$$\begin{aligned}U_\varphi(t)\xi &\overset{\mathrm{def}}{=} \varphi^{\mathrm{i}t}\xi \in \mathfrak{H}(t), \quad \xi \in \mathfrak{H};\\ V_\psi(t)\eta &\overset{\mathrm{def}}{=} \eta\psi^{\mathrm{i}t} \in \mathfrak{H}(t), \quad \eta \in \mathfrak{H}.\end{aligned}$$

Then we have

$$V_\psi(t)^*U_\varphi(t) = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi^\circ}\right)^{\mathrm{i}t}.$$

We then define a multilinear map: $\mathcal{M}(r) \times \mathfrak{H}(s) \times \mathcal{N}(t) \mapsto \mathfrak{H}(r+s+t)$ as follows:

$$\begin{aligned}(x\varphi^{\mathrm{i}r}, \varphi^{\mathrm{i}s}\xi, y\psi^{\mathrm{i}t}) \in \mathcal{M}(r) \times \mathfrak{H}(s) \times \mathcal{N}(t) &\mapsto x\varphi^{\mathrm{i}r}\varphi^{\mathrm{i}s}\xi y\psi^{\mathrm{i}t}\\ x\varphi^{\mathrm{i}r}\varphi^{\mathrm{i}s}\xi y\psi^{\mathrm{i}t} &= \varphi^{\mathrm{i}(r+s)}\sigma^\varphi_{-(r+s)}(x)\xi y\psi^{\mathrm{i}t}\\ &= x\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi^\circ}\right)^{\mathrm{i}(r+s)}\xi\sigma^\psi_{r+s}(y)\psi^{\mathrm{i}(r+s+t)}.\end{aligned}$$

This trilinear map does not depend on the choice of faithful semi-finite normal weight $\varphi \in \mathfrak{W}_0(\mathcal{M})$ and $\psi \in \mathfrak{W}_0(\mathcal{N})$ and is associative whenever the consequtive product is possible.

Let $\mathcal{G}$ be the disjoint union of $\mathfrak{H}(t)$:

$$\mathcal{G} = \dot{\bigcup}\{\mathfrak{H}(t) : t \in \mathbb{R}\}$$

Now with the above preparation, we will construct a Hilbert space on which $\Gamma^1(\mathcal{F}(\mathcal{M}))$ and $\Gamma^1(\mathcal{F}(\mathcal{N}))$ act from the left and the right. Making use of the maps $\{U_\varphi(t) : t \in \mathbb{R}\}$ we can transplant the product topology of $\mathfrak{H} \times \mathbb{R}$

to $\mathcal{G}$. Let $\widetilde{\mathfrak{H}} = \Gamma^2(\mathcal{G})$ be the Hilbert space of square integrable cross-sections. To make the Hilbert space $\widetilde{\mathfrak{H}}$ a left and right bimodule over $\Gamma^1(\mathcal{F}(\mathcal{M}))$ and $\Gamma^1(\mathcal{F}(\mathcal{N}))$. We need to discuss the opposit algebra of $\Gamma^1(\mathcal{F}(\mathcal{N}))$. Recall $\mathcal{N}$ is defined to be the opposit algebra $(\mathcal{M}')^\circ$. We set the correspondence as follows:

$$\begin{gathered}
y \in \mathcal{N} \quad \longleftrightarrow \quad y^\circ \in \mathcal{M}'; \\
\psi \in \mathfrak{W}_0(\mathcal{N}) \quad \longleftrightarrow \quad \psi^\circ \in \mathfrak{W}_0(\mathcal{M}'); \\
\psi^\circ(y^\circ) = \psi(y), \quad y \in \mathcal{N}; \\
\sigma^{\psi^\circ}_{-t}(y^\circ) = \sigma^\psi_t(y)^\circ; \\
y^{\circ\circ} = y,\ y \in \mathcal{N}, \quad \psi^{\circ\circ} = \psi.
\end{gathered}$$

The $\circ$-operation on $\mathcal{F}(\mathcal{N})$ and $\mathcal{F}(\mathcal{M}')$ is then given by the following:

$$\left(y\psi^{\mathrm{i}t}\right)^\circ = (\psi^\circ)^{-\mathrm{i}t} y^\circ = \sigma^{\psi^\circ}_{-t}(y)(\psi^\circ)^{-\mathrm{i}t}, \quad y \in \mathcal{N}, \psi \in \mathfrak{W}_0(\mathcal{N}).$$

So we set

$$y^\circ(t) = y(-t)^\circ, \quad y \in \Gamma^1(\mathcal{F}(\mathcal{N})) \cup \Gamma^1(\mathcal{F}(\mathcal{M}')),$$

and we get

$$\Gamma^1(\mathcal{F}(\mathcal{N}))^\circ = \Gamma^1(\mathcal{F}(\mathcal{M}')),$$

the $\circ$-operation is an anti-isomorphism of the modular cross-section algebra. Now we let $\Gamma^1(\mathcal{F}(\mathcal{M}))$ and $\Gamma^1(\mathcal{F}(\mathcal{N}))$ act from the left and right as follows:

$$\begin{aligned}
(x\xi)(t) &= \int_{\mathbb{R}} x(s)\xi(t-s)\mathrm{d}s, \quad x \in \Gamma^1(\mathcal{F}(\mathcal{M})); \\
(\xi y)(t) &= \int_{\mathbb{R}} \xi(s)y(t-s)\mathrm{d}s, \quad y \in \Gamma^1(\mathcal{F}(\mathcal{N})).
\end{aligned}$$

Also $\Gamma^1(\mathcal{F}(\mathcal{M}'))$ acts on $\widetilde{\mathfrak{H}}$ from the left:

$$y\xi = \xi y^\circ, \quad y \in \Gamma^1(\mathcal{F}(\mathcal{M}')),$$

which means that the action is given by the following:

$$\begin{aligned}
(y\xi)(t) = (\xi y^\circ)(t) &= \int_{\mathbb{R}} \xi(s)y^\circ(t-s)\mathrm{d}s \\
&= \int_{\mathbb{R}} \xi(s)(y(s-t))^\circ\mathrm{d}s = \int_{\mathbb{R}} \xi(s+t)y(s)^\circ\mathrm{d}s
\end{aligned}$$

Lecture 10: The Non-Commutative Flow of Weights on a von Neumann algebra $\mathcal{M}$.

Theorem 10.1. *The von Neumann algebra* $\left\{\widetilde{\mathcal{M}}, \widetilde{\mathfrak{H}}\right\}$ *generated by the left action of* $\Gamma^1(\mathcal{F}(\mathcal{M}))$ *on* $\widetilde{\mathfrak{H}}$ *has the commutant* $\widetilde{\mathcal{M}}'$ *generated by* $\Gamma^1(\mathcal{F}(\mathcal{M}'))$.

