§ 3b: From Tannaka-Krein to easy QG

- Let $\mathcal{C} \subseteq P$ be a category of partitions,

(=4.1) $\left[\mathcal{C} = (\mathcal{C}(r,s))_{r,s \in \mathbb{N}_0}, \ \mathcal{C}(r,s) \subseteq P(r,s), \ \mathcal{C} \text{ closed under } \otimes, \text{comp., invol.}, | \in \mathcal{C}, \sqcap \in \mathcal{C} \right]$

- To $p \in P(r,s)$ associate $T_p : (\mathbb{C}^n)^{\otimes r} \to (\mathbb{C}^n)^{\otimes s}$ lin. Hence $T_p \otimes T_q = T_{p \otimes q}$ etc. (Prop 4.5)

(=4.3, 4.5) $\Rightarrow \left(\mathrm{Span}\{T_p \mid p \in \mathcal{C}(k,l)\} \right)_{k,l \in \mathbb{N}_0}$ closed under $\otimes$, comp., invol., $\mathrm{id}_{(\mathbb{C}^n)^{\otimes k}} \in (\mathrm{Span}\{\ldots\})$ $\forall k$.

- Define $\mathcal{R} := (R, (H_r)_{r \in R}, (\mathrm{Mor}(r,s))_{r,s \in R}, \cdot, f)$ by
  $R := \mathbb{N}_0$, $H_r := (\mathbb{C}^n)^{\otimes r}$, $\mathrm{Mor}(r,s) := \mathrm{Span}\{T_p \mid p \in \mathcal{C}(r,s)\}$, $r \cdot s := r+s$, $f = 1$.
  This is a concrete monoidal W^*-category (Choose $n \in \mathbb{N}$) ($\mathcal{R} = \mathcal{R}(n, \mathcal{C})$ ...)

  Γ (i) $T_1 \in \mathrm{Mor}(r_1, s_1)$, $T_2 \in \mathrm{Mor}(r_2, s_2) \Rightarrow T_1 \otimes T_2 \in \mathrm{Mor}(\underbrace{r_1 r_2}_{= r_1 + r_2}, \underbrace{s_1 s_2}_{= s_1 + s_2})$

  (ii) $T_1 \in \mathrm{Mor}(r,s)$, $T_2 \in \mathrm{Mor}(s,t) \Rightarrow T_2 T_1 \in \mathrm{Mor}(r,t)$

  (iii) $T \in \mathrm{Mor}(r,s) \Rightarrow T^* \in \mathrm{Mor}(s,r)$

(=4.6) (iv) $\mathrm{id} \in \mathrm{Mor}(r,r)$ $\forall r$

  (v) $H_r = H_s$, $\mathrm{id} \in \mathrm{Mor}(r,s) \Rightarrow r = s$

  (vi) $(rs)t = r(st)$

  L (vii) $\exists 1 \in R : H_1 = \mathbb{C}$, $1r = r1 = r$ (here $1 := 0$)

- $\bar{r}$ is conjugate of r, if $\exists j : H_r \to H_{\bar{r}}$ invertible, antilinear: $t_j \in \mathrm{Mor}(1, r\bar{r})$, $\bar{t}_j \in \mathrm{Mor}(\bar{r}r, 1)$
  where $t_j : \mathbb{C} \to H_r \otimes H_{\bar{r}}$, $1 \mapsto \sum e_i \otimes j(e_i)$, e_i ONB of H_r, $\bar{t}_j(x \otimes y) := \langle j^{-1}(x), y \rangle$

  We have $f = \bar{f}$ since $j : \mathbb{C}^n \to \mathbb{C}^n$, $\sum \alpha_i e_i \mapsto \sum \bar{\alpha}_i e_i$ yields $t_j(1) = \sum_i e_i \otimes e_i$ which is in $\mathrm{Mor}(0,2)$.
  $= T_{\sqcap}(1)$

- $Q \underset{\text{fin}}{\subseteq} R$ generates R, if $\forall s \in R$ $\exists b_k \in \mathrm{Mor}(\underbrace{q^{(k)}_{r_1}}_{\in Q} \cdots \underbrace{q^{(k)}_{r_{n_k}}}_{\in Q}, s)$, $k = 1, \ldots, m$: $\sum_k b_k b_k^* = \mathrm{id}_s$

