§6 Overview on the applications of easy QG

6.1 For quantum groups:

- Easy QG give rise to many examples of CMQG. Wang's "liberation" and beyond.
- machine: result for $S_n/O_n/U_n \rightsquigarrow$ result for $S_n^+/O_n^+/U_n^+ \rightsquigarrow$ all easy QG $\rightsquigarrow$ all CMQG (uniform proof)
- philosophy: Every quantum algebraic property is visible in the combinatorics of partitions.

  Ex. 1: $S_n \subseteq G \subseteq H_n^{[\infty]}$, where $H_n^{[\infty]} \subseteq H_n^+$ is the easy QG associated to $\langle \quad \rangle$.
  (Equivalently we may say $S_n \subseteq G \subset O_n^+$ such that u_{ij}^2 are central projections).
  Then $G \cong \hat{\Gamma} \bowtie S_n$ for $\mathbb{Z}_2^{*n} \twoheadrightarrow \Gamma$, i.e. $C(G) \cong C^*(\Gamma) \otimes C(S_n)$, $(u_{ij}) \mapsto (a_{g_i} v_{ij})$, (v_{ij})
  (if G is in its maximal version)
  This is inspired from Th. 5.9(c) for easy QG! [Raum-Weber 13]

  Ex. 2: Representation theory of easy QG may be described in terms of partitions.
  Fusion rules: (a) Find all irred. rep. $(u_\alpha)_{\alpha \in I}$ (b) How does $u_\alpha \otimes u_\beta = \sum_{\gamma \text{ irred.}} u_\gamma$ decompose?
  Here: For $p \in \mathcal{C}(k,k)$ with $p = p^* = pp$, define $P_p := T_p - \bigvee_{\substack{q \in \mathcal{C}(k,k) \\ q = q^* = qq \\ pq = qp = q \neq p}} T_q$ (up to some normalization)
  and put $u_p := P_p u^{\otimes k} = u^{\otimes k} P_p \subseteq u^{\otimes k}$ block of the matrix.
  Then $u_p \otimes u_q = \sum_{m \in X_{\mathcal{C}}(p,q)} u_m$, $X_{\mathcal{C}}(p,q)$ can be described explicitly. Likewise $u_p \cong u_q$.
  If $\mathcal{C} \subseteq NC$, then u_p irreducible, otherwise these are only "rough fusion rules", but we have a refinement.
  [Freslon-Weber 13]

  Ex. 3: The Haar state of easy QG may be expressed in terms of partitions:
  $h(u_{i_1 j_1} \cdots u_{i_k j_k}) = \sum_{p,q \in \mathcal{C}(0,k)} \delta_p(i) \delta_q(j) W_{kn}(p,q)$, $W_{kn} = (G^{-1})_{kn}$, $G_{kn}(p,q) = n^{\#\text{blocks}(p \vee q)}$
  "Weingarten calculus"
  [Collins, Banica, Banica-Curran-Speicher]

6.2 For von Neumann algebras:

- $C^*_{red}(O_n^+)$, $C^*_{red}(U_n^+)$ non-nuclear, exact, simple, MAP [Banica 97, Vaes-Vergnioux, Brannan]
- LO_n^+, LU_n^+ strongly solid, non-injective, full, prime II_1 factors, HAP, no Cartan, $LU_2^+ \cong LF_2$ [B, W, Br, Isono, ...]
- O_n^+, U_n^+ fulfill Baum-Connes, $K_0 C(O_n^+) = \mathbb{Z}$, $K_1 C(O_n^+) = \mathbb{Z}$, $K_0 C(U_n^+) = \mathbb{Z}$, $K_1 C(U_n^+) = \mathbb{Z}^2$,
  $K_0(C(S_n^+)) = \mathbb{Z}^{n^2-2n+2}$, $K_1 C(S_n^+) = \mathbb{Z}$ [Voigt, Vergnioux-Voigt]
- O_n^+, U_n^+ weakly amenable, Akemann-Ostrand, rapid decay [Freslon, Vergnioux]

6.3 For Free Probability:

Motivation: Have $\mathbb{F}_n \not\cong \mathbb{F}_m$ for $n \neq m$
$\mathbb{C}\mathbb{F}_n \not\cong \mathbb{C}\mathbb{F}_m$ for $n \neq m$
$C^*_r \mathbb{F}_n \not\cong C^*_r \mathbb{F}_m$ for $n \neq m$
but $L\mathbb{F}_n \not\cong L\mathbb{F}_m$ for $n \neq m$? Open!

Freeness: $G_1 := \mathbb{F}_n$, $G_2 := \mathbb{F}_m$, $G := \mathbb{F}_{n+m}$. Then $G_1, G_2 \subseteq G$ "free" in the sense that $g_j \in G_{i(j)}, g_j \neq e, i(j) \neq i(j+1) \Rightarrow g_1 \cdots g_k \neq e$

Also $\mathbb{C}G_1, \mathbb{C}G_2 \subseteq \mathbb{C}G$ "free" with
$a_j := \sum \alpha_g g \in \mathbb{C}G_{i(j)}, \alpha_e = 0, i(j) \neq i(j+1) \Rightarrow a_1 \cdots a_k = \sum \beta_g g$ with $\beta_e = 0$

How to formulate for $L\mathbb{F}_n, L\mathbb{F}_m \subseteq L\mathbb{F}_{n+m}$ "free"? Use the trace!