Definition 10.2. The von Neumann algebra $\widetilde{\mathcal{M}}$ on $\widetilde{\mathfrak{H}}$ generated by the left action of the modular cross-section algebra $\Gamma^1(\mathcal{F}(\mathcal{M}))$ is called the **canonical core** of $\mathcal{M}$, or simply the **core** of $\mathcal{M}$. On the Hilbert space $\Gamma^2(\mathcal{F}(\mathcal{M}))$, we define a one parameter unitary group:

$$(U(s)\xi)(t) = e^{-\mathrm{i}st}\xi(t), \quad \xi \in \Gamma^2(\mathcal{F}(\mathcal{M})),\ s, t \in \mathbb{R}.$$

Theorem 10.3. *For a von Neumann algebra* $\{\mathcal{M}, \mathfrak{H}\}$ *the core represnted on* $\widetilde{\mathfrak{H}}$ *through the above procedure has the following properties*:

i) *The one parameter unitary group* $\{U(s) : s \in \mathbb{R}\}$ *defined above gives rise to a one parameter automorphism* $\{\theta_s : s \in \mathbb{R}\}$ *of* $\widetilde{\mathcal{M}}$ *such that the original von Neumann algebra is precisely the fixed point subalgebra* $\widetilde{\mathcal{M}}^\theta$ *of the one parameter automorphism group* θ.

ii) *For each* $\varphi \in \mathfrak{W}_0(\mathcal{M})$, *there is a natural isomorphism* Π_φ *from* $\widetilde{\mathcal{M}}$ *to the crossed-product* $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$, *which conjugate the dual action* $\widehat{\sigma^\varphi}$ *on* $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ *and* θ.

iii) *The integral* I_θ *along the one parameter automorphism group* θ:

$$I_\theta(x) = \int_{\mathbb{R}} \theta_s(x)\mathrm{d}s, \quad x \in \widetilde{\mathcal{M}}_+,$$

is a faithful normal semi-finite $\mathcal{M}$*-valued operator valued weight and the compostion*:

$$\widetilde{\varphi} = \varphi \circ I_\theta$$

corresponds to the dual weight $\widehat{\varphi}$ *on* $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ *and* $\sigma_t^{\widetilde{\varphi}} = \mathrm{Ad}\big(\varphi^{\mathrm{i}t}\big), t \in \mathbb{R}$.

iv) *Viewing the logarithmic generator of* $\big\{\varphi^{\mathrm{i}t} : t \in \mathbb{R}\big\}$, *i.e.*,

$$\varphi = \exp\left(\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi^{\mathrm{i}t}\big)\bigg|_{t=0}\right),$$

the weight τ *defined by the following*:

$$\tau(x) = \widetilde{\varphi}\big(\varphi^{-1}x\big) = \lim_{\varepsilon \searrow 0} \widetilde{\varphi}\Big((\varphi + \varepsilon)^{-\frac{1}{2}} x (\varphi + \varepsilon)^{-\frac{1}{2}}\Big), \quad x \in \widetilde{\mathcal{M}}_+,$$

is a faithful semi-finite normal trace on $\widetilde{\mathcal{M}}$.

v) *The trace* τ *does not depend on the choice of* $\varphi \in \mathfrak{W}_0(\mathcal{M})$ *and scaled by* θ:

$$\tau \circ \theta_s = e^{-s}\tau, \quad s \in \mathbb{R}.$$

vi)

$$\widetilde{\mathcal{M}} \rtimes_\theta \mathbb{R} \cong \mathcal{M} \overline{\otimes} \mathcal{L}\big(L^2(\mathbb{R})\big).$$

Theorem 10.4. i) *The correspondence:* $\mathcal{M} \longrightarrow \left\{\widetilde{\mathcal{M}}, \tau, \mathbb{R}, \theta\right\}$ *is a functor from the category of von Neumann algebra with isomorphisms as morphisms to the category of semi-finite von Neumann algebras equipped with a trace scaling one parameter automorphism group* θ *with conjugation as morphism which transforms the trace one to another.*

ii) *The correspondence* $\{\mathcal{M}, \mathfrak{H}\} \longrightarrow \left\{\widetilde{\mathcal{M}}, \tau, \mathbb{R}, \theta, \widetilde{\mathfrak{H}}\right\}$ *is a functor from the category of spatial von Neumann algebras with spatial isomorphisms as morphisms to the category of semi-finite von Neumann algebras equipped with trace scaling one parameter automorphis group* θ *represented on a Hilbert space with spatial isomorphisms intertwining* θ *and* τ.

Definition 10.5. i) The covariant system $\left\{\widetilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\right\}$ for $\mathcal{M}$ is called the **noncommutative flow of weights** on $\mathcal{M}$.

ii) The group $\widetilde{\mathcal{U}}(\mathcal{M}) = \left\{u \in \mathcal{U}(\widetilde{\mathcal{M}}) : u\mathcal{M}u^* = \mathcal{M}\right\}$, the normalizer of $\mathcal{M}$, is called the **extended unitary group** of $\mathcal{M}$. For each $u \in \widetilde{\mathcal{U}}(\mathcal{M})$, we write

$$\widetilde{\mathrm{Ad}}(u) = \mathrm{Ad}(u)|_{\mathcal{M}} \in \mathrm{Aut}(\mathcal{M})$$

and set

$$\mathrm{Cnt_r}(\mathcal{M}) = \left\{\widetilde{\mathrm{Ad}}(u) : u \in \widetilde{\mathcal{U}}(\mathcal{M})\right\} \triangleleft \mathrm{Aut}(\mathcal{M}).$$

Theorem 10.6. (Connes - Takesaki Relative Commutant Theorem). *Let* $\left\{\widetilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\right\}$ *be the core of a von Neumann algebra* $\mathcal{M}$. *Then the relative commutant of* $\mathcal{M}$ *in* $\widetilde{\mathcal{M}}$ *is the center of the core* $\widetilde{\mathcal{M}}$:

$$\mathcal{M}' \cap \widetilde{\mathcal{M}} = \mathcal{C} = \text{The Center of } \widetilde{\mathcal{M}}.$$

Theorem 10.7 (The Stability of Non-Commutative Flow). *Suppose that* $\left\{\widetilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\right\}$ *is a semi-finite von Neumann algebra equipped with a trace* τ *scaling one parameter automorphism group* θ. *Then the core of the fixed point* $\mathcal{M} = \widetilde{\mathcal{M}}^\theta$ *is conjugate to the original system. Every* θ*-cocycle is a coboundary.*

Definition 10.8. Each $\alpha \in \mathrm{Aut}(\mathcal{M})$ is extended to an automorphism $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{\mathcal{M}})$ such that

$$\widetilde{\alpha}{\circ}\theta_s = \theta_s{\circ}\widetilde{\alpha}, \quad s \in \mathbb{R}, \quad \tau{\circ}\widetilde{\alpha} = \tau.$$

The restriction of $\widetilde{\alpha}$ to the center $\mathcal{C}$ of the core $\widetilde{\mathcal{M}}$ is called the **module** of α and denoted by mod $(\alpha) \in \mathrm{Aut}(\mathcal{C})$ which commutes with the flow of weights $\theta|_{\mathcal{C}}$.

Theorem 10.9. *The automorphims group* $\mathrm{Aut}(\mathcal{M})$ *is identified with the following subgroup* $\mathrm{Aut}_{\tau,\theta}(\mathcal{M})$ *of* $\mathrm{Aut}(\mathcal{M})$:

$$\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}_{\tau,\theta}(\mathcal{M}) = \left\{\sigma \in \mathrm{Aut}(\widetilde{\mathcal{M}}) : \sigma{\circ}\theta_s = \theta_s{\circ}\sigma, \tau{\circ}\sigma = \tau\right\}.$$

In other words, each $\widetilde{\alpha}, \alpha \in \mathrm{Aut}(\mathcal{M})$, *belongs to* $\mathrm{Aut}_{\tau,\theta}(\mathcal{M})$ *and if an element* $\sigma \in \mathrm{Aut}_{\tau,\theta}(\mathcal{M})$ *leaves each element of* $\mathcal{M}$ *invariant, then* $\sigma = \mathrm{id}$.

Theorem 10.10 (Characteristic Square). *To each factor* $\mathcal{M}$ *there corresponds a commutative square of equivariant exact sequences relative to the action of* $\mathrm{Aut}(\mathcal{M}) \times \mathbb{R}$:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathbb{T} \arrow[r] \arrow[d] & \mathcal{U}(\mathcal{C}) \arrow[r, "\partial"] \arrow[d] & \mathrm{B}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathcal{U}(\mathcal{M}) \arrow[r] \arrow[d] & \widetilde{\mathcal{U}}(\mathcal{M}) \arrow[r, "\partial_\theta"] \arrow[d, "\widetilde{\mathrm{Ad}}"'] & \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Int}(\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{Cnt_r}(\mathcal{M}) \arrow[r, "\dot{\partial}_\theta"] \arrow[d] & \mathrm{H}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

where $\mathcal{C}$ *is the center of the non-coummutative flow of weights* $\left\{\widetilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\right\}$ *of* $\mathcal{M}$ *and* $\widetilde{\mathcal{U}}(\mathcal{M})$ *is the extended unitary group of* $\mathcal{M}$.

Extended Lecture I: The Non-Commutative Flow of Weights on a von Neumann algebra . Continued.

§**E.1. Grading and $L^p(\mathcal{M}), 1 < p \leq +\infty$ and Duality.** First we need a definition:

Definition E.1. Let $\{\mathcal{N}, \tau, \mathfrak{H}\}$ be a semi-finite von Neumann algebra equipped with a faithful semi-finite normal trace τ represented on a Hilbert space $\mathfrak{H}$. A densely defined closed operator T is said to be **affiliaed to** $\mathcal{N}$ if

$$uTu^* = T, \quad u \in \mathcal{U}(\mathcal{N}').$$

It is said to be ***τ*-tamed** if

$$\lim_{\lambda \to +\infty} \tau\Big(\chi_{[\lambda,+\infty)}\Big((T^*T)^{\frac{1}{2}}\Big)\Big) = 0,$$

where $\chi_{[\lambda,+\infty)}$ means the characteristic function of the half line $[\lambda, +\infty) \subset \mathbb{R}$. We will write

$$E_\lambda(|T|) = \chi_{[\lambda,+\infty)}(|T|),$$

Note that

$$\bigcup_{\lambda>0} (1 - E_\lambda(|T|)\mathfrak{H} \subset \mathfrak{D}(T).$$

Lemma E.2. *Suppose $\{e_n; n \in \mathbb{N}\}$ and $\{f_n : n \in \mathbb{N}\}$ are decreasing sequences of projections in $\mathcal{N}$ such that*

$$\lim_{n\to\infty} \tau(e_n) = \lim_{n\to\infty} \tau(f_n) = 0.$$

Then

$$\lim_{n\to\infty} \tau(e_n \vee f_n) = 0.$$

Let $\mathfrak{M}(\mathcal{N}, \tau)$ be the set of all τ-tamed operators affiliated to $\mathcal{N}$ on $\mathfrak{H}$.
The following facts makes the significance of the τ-tame property:

i) A τ-tame operator T has no proper extension as a closed operator: a property called **hypermaximality.**
ii) The sum $S + T$ and the product ST are both preclosed and their closures are also affiliated to $\mathcal{N}$ and τ-tamed.
iii) The adjoint T^* of a τ-tamed operator is also τ-tamed.
iv) Consequently the set $\mathfrak{M}(\mathcal{N}, \tau)$ of τ-tamed operators forms an involutive algebra over $\mathbb{C}$ if we replace the sum and the product of τ-tamed operators by their closurs.
v) If the trace τ is finite, then every densely defined closed operator affiliated to $\mathcal{N}$ is τ-tamed.

Example E.3. *Let $\mathcal{N} = L^\infty(\mathbb{R})$ and τ be the trace obtained by the integration relative to the Lebesgue measure on $\mathbb{R}$. We then represent $\mathcal{N}$ on $L^2(\mathbb{R})$ by multiplication. Each densely defined closed operator T affiliated to $\mathcal{N}$ is easily identifiedd with the multiplication operator by a measurable function f on $\mathcal{R}$ relative to the Lebesgue measure. Now T is τ-tamed if and only if*

$$\operatorname{ess.}\limsup_{\lambda\to-\infty} |f(\lambda)| < \infty \quad \textit{and} \quad \operatorname{ess.}\limsup_{\lambda\to+\infty} |f(\lambda)| < \infty,$$

So the mutiplication operator given by the identity function: $\lambda \in \mathbb{R} \to \lambda \in \mathbb{R}$ is not τ-tamed while the one by $\lambda \in \mathbb{R} \to \frac{1}{\lambda} \in \mathbb{R}, \lambda \neq 0$, is τ-tamed, where the singular point 0 of the function does not matter as it has measure zero.

Let $\left\{\widetilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\right\}$ be the canonical core of $\mathcal{M} = \widetilde{\mathcal{M}}^\theta$. Each faithful semi-finite normal weight $\varphi \in \mathfrak{W}_0(\mathcal{M})$ gives rise to a one parameter unitary group $\{\varphi^{\mathrm{i}t} : t \in \mathbb{R}\}$ in $\widetilde{\mathcal{U}}(\mathcal{M}) \subset \mathcal{U}(\widetilde{\mathcal{M}})$ as seen already. Then we view φ as a densely defined self-adjoint positive operator affiliated to $\widetilde{\mathcal{M}}$. The trace τ-scaling automorphism group θ transforms $\varphi^{\mathrm{i}t}$ in the following way:

$$\begin{gathered}\theta_s(\varphi^{\mathrm{i}t}) = e^{-\mathrm{i}st}\varphi^{\mathrm{i}t}, \quad s, t \in \mathbb{R};\\ \theta_s(\varphi) = e^{-s}\varphi\end{gathered}$$

We denote by $\mathcal{A}_\varphi$ by the von Neumman subalgera of $\widetilde{\mathcal{M}}$ generated by the one parameter unitary group $\{\varphi^{\mathrm{i}t} : t \in \mathbb{R}\}$. The uniqueness of the Heisenberg covariant system implies the existence of a unique isomorphism π^φ from the covariant system $\{L^\infty(\mathbb{R}), \mathbb{R}, \rho\}$ to the covariant system $\{\mathcal{A}_\varphi, \mathbb{R}, \theta\}$ conjugating ρ and θ, where

$$\begin{aligned}(\rho_s f)(x) &= f(x+s), \quad f \in L^\infty(\mathbb{R});\\ \Big((\pi^\varphi)^{-1}(\varphi)\Big)(x) &= e^{-x}, \qquad\qquad s, x \in \mathbb{R}, \quad \lambda \in \mathbb{R}_+.\\ E_\lambda(\varphi)(&= \pi^\varphi\big(\chi_{(-\infty, -\log\lambda]}\big),\end{aligned}$$

Proposition E.4. *The restriction of the trace τ to $\mathcal{A}_\varphi$ either purely infinite or semi-finite which corresponds to the integration on $\mathbb{R}$ shown below:*

$$\tau(\pi^\varphi(f)) = \int_{\mathbb{R}} f(x)e^x \mathrm{d}x, \quad f \in L^\infty(\mathbb{R}).$$

Furthermore, φ is τ-tamed if and only if φ is bounded, i.e., $\varphi(1) = \|\varphi\| < +\infty$.

Remark E.5. *So far we discussed only faithful semi-finite normal weights. But we extend the above theory to non-faithful semi-finite normal weights with a little bit of extra work by considering the reduced von Neumann algebra*

$\mathcal{M}_{s(\varphi)}$. *So we then identify the set* $\left\{\varphi^{\mathrm{i}t} : t \in \mathbb{R}\right\}$ *of one parameter set of partial isometries such that*

$$\varphi^{\mathrm{i}s}\varphi^{\mathrm{i}t} = \varphi^{\mathrm{i}(s+t)}, \quad s, t \in \mathbb{R};$$
$$\varphi^{\mathrm{i}s}\varphi^{-\mathrm{i}s} = \varphi^{-\mathrm{i}s}\varphi^{\mathrm{i}s} = s(\varphi), \quad s \in \mathbb{R}.$$

With this remark, we may discuss the τ-tamed operator φ affiliated with $\widetilde{\mathcal{M}}$ which corresponds to a normal positive functional on $\mathcal{M}$, i.e., an element of $\mathcal{M}_*^+$.