$\tau: \mathbb{C}G \to \mathbb{C}$, $\sum \alpha_g g \mapsto \alpha_e$ yields $a_j \in \mathbb{C}G_{i(j)}, \tau(a_j) = 0, i(j) \neq i(j+1) \Rightarrow \tau(a_1 \cdots a_k) = 0$.

Def (Voiculescu 85): $A_1, \dots, A_n \subseteq A$ subalgebras, $1 \in A_i$, A unital algebra, ((A,φ) "noncommutative prob. space") $\varphi: A \to \mathbb{C}$ unital lin. functional. $A_1, \dots, A_n \subseteq A$ are <u>free</u>, if
$$a_j \in A_{i(j)}, \varphi(a_j) = 0 \ \forall j, \ i(j) \neq i(j+1) \Rightarrow \varphi(a_1 \cdots a_k) = 0$$

Free "Probability": $(\Omega, \Sigma, \mathbb{P})$ classical probability space. Then $L^\infty(\Omega, \mathbb{P})$ algebra of random variables and $\mathbb{E}X := \int_\Omega X \, d\mathbb{P}$ linear functional.

Put $A := L^\infty(\Omega, \mathbb{P})$, $\varphi := \mathbb{E}$. Similar to the above structure!

Moreover, what is independence? X, Y independent $\Rightarrow \varphi(X^n Y^m) = \varphi(X^n)\varphi(Y^m)$

Thus: independence is a rule for computing mixed moments and many distributions are completely determined by their moments (like the Gaussian),

i.e. distribution of X $\leftrightarrow \{\varphi(X^n) \mid n \in \mathbb{N}\}$.

distribution of X, Y $\leftrightarrow \{\varphi(X^n Y^m) \mid n, m \in \mathbb{N}\}$

Know the distr. of X, and of Y; X, Y indep. $\Rightarrow$ know joint distr. of X, Y

Likewise in free prob.: $a \in A_1$, $b \in A_2$, $A_1, A_2 \subseteq A$ free. Then
$$0 = \varphi[(a - \varphi(a)1)(b - \varphi(b)1)(a - \varphi(a)1)(b - \varphi(b)1)]$$
$$= \varphi(abab) - \varphi(a)\varphi(bab) + \dots$$

Inductively: $\varphi(abab) = \sum \underbrace{\varphi(\dots)\varphi(\dots) \cdot \varphi(\dots)}_{\text{lower order moments}} = \sum$ Moments $\varphi(a^k), \varphi(b^k)$

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We identify {distribution} = {moments}

Note: $A = W^*(a_1, \dots a_n)$, $B = W^*(b_1, \dots b_n)$, $a_i = a_i^*$, $b_i = b_i^*$, $\varphi : A \to \mathbb{C}$, $\psi : B \to \mathbb{C}$ faithful normal states,

moments $(a_1, \dots a_n)$ = moments $(b_1, \dots b_n)$ [i.e. $\varphi(a_{i_1} \dots a_{i_k}) = \psi(b_{i_1} \dots b_{i_k})$]

$\Rightarrow A \cong B$ via $a_i \mapsto b_i$, hence moments determine vN algebra.

Freeness is useful:
- Isomorphisms of vN algebras?
- large random matrices behave like free elements
- random matrices as a tool in vN alg. $\Rightarrow$ results for $L(F_n)$
- Links to combinatorics, complex analysis, wireless communications etc.

Easy QG as symmetries in free prob. — the de Finetti theorems:

NC partitions in free prob: Let (A, φ) be a noncomm. prob. space.

Speicher 90's: $\varphi(a_1 \cdot \ldots \cdot a_k) = \sum_{p \in NC(0,n)} \kappa_p(a_1, \dots a_k)$ "free cumulants" κ_p (multilin., multipl. functionals)

Then $A_1, A_2 \subseteq A$ free $\Leftrightarrow$ $\kappa_p(a_1, \dots a_k) = 0$ if $a_i \in A_1$, $a_j \in A_2$ exist.
"vanishing of mixed cumulants"

In classical probability $\varphi \dots = \sum_{p \in P(0,n)} \dots$ is the right approach.

Very often " $P \rightsquigarrow NC$ $\Leftrightarrow$ classical prob. $\rightsquigarrow$ free prob."

De Finetti (30's): $(x_n)_{n \in \mathbb{N}}$ classical random variables, $x_i x_j = x_j x_i$.

distribution invariant under S_n $\forall n \in \mathbb{N}$ $\Leftrightarrow$ (x_n) iid over the tail σ-algebra
(i.e. $S_n \curvearrowright (x_1, \dots x_n)$)
i.e. with respect to
$E: L^\infty(\Omega, \mathbb{P}) \to L^\infty(\Omega, \mathbb{P}_{tail})$
$\Sigma_{tail} := \bigcap_{n \in \mathbb{N}} \sigma(x_k, k \geq n)$

Noncommutative de Finetti [Köstler-Speicher 2009]: $(x_n)_{n \in \mathbb{N}}$ elements in a n.c.p.s. (A, φ).

distr. invariant under S_n^+ $\forall n \in \mathbb{N}$ $\Leftrightarrow$ (x_n) $\underset{(free)}{fid}$ over the tail vN algebra
$E: A \to \bigcap_{n \in \mathbb{N}} W^*(x_k, k \geq n)$

$\varphi(x_{i_1} \dots x_{i_m}) 1 = \sum_{j_1, \dots j_m} u_{j_1 i_1} \dots u_{j_m i_m} \varphi(x_{j_1} \dots x_{j_m})$ ($x_i \mapsto \sum_j u_{ji} \otimes x_j$ then φ on both sides)