Defnition E.6. For each $\alpha \in \mathbb{C}$ and a densely defined closed operator T affiliated to $\widetilde{\mathcal{M}}$, we say that the operator T has **grade $\boldsymbol{\alpha}$** if

$$\theta_s(T) = e^{-\alpha s}T, \quad s \in \mathbb{R},$$

and write $\boldsymbol{\alpha} = \mathbf{grad}(\boldsymbol{T})$. We then set

$$\mathfrak{M}(\alpha) = \left\{T \ \eta \ \widetilde{\mathcal{M}} : \mathrm{grad}(T) = \alpha \quad \text{and} \quad T \in \mathfrak{M}(\widetilde{\mathcal{M}}, \tau)\right\}.$$

The algebra $\mathfrak{M}\left(\widetilde{\mathcal{M}}, \tau\right)$ is "graded".

Now we can state the signicance of the canonical core:

Theorem E.7. *The grading on* $\mathfrak{M}(\widetilde{\mathcal{M}}, \tau)$ *has the following propeties*:

i) *If* $\Re\alpha < 0$, *then* $\mathfrak{M}(\alpha) = \{0\}$ *and* $\mathfrak{M}(0) = \mathcal{M}$.

ii) *If* $p = \Re\alpha > 0$, *then for every* $T \in \mathfrak{M}(\alpha)$ *the* $1/p$*-th power* $|T|^{\frac{1}{p}}$ *of the absolute value of* T *is given by* $\varphi \in \mathcal{M}_*^+$.

iii) *For any* $a \in \widetilde{\mathcal{M}}_+$ *with* $I_\theta(a) = 1$,

$$a^{\frac{1}{2}}\mathfrak{M}(1)a^{\frac{1}{2}} \subset L^1\left(\widetilde{\mathcal{M}}, \tau\right),$$

the value $\tau\left(a^{\frac{1}{2}}Ta^{\frac{1}{2}}\right)$ *is independent of the choice of such* a. *We will write this value* $\int T$ *for* $T \in \mathfrak{M}(1)$.

iv) *The positive part* $\mathfrak{M}(1)_+$ *of* $\mathfrak{M}(1)$ *is precisely the positive part* $\mathcal{M}_*^+$ *of the predual* $\mathcal{M}_*$ *when* $\mathcal{M}_*^+$ *is embedded in* $\mathfrak{M}^1\left(\widetilde{\mathcal{M}}, \tau\right)$.

If $\varphi \in \mathfrak{M}(1), \varphi \in \mathcal{M}_*^+$, *then*

$$\|\varphi\| = \varphi(1) = \int \varphi.$$

The duality between $\mathcal{M}$ *and* $\mathcal{M}_*$ *is given by the integral*:

$$\int xT = \langle x, T\rangle, \quad x \in \mathcal{M}, \quad T \in \mathfrak{M}(1).$$

Thus we conclude that

$$\mathcal{M}_* = \mathfrak{M}(1).$$

Discussion of the Theorem. Suppose $h \in \mathfrak{M}(1)_+$ and $e = s(h)$. Setting $h^{\mathrm{it}}(1-e) = 0$, we consider, with $\varphi \in \mathfrak{W}_0(\mathcal{M})$ fixed,

$$u(t) = h^{\mathrm{it}}\varphi^{-\mathrm{it}} \in \mathcal{M}.$$

Then we have

$$u(s+t) = u(s)\sigma_s^{\varphi}(u(t)), \quad s, t \in \mathbb{R};$$
$$u(s)^* u(s) = \sigma_s^{\varphi}(e) \quad \text{and} \quad u(s)u(s)^* = e.$$

So there exists $\omega \in \mathfrak{W}(\mathcal{M})$ such that

$$(\mathrm{D}\omega : \mathrm{D}\varphi)_t = u(t) = h^{\mathrm{it}}\varphi^{-\mathrm{it}}, \quad t \in \mathbb{R}.$$

We then have

$$\widehat{\omega}(x) = \omega(I_\theta(x)) = \tau\left(h^{\frac{1}{2}} x h^{\frac{1}{2}}\right), \quad x \in \widetilde{\mathcal{M}}_+.$$

Consider the von Neumann subalgebra $\mathcal{A}$ of $\widetilde{\mathcal{M}}_e$ generated by h. Then it is identified with $L^\infty(\mathbb{R})$ with the action θ_s given by the following:

$$(\theta_s(f))(p) = f(p+s), \quad f \in \mathcal{A}.$$

The τ-tamed property of h and the trace scaling property of θ yield that τ is semi-finite on $\mathcal{A}$ and τ is given by the integral:

$$\tau(f) = C\int_{\mathbb{R}} f(p)e^p \mathrm{d}p, \quad \text{for some constant } C > 0;$$
$$h(p) = e^{-p};$$
$$\widehat{\omega}(f) = \tau\left(h^{\frac{1}{2}} f h^{\frac{1}{2}}\right) = C\int_{\mathbb{R}} f(p)\mathrm{d}p.$$

The semi-finiteness of τ on $\mathcal{A}$ guarantees the existence of the conditional expectation $\mathcal{E}$ from $\widetilde{\mathcal{M}}_e$ to $\mathcal{A}$. Futhermore, the scaling property of τ under θ implies that $\mathcal{E}$ commutes with the flow θ, i.e., for every $f \in \mathcal{A}$ we have

$$\begin{aligned}
\tau(\mathcal{E}(\theta_s(x))f) &= \tau(\theta_s(x)f) = \tau\big(\theta_s\big(x\theta_s^{-1}(f)\big)\big) \\
&= e^{-s}\tau\big(x\theta_s^{-1}(f)\big) = e^{-s}\tau\big(\mathcal{E}(s)\theta_s^{-1}f\big) \\
&= e^{-s}\tau\big(\theta_s^{-1}(\theta_s(\mathcal{E}(x))f)\big) = \tau(\theta_s(\mathcal{E}(x))f); \\
\therefore \quad \mathcal{E}(\theta_s(x)) &= \theta_s(\mathcal{E}(x)), \quad x \in \widetilde{\mathcal{M}}_e.
\end{aligned}$$

Let $e_0 = \chi_{[0,1]} \in L^\infty(\mathbb{R}) = \mathcal{A}$. Then we have

$$\begin{aligned} I_\theta(e_0)(p) &= \int_{\mathbb{R}} \chi_{[0,1]}(p+s)\mathrm{d}s = 1; \\ \widehat{\omega}(e_0) &= C\int_{\mathbb{R}} e_0(p)\mathrm{d}p = C < +\infty; \\ \widehat{\omega}(e_0) &= \omega(I_\theta(e_0)) = \omega(e) = \omega(1) < +\infty. \end{aligned}$$

Suppose $a \in \widetilde{\mathcal{M}}_+$ and $I_\theta(a) = 1$. Then we have

$$\begin{aligned} 1 = \mathcal{E}(I_\theta(a)) = \mathcal{E}\left(\int_{\mathbb{R}} \theta_s(a)\mathrm{d}s\right) &= \int_{\mathbb{R}} \mathcal{E}(\theta_s(a))\mathrm{d}s \\ = \int_{\mathbb{R}} \theta_s(\mathcal{E}(a))\mathrm{d}s &= \int_{\mathbb{R}} (\mathcal{E}(a))(p+s)\mathrm{d}s. \end{aligned}$$

Hence we have $\mathcal{E}(a) \in L^1(\mathbb{R})$. Now we evaluate the trace of $a^{\frac{1}{2}}ha^{\frac{1}{2}}$ in the following:

$$\tau\left(a^{\frac{1}{2}}ha^{\frac{1}{2}}\right) = \tau\left(h^{\frac{1}{2}}ah^{\frac{1}{2}}\right) = \widehat{\omega}(a) = \omega(I_\theta(a)) = \omega(1) < +\infty.$$

Thus we conclude that $a^{\frac{1}{2}}\mathfrak{M}(1)a^{\frac{1}{2}} \subset L^1\left(\widetilde{\mathcal{M}}, \tau\right)$ and the value $\tau\left(a^{\frac{1}{2}}ha^{\frac{1}{2}}\right)$ is independent of the choice of $a \in \widetilde{\mathcal{M}}_+$ with $I_\theta(a) = 1$. ♡

Theorem E.8. *For $\alpha \in \mathbb{C}$ with $p = \Re\alpha \geq 1$, the quontity:*

$$\|T\| = \left(\int |T|^{\frac{1}{p}}\right)^p, \quad T \in \mathfrak{M}(\alpha),$$

makes $\mathfrak{M}(\alpha)$ a Banach space.

Unlike the usual integration, the intersection of different $\mathfrak{M}(\alpha)$'s is $\{0\}$. So the norm $\|T\|$ is defined only for one value of α for a homogenous $T \in \mathfrak{M}$ relative to the grading given by θ.

Theorem E.9. *If $\{T_1, \cdots, T_n\} \subset \mathfrak{M}\left(\widetilde{\mathcal{M}}, \tau\right)$ and*

$$\mathrm{grad}(T_1) + \cdots + \mathrm{grad}(T_n) = 1,$$

then $T_1 \cdots T_n \in \mathfrak{M}(1)$ and

$$\begin{aligned} \|T_1 \cdots T_n\| &\leq \|T_1\| \cdots \|T_n\|; \\ \int T_1 \cdots T_n &= \int T_2 \cdots T_{n-1}T_1. \end{aligned}$$

Theorem E.10. *If $\alpha \in \mathbb{C}$ satisfies the inequality $1 \leq \Re\alpha \leq 1$, then Banach spaces $\mathfrak{M}(\alpha)$ and $\mathfrak{M}(1-\alpha)$ are in duality under the biliner form:*

$$\langle S, T\rangle = \int ST, \quad S \in \mathfrak{M}(\alpha), \quad T \in \mathfrak{M}(1-\alpha).$$

§**E.2 Local Characteristic Square.** For each non-zero semi-finite normal weight $\varphi \in \mathfrak{W}$, let π^φ be the equivariant isomorphism of $\{L^\infty(\mathbb{R}), \mathbb{R}, \rho\}$ into the non commutative flow of weights $\left\{\widetilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\right\}$ such that φ corresponds to the function $x \in \mathbb{R} \mapsto e^{-x} \in \mathbb{R}_+$ and set

$$\mathcal{A}^\varphi = \pi^\varphi(L^\infty(\mathbb{R})), \quad \mathcal{D}^\varphi = \mathcal{A}^\varphi \vee \mathcal{C}.$$

Proposition E11. *We have the following*:

$$\begin{gathered}(\mathcal{A}^\varphi)' \cap \mathcal{M} = \mathcal{M}_\varphi;\\ \mathcal{D}^\varphi \cap \mathcal{M} \subset \mathcal{C}_\varphi = \textit{The center of } \mathcal{M}_\varphi;\\ \{\mathcal{D}^\varphi, \mathbb{R}, \theta\} = \{(\mathcal{M} \cap \mathcal{D}^\varphi)\overline{\otimes}\mathcal{A}^\varphi, \mathbb{R}, \mathrm{id} \otimes \theta\}.\end{gathered}$$

$$\begin{gathered}\mathcal{C} = L^\infty(X, \mu), \quad (\theta_s(a))(x) = a\big(T_s^{-1}x\big), \quad x \in X, s \in \mathbb{R}.\\ c(s+t; x) = c(s; x)c\big(t; T_s^{-1}x\big)\end{gathered}$$

$$\begin{gathered}\mathcal{A}^\varphi = L^\infty(\mathbb{R}), \quad \varphi(p) = e^{-p}\\ (\theta_s(f))(p) = f(p+s), \quad f \in L^\infty(\mathbb{R});\\ (\theta_s(\varphi))(p) = e^{-(p+s)} = e^{-s}\varphi(p).\end{gathered}$$

Lemma EII.1. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two commutative von Neumann algebras and that π_1 and π_2 are faithful normal representations of $\mathcal{A}_1$ and $\mathcal{A}_2$ such that*

$$\pi_1(\mathcal{A}_1) \subset \pi_2(\mathcal{A}_2)'.$$

Then there exists a measure μ on $X = X_1 \times X_2$ such that

i)
$$L^\infty(X_1 \times X_2, \mu) \cong \pi_1(\mathcal{A}_1) \vee \pi_2(\mathcal{A}_2);$$

ii) *The projections* pr_1 *and* pr_2 *from X to the respective cooridnates carry the measure μ into μ_1 and μ_2 of X_1 and X_2 respectively.*

For each $c \in \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C}))$ and setting

$$(b_\varphi(c))(p, x) = c(p, T_p x), \quad b \in \mathcal{D}^\varphi,$$

we compute

$$\begin{aligned}(\partial_\theta b_\varphi(c))_t(p, x) &= \overline{b(p, x)}b\big(p+t, T_t^{-1}x\big)\\ &= \overline{c(p, T_p x)}c\big(p+t, T_{p+t}T_t^{-1}x\big)\\ &= \overline{c(p, T_p x)}c(p+t, T_p x)\\ &= \overline{c(p, T_p x)}c(p, T_p x)c\big(t, T_p^{-1}T_p x\big)\\ &= c(t, x).\end{aligned}$$

Hence $(\partial_\theta b)_t \in \mathcal{U}(\mathcal{C})$, so that $b_\varphi(c) \in \widetilde{\mathcal{U}}(\mathcal{M})$ and

$$c(t, \cdot) = (\partial_\theta b)_t.$$

Observe that the map $b_\varphi : c \in \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \mapsto b_\varphi(c) \in \mathcal{U}(\mathcal{D}^\varphi)$ is a group homomorphism.

We now observe the following cobounary operation:

$$\partial_\theta\big(\varphi^{\mathrm{it}}\big)(s) = \varphi^{-\mathrm{it}}\theta_s\big(\varphi^{\mathrm{it}}\big) = e^{-\mathrm{i}st}, \quad s, t \in \mathbb{R}$$

Setting

$$c_t(s; x) = e^{-\mathrm{i}st}, \quad x \in X,$$

we have a cocycle $c_t \in \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C}))$. We then check $b_\varphi(c_t)$ in the following:

$$(b_\varphi(c_t))(p, x) = c_t(p, T_p x) = e^{-\mathrm{i}pt} = \varphi^{\mathrm{it}}(p),$$

so that $(b_\varphi(c_t)) = \varphi^{\mathrm{it}} \in \mathcal{A}^\varphi$.

Theorem E.12. *To each faithful semi-finite normal weight $\varphi \in \mathfrak{W}_0(\mathcal{M})$ there corresponds a right inverse: $c \in \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \mapsto b_\varphi(c) \in \mathcal{D}^\varphi \cap \widetilde{\mathcal{U}}(\mathcal{M})$ of the coboundary map $\partial : u \in \mathcal{D}^\varphi \cap \widetilde{\mathcal{U}}(\mathcal{M}) \mapsto \partial(u)_s = u^*\theta_s(u), s \in \mathbb{R}$ such that*

i) $\partial_\theta(b_\varphi(c)) = c, \quad c \in \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C}))$.

ii) $b_\varphi : c \in \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \mapsto b_\varphi(c) \in \widetilde{\mathcal{U}}(\mathcal{M})) \cap \mathcal{D}^\varphi$ *is a continuous homomorphism.*

iii)
$$b_\varphi(c_t) = \varphi^{\mathrm{it}}.$$

iv) *For every $\alpha \in \mathrm{Aut}(\mathcal{M})$, we have the covariance:*

$$b_{\varphi\circ\alpha^{-1}} = \widetilde{\alpha}\circ b_\varphi\circ\widetilde{\alpha}^{-1}$$

v) *If $\varphi \in \mathfrak{W}_0(\mathcal{M})$ is dominant, then*

$$\widetilde{\mathrm{Ad}}(b_\varphi(c)) = \sigma^\varphi_c \quad \text{the extended modular automorphism.}$$

vi) *For a pair $\varphi, \psi \in \mathfrak{W}_0(\mathcal{M})$ of dominant weight, we have*

$$b_\varphi(c)b_\psi(c)^* = (\mathrm{D}\varphi : \mathrm{D}\psi)_c \in \mathcal{U}(\mathcal{M}), \quad c \in \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})),$$

and for a general pair $\varphi, \psi \in \mathfrak{W}_0(\mathcal{M})$, the left side of the above expression belongs to $\mathcal{U}(\mathcal{M})$ and therefore we write

$$(\mathrm{D}\varphi : \mathrm{D}\psi)_c \stackrel{\text{def}}{=} b_\varphi(c)b_\psi(c)^*, c \in \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})).$$

vii) *For a pair $c_1, c_2 \in \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C}))$, we have*

$$(\mathrm{D}\varphi : \mathrm{D}\psi)_{c_1c_2} = (\mathrm{D}\varphi : \mathrm{D}\psi)_{c_1}\sigma^\psi_{c_1}\big((\mathrm{D}\varphi : \mathrm{D}\psi)_{c_2}\big).$$

Definition E.13. For each $\varphi \in \mathfrak{W}_0(\mathcal{M})$ and $c \in \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C}))$, the automorphism

$$\sigma^\varphi_c = \widetilde{\mathrm{Ad}}(b_\varphi(c)) \in \mathrm{Aut}(\mathcal{M})$$

is called the **extended modular automorphism corresponding to a cocycle**

$$\boldsymbol{c \in \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C}))}.$$

Definition E.14. i) The group defined by the folloing:

$$\mathrm{Mod}^\varphi(\mathcal{M}) = \left\{\widetilde{\mathrm{Ad}}(u) : u \in \mathcal{D}^\varphi \cap \widetilde{\mathcal{U}}(\mathcal{M})\right\}$$

is called the **extended modular group** of $\varphi \in \mathfrak{W}_0(\mathcal{M})$.

ii) The abelian von Neumann subalgebra $\mathcal{D}_\varphi = \mathcal{D}^\varphi \cap \mathcal{M}$ is called the **strong center** of the centralizer $\mathcal{M}_\varphi, \varphi \in \mathfrak{W}_0(\mathcal{M})$.

iii) We set $\mathrm{Mod}^\varphi_0(\mathcal{M}) = \{\mathrm{Ad}(u) : u \in \mathcal{U}(\mathcal{D}_\varphi)\}$.

With these notations, we get the following local characteristic square:

Theorem E.15 (Falcone-Takesaki). *To each faithful semi-finite normal weight $\varphi \in \mathfrak{W}_0$, there corresponds a commutative square of exact sequences*:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathcal{U}(\mathcal{C}) & \xrightarrow{\partial} & \mathrm{B}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{U}(\mathcal{D}_\varphi) & \longrightarrow & \mathcal{U}(\mathcal{D}^\varphi) \cap \widetilde{\mathcal{U}}(\mathcal{M}) & \xrightarrow{\partial_\theta} & \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Mod}^\varphi_0(\mathcal{M}) & \longrightarrow & \mathrm{Mod}^\varphi(\mathcal{M}) & \xrightarrow{\dot{\partial}_\theta} & \mathrm{H}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

and

$$\mathcal{U}(\mathcal{D}^\varphi) \cap \widetilde{\mathcal{U}}(\mathcal{M}) \cong \mathcal{U}(\mathcal{D}_\varphi) \times \mathrm{Z}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})) \quad \text{by the isomorphism } b_\varphi.$$

Definition E.16. We call the last square the **the local characteristic square.**

Theorem E.17 (Connes-Kawahigashi-Sutherland-Takesaki). *If $\mathcal{R}$ is an* AFD *factor, then the auotomorphism group* $\mathrm{Aut}(\mathcal{R})$ *has the following normal subgroups*:

i)

$$\mathrm{Ker}(\mathrm{mod}) = \overline{\mathrm{Int}}(\mathcal{R}).$$

ii) *Let C be the set of all bounded sequences $\{x_n\} \in \ell^\infty(\mathcal{R}, \mathbb{N})$ such that*

$$\lim_{n\to\infty} \|\omega x_n - x_n\omega\| = 0, \quad \omega \in \mathcal{M}_*.$$

Then group $\mathrm{Cnt_r}(\mathcal{R})$ is precisely the group of all automorphisms $\alpha \in \mathrm{Aut}(\mathcal{M})$ such that

$$\sigma\text{-strong}^* \lim_{n\to\infty} (\alpha(x_n) - x_n) = 0 \text{ for every } \{x_n\} \in C.$$

iii) *The exact sequence*:

$$1 \longrightarrow \mathrm{Ker(mod)} \xrightarrow{\ i\ } \mathrm{Aut}(\mathcal{R}) \xrightarrow{\mathrm{mod}} \mathrm{Aut}_\theta(\mathcal{C}) \longrightarrow 1$$

splits. Hence we have

$$\mathrm{Aut}(\mathcal{R}) \cong \mathrm{Ker(mod)} \rtimes \mathrm{Aut}_\theta(\mathcal{C}).$$

Note The last assertion (iii) was proven by R. Wong in his thesis but never published. So the proof was presented in [ST3].

§E.3. A Factor of Type III_1: The Mother of Factors of Other Types. Suppose that $\mathcal{M}$ is a separable factor of type III_1. Then its core $\widetilde{\mathcal{M}}$ is a factor of type II_∞ equipped with a trace scaling one parameter automorphism group θ.

Theorem E.18. *In the above setting, to each closed subgroup $T\mathbb{Z}, T > 0$, of the addtivie group $\mathbb{R}$ a factor $\widetilde{\mathcal{M}}^{\theta_T}$ of type III_λ with $\lambda = e^{-T}$ corresponds. If $T_1\mathbb{Z} \subset T_2\mathbb{Z}$, i.e., if $T_1/T_2 \in \mathbb{Z}$, then we have*

$$\widetilde{\mathcal{M}}^{\theta_{T_1}} \supset \widetilde{\mathcal{M}}^{\theta_{T_2}} \supset \cdots \supset \mathcal{M} = \widetilde{\mathcal{M}}^{\theta}.$$

Let $\{\mathcal{A}, \mathbb{R}, \alpha\}$ is an ergodic flow. We represent the covariant system $\{\mathcal{C}, \mathbb{R}, \alpha\}$ by an ergodic flow of non-singular transformations:

$$\begin{gathered}\mathcal{C} = L^\infty(X, \mu), \quad (\alpha_s(f))(x) = f\big(T_s^{-1}x\big), \quad s \in \mathbb{R},\ x \in X,\ f \in \mathcal{C};\\ \mu(N) = 0 \quad \Leftrightarrow \quad \mu(T_s(N)) = 0, \quad N \subset X;\\ \rho(s, x) = \frac{\mathrm{d}\mu{\circ}T_s}{\mathrm{d}\mu}(x), \quad s \in \mathbb{R},\ x \in X.\end{gathered}$$

Now consider the tensor product: $\mathcal{N} = \mathcal{C}\overline{\otimes}\widetilde{\mathcal{M}}$ and view it as the von Neumann algebra $L^\infty\Big(\widetilde{\mathcal{M}}, X, \mu\Big)$ of all essentially bounded $\widetilde{\mathcal{M}}$-valued measurable functions on X. We are going to build a one parameter automorphism group $\Big\{\tilde{\theta}_s : s \in \mathbb{R}\Big\}$ which scales a semifinite normal trace on $\mathcal{N}$ down and the restriction of $\tilde{\theta}$ to $\mathcal{C}$ is exactly the given ergodic flow α. Set

$$\Big(\tilde{\theta}_s(a)\Big)(x) = \theta_{s-\log\rho(s^{-1},x)}\big(a\big(T_s^{-1}x\big)\big), \quad a \in \mathcal{N} = L^\infty\Big(\widetilde{\mathcal{M}}, X, \mu\Big).$$

We then compute for $a \in \mathcal{N}_+$:

$$
\begin{aligned}
(\tau \otimes \mu)\Big(\tilde{\theta}_s(a)\Big) &= \int_X \tau\big(\theta_{s-\log\rho(s^{-1},x)}\big(a\big(T_s^{-1}x\big)\big)\big)\mathrm{d}\mu(x) \\
&= \int_X e^{-s}\rho\big(s^{-1},x\big)\tau\big(a\big(T_s^{-1}x\big)\big)\mathrm{d}\mu(x) \\
&= e^{-s}\int_X \rho\big(s^{-1},x\big)\tau\big(a\big(T_s^{-1}x\big)\big)\mathrm{d}\mu(x) \\
&= e^{-s}\int_X \tau(a(x))\mathrm{d}\mu(x) \\
&= e^{-s}(\tau \otimes \mu)(a).
\end{aligned}
$$

Therefore, the one parameter automorphism group $\tilde{\theta}$ transforms the trace $\tilde{\tau} = \tau \otimes \mu$ exactly in the way we want. Hence the covariant system $\Big\{\mathcal{N}, \mathbb{R}, \tilde{\theta}, \tilde{\tau}\Big\}$ is the core of the factor $\mathcal{M}_0 = \mathcal{N}^{\tilde{\theta}}$. The flow of weights on $\mathcal{M}_0$ is exactly the one we started, i.e., $\{\mathcal{C}, \mathbb{R}, \alpha\}$. If the flow $\{\mathcal{C}, \mathbb{R}, \alpha\}$ is properly ergodic, then the factor $\mathcal{M}_0$ is of type III_0.

Theorem E.19. *Every erogic flow $\{\mathcal{C}, \mathbb{R}, \alpha\}$ appears as the flow of weights on a factor of type* III_0*. If the original factor $\widetilde{\mathcal{M}}$ is* AFD*, then the factor $\mathcal{M}_0$ constructed above is also* AFD*. The flow of weights on an* AFD *factor of type* III_0 *is a complete algebraic invariant. Hence all* AFD *factors are isomorphic to the ones obtained through the above construction.*

Fix an ergodic flow $\{\mathcal{C}, \mathbb{R}, \theta\}$. Let $\mathrm{Aut}_\theta(\mathcal{C})$ be the group of automorphisms of $\mathcal{C}$ commuting with θ. Each $\alpha \in \mathrm{Aut}_\theta(\mathcal{C})$ gives rise to a non-singular transformation T_α of the measure space $\{X, \mu\}$:

$$(\alpha(a))(x) = a\big(T_\alpha^{-1}x\big), \quad a \in \mathcal{C}, \alpha \in \mathrm{Aut}_\theta(\mathcal{C}).$$

We have also the ρ-function:

$$\rho(\alpha, x) = \frac{\mathrm{d}\mu\circ T_\alpha}{\mathrm{d}\mu}(x), \quad \alpha \in \mathrm{Aut}_\theta(\mathcal{C}).$$

We then set

$$(\widetilde{\alpha}(a))(x) = \theta_{-\log\big(\rho(\alpha^{-1}_, x)\big)}\big(a\big(T_\alpha^{-1}x\big)\big).$$

Then the map: $\alpha \in \mathrm{Aut}_\theta(\mathcal{C}) \mapsto \widetilde{\alpha} \in \mathrm{Aut}_{\tau,\theta}(\mathcal{N})$ is an injective homorphism and gives rise to an automorphism $\bar{\alpha} \in \mathrm{Aut}(\mathcal{M}_0)$ such that $\mathrm{mod}(\bar{\alpha}) = \alpha$.

Theorem E.20. *The group* $\mathrm{Aut}(\mathcal{M}_0)$ *is splits as the semi-direct product:*

$$\mathrm{Aut}(\mathcal{M}_0) \cong \mathrm{Ker}(\mathrm{mod}) \rtimes \mathrm{Aut}_\theta(\mathcal{C}).$$

Note on the Paper of R. Wong. *For an* AFD *factor* $\mathcal{R}$ *of type* III_0*, the above result was proven by R. Wong in his thesis, although his method was much more based on the ergodic theory. But his papaer was never published. So the proof was given in* [ST3]*. The reason why his paper was never published was that when his paper was ready to go printing, the Transaction of American Mathematical Society needed his agreement on the copy right, which was never sent back to the AMS office: he had left the Univ. of New South Wales before the copy right signature request was sent to him. His supervioser was Prof. Colin E. Sutherland. Even he was unable to trace whereabout of Dr. R. Wong. Probably he had left the UNSW in a despair on the academic job market at that time.*

Concluding Remark

Now we come to the end of the lecture series. As seen through the lectures, the theory of von Neumann algebras is very much like a number theory in analysis. There are many mathematical problems which associate operators on a Hilbert space. The theory of operator algebras looks at the enviroment of the operators in quesion and give a guide how to attack the problem. Namely, the theory of operator algebras will give us the framework for how to attack the problem. For instance, the unitary operators on $L^2(\mathbb{R})$ of the translation operator by one and the mutiplication operator by trigonometric funcions discussed Example 7.12 is not exotic at all and has a close relation to difference equations. I believe that the theory of operator algebras has been sufficently developped that one can now apply them in many areas of mathematics and mathematical physics. One should try to rephrase the existing mathematical problems in terms of operator algebras. The non-commutative geometry of Alain Connes is exactly one of these attempts.

References

[Ark1] H. Araki, *Type of von Neumann algebra associated with free field*, Progr. Theoret. Phys. **32** (1964), 956-965.

[Ark2] H. Araki, *A classification of factors.* II, Publ. Res. Inst. Math. Sci. Ser. A 4 (1968/1969), 585-593.

[AkWd] H. Araki and E.J. Woods, *A classification of factors*, Publ. Res. Inst. Math. Sci. Ser. A 4 (1968/1969), 51-130.

[Arv] W. Arveson, *On groups of automorphisms of operator algebras*, J. Functional Analysis **15** (1974), 217-243.

[Brch] H.-J. Borchers, *Energy and momentum as observables in quantum field theory*, Comm. Math. Phys.. **2** (1966), 49-54.

[Brw] K.S. Brown, *Cohomology of groups,* Graduate Text in Mathematics, 87, Springer Verlag, Berline, New York, Heidelberg, Hong Kong, Tokyo, 1994, pp. 306+x.

[Cnn1] A. Connes, *Une classification des facteurs de type* III, Ann. Scient. Ecole Norm. Sup. **4ème Sèrie, 6** (1973), 133-252.

[Cnn2] ———, *Almost periodic states and factors of type* III_1, J. Funct. Anal., **16** (1974), 415-445.

[Cnn3] ———, *Periodic automorphisms of the hyperfinite factor of type* II_1, Acta Math. Szeged, **39** (1977), 39-66.

[Cnn4] ———, *Outer conjugacy of automorphisms of factors*, Symposia Mathematica, **20**, 149-159.

[Cnn5] ———, *Classification of injective factors*, Ann. of Math., **104** (1976), 73-115..

[Cnn6] ———, *Outer conjugacy classes of automorphisms of factors*, Ann. Sci. École Norm. Sup., **4éme Série, 8,** (1975), 383-419.

[Cnn7] ———, *Outer conjugacy of automorphisms of factors*, Symposia Mathematica, **20**, 149-159.

[Cnn8] ———, *Spatial theory of von Neumann algebras*, J. Functional Analyis, **35** (1980), 153-164.

[CT] A. Connes and M. Takesaki, *The flow of weights on factors of type III*, Tôhoku Math. J., **29** (1977), 473-575.

[EnSw] M. Enock, J-M. Schwartz, *Kac algebras and duality of locally compact groups. With a preface by Alain Connes. With a postface by Adrian Ocneanu*, Springer-Verlag, Berlin, (1992), pp. x+257.

[FT1] A.J. Falcone and M. Takesaki, *Operator valued weights without structure theory*, Trans. Amer. Math. Soc., **351** (1999), 323–341.

[FT2] ———, *Non-commutative flow of weights on a von Neumann algebra*, J. Functional Analysis, **182** (2001), 170 - 206.

[HHW] H.M. Hugenhotz, R. Haage, M. Winnink, *On the equilibrium states in quantum statistical mechanics*, Comm. Math. Phys. **5** (1967), 215-236.

[Hu] U.K. Hui, *Cocycle conjugacy of one parameter automorphism groups of* AFD *factors of type* III, International J. Math., (2002), 579-603.

[Jn] V.F.R. Jones, *Actions of finite groups on the hyperfinite type* II_1 *factor*, Amer. Math. Soc. Memoirs, **237** (1980).

[JT] V.F.R. Jones and M. Takesaki, *Actions of compact abelian groups on semifinite injective factors*, Acta Math., **153** (1984), 213-258.

[KST] D. Kastler, *The C^*-algebras of a free Boson field.* I. Discussion of the basic facts., Comm. Math. Phys. **1** (1965), 1448.

[KtST1] Y. Katayama, C.E. Sutherland and M. Takesaki, *The characteristic square of a factor and the cocycle conjugacy of discrete amenable group actions on factors*, Invent. Math., **132** (1998), 331-380.

[KtST2] ———, *The structure of the automorphism group of a factor and cocycle conjugacy of discrete group actions*, Proceeding of Conference on Operator Algebras and Quantum Field Theory, (1997), International Press, Cambridge, MA, 166-198.

[KtT1] Y. Katayama and M. Takesaki, *Outer actions of a countable discrete amenable group on* AFD *factors*, Contemporary Mathematics, **335** (2003), 163 - 171.

[KtT2] ———, *Outer actions of a countable discrete amenable group on approximately finite dimensional factors* I, *General Theory Contemporary Mathematics* **365** (2004), 181 - 237.

[KtT3] ———, *Outer actions of a countable discrete amenable group on approximately finite dimensional factors* II *, The* III_λ*-Case,* $\lambda \neq 0$, Math. Scand., **100** (2007), 75-129.

[Kw1] Y. Kawahigashi, *Centrally ergodic one-parameter automorphism groups on semi finite injective von Neumann algebras*, Math. Scand., **64 no. 2,** (1989), 285-299..

[Kw2] ———, *One-parameter automorphism groups of the injective* II_1 *factor arising from the irrational rotation C^*-algebra*, Amer. J. Math., **112, no. 4,** (1990), 499–523.

[Kw3] ———, *One-parameter automorphism groups of the injective factor of type* II_1 *with Connes spectrum zero*, Canad. J. Math., **43, no. 1,** (1991), 108–118.

[Kw4] ———, *One-parameter automorphism groups of the hyperfinite type* II_1 *factor*, J. Operator Theory, **25, no. 1,** (1991), 37-59.

[KwST] Y. Kawahigashi, C.E. Sutherland and M. Takesaki, *The structure of the automorphism group of an injective factor and the cocycle conjugacy of discrete abelian group actions*, Acta Math., **169** (1992), 105-130.

[KsK1] H. Kosaki, *Applications of the complex interpolation method to a von Neumann algebra: noncommutative L^p -spaces*, J. Funct. Anal. **56** (1984), 29-78..

[KsK2] ———, *Applications of the complex interpolation method to a von Neumann algebra: noncommutative Lp-spaces*, J. Funct. Anal., **56** (1984,), 29-78.

[Ocn] A. Ocneanu, *Actions of discrete amenable groups on factors*, Lecture Notes in Math. No. **1138,** (1985), Springer, Berlin.

[OPT] D. Olesen, G. K. Pedersen and M. Takesaki, *Ergodic actions of compact abelian groups*, J. Operator Theory, **3** (1980), 237-269.

[PdTk] G.K. Pedersen, M. Takesaki,, *The Radon-Nikodym theorem for von Neumann algebras*, Acta Math., **130** (1973), 53-87.

[RTPW] R.T. Powers, *Representations of uniformly hyperfinite algebras and their associated von Neumann rings*, Bull. Amer. Math. Soc. **73** (1967), 572-575.

[Pksky] L. Puknszky, *Some examples of factors*, Publ. Math. Debrecen **4** (1956,), 135-156..

[Rc] J.G. Ratcliffe, *Crossed extensions*, Trans. Amer. Math. Soc., **237,** (1980), 73 - 89.

[Sk] S. Sakai, *On topological properties of W^*-algebras*, Proc. Japan Acad. **33,** (1957), 439-444..

[St1] C.E. Sutherland, *Cohomology and extensions of von Neumann algebras,* I *and* II , Publ. RIMS., Kyoto Univ., **16**, 105 - 133; 135 - 174.

[St2] C.E. Sutherland, *A Borel parametrization of Polish groups*, Publ. RIMS., Kyoto Univ., **21** (1985), 1067 - 1086.

[ST1] C.E. Sutherland and M. Takesaki, *Actions of discrete amenable groups and groupoids on von Neumann algebras*, Publ Res. Inst. Math. Sci. **21** (1985), 1087-1120.

[ST2] C.E. Sutherland and M. Takesaki, *Actions of Discrete amenable groups on injective factors of type* III_λ *,* $\lambda \neq 1$, Pacific J. Math. **137** (1989), 405-444.

[ST3] C.E. Sutherland and M. Takesaki, *Right inverse of the module of approximately finite dimensional factors of type* III *and approximately finite ergodic principal measured groupoids. Operator algebras and their applications* II , Fields Institute Comm., **20** (1998), 149-159.

[Tk0] M. Takesaki, *Tomita's theory of modular Hilbert algebras and its applications. Lecture Notes in Mathematics*, vol. 128, Springer-Verlag, Berlin-New York, 1970, pp. ii+123.

[Tk1] M. Takesaki, *Theory of Operator Algebras* I, Springer - Verlag, Heidelberg, New York, Hong Kong, Tokyo, 1979.

[Tk2] ———, *Theory of Operator Algebras* II, Springer - Verlag, Heidelberg, New York, Hong Kong, Tokyo, 2002.

[Tk3] ———, *Theory of Operator Algebras* III, Springer - Verlag, Heidelberg, New York, Hong Kong, Tokyo, 2002.

[Tk4] ——— Duality and von Neumann algebras, Bull. Amer. Math. Soc. **77** (1971), 553-557.

Department of Mathematics, UCLA, PO Box 951555, Los Angeles, California 90095-1555. Postal Mail Address: 3-10-39 Nankohdai, Izumi-ku, Sendai, 981-8003 Japan

E-mail address: `mt@math.ucla.edu`