  Here: $\{f\}$ generates R, since $\underbrace{f \cdots f}_{s\text{-times}} = 1 + \ldots + 1 = s$. For $s \in R$, take $b_k = \mathrm{id}_s \in \mathrm{Mor}(\underbrace{f \cdots f}_{s\text{-times}}, s)$

- Due to [24, Th. 1.3] = Th. 3.15 there is a CMQG $G = (A, u)$ such that
  (a) $(A, (u^r)_{r \in R})$ is a model of R with $u^f = u$, i.e. $u^r \in B(H_r) \otimes A$ unitary

(=3.15) and $u^{rs} = u^r \otimes u^s$, $u^r(t \otimes \mathrm{id}) = (t \otimes \mathrm{id}) u^s$ $\forall r,s \in R$, $t \in \mathrm{Mor}(s,r)$
  (here: $u^{\otimes k} T_p = T_p u^{\otimes l}$ $\forall p \in \mathcal{C}(l,k)$) (here $u = \bar{u}$ because of $f = \bar{f}$)

  (b) A is the smallest C^*-algebra containing the matrix elements of u and it is universal in the sense that $A \to B$, $u \mapsto u'$ whenever $(B, (u'^r)_{r \in R})$ is a model of R with $u'^f = u'$

  (c) $\mathrm{Rep}\, G = \hat{R}$ for the completion of R (equivalences, subrepr., direct sums).

• Def. (Banica, Speicher 2009): A CMQG is called <u>easy</u>, if it is obtained this way.

(=4.7, 4.8) Hence A is the universal C^*-algebra generated by elements $u_{ij} = u_{ij}^*$, $1 \le i,j \le n$ such that the relations "$T_p u^{\otimes k} = u^{\otimes l} T_p$" are fulfilled for all $p \in \mathcal{C}(k,l)$, i.e.

⌈ As $u^{\otimes k} = \sum e_{\alpha_1\beta_1} \otimes \ldots \otimes e_{\alpha_k\beta_k} \otimes u_{\alpha_1\beta_1} \ldots u_{\alpha_k\beta_k} \in M_{n^k}(A)$, we may let $u^{\otimes k}$ act as $u^{\otimes k}(e_{i_1} \otimes \ldots \otimes e_{i_k} \otimes 1) = \sum_{\alpha_1 \ldots \alpha_k} e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_k} \otimes u_{\alpha_1 i_1} \ldots u_{\alpha_k i_k}$

Thus $T_p u^{\otimes k}(e_{i_1} \otimes \ldots \otimes e_{i_k} \otimes 1) = \sum_{t_1 \ldots t_k} T_p(e_{t_1} \otimes \ldots \otimes e_{t_k}) \otimes u_{t_1 i_1} \ldots u_{t_k i_k}$

$$= \sum_{s_1 \ldots s_l} e_{s_1} \otimes \ldots \otimes e_{s_l} \otimes \Big(\sum_{t_1 \ldots t_k} \delta_p(t,s)\, u_{t_1 i_1} \ldots u_{t_k i_k} \Big)$$

and $u^{\otimes l} T_p (e_{i_1} \otimes \ldots \otimes e_{i_k} \otimes 1) = \sum_{s_1 \ldots s_l} e_{s_1} \otimes \ldots \otimes e_{s_l} \otimes \Big(\sum_{j_1 \ldots j_l} \delta_p(i,j)\, u_{s_1 j_1} \ldots u_{s_l j_l} \Big)$
⌊

$\forall\, s_1 \ldots s_l,\ i_1 \ldots i_k \in \{1, \ldots, n\}$: $\sum_{t_1 \ldots t_k} \delta_p(t,s)\, u_{t_1 i_1} \ldots u_{t_k i_k} = \sum_{j_1 \ldots j_l} \delta_p(i,j)\, u_{s_1 j_1} \ldots u_{s_l j_l}$

with lhs $= \delta_p(0,s)$, if $k=0$, same for rhs

• In this sense $\quad P \longleftrightarrow S_n \qquad P_2 \longleftrightarrow O_n$

(=4.7) $\qquad NC \longleftrightarrow S_n^+ \qquad NC_2 \longleftrightarrow O_n^+$

⌈ $\sqcap$: $\delta_{s_1 = s_2} = \delta(0, s_1 s_2) = \sum_{j_1 j_2} \delta_p(0, j_1 j_2)\, u_{s_1 j_1} u_{s_2 j_2} = \sum_{j_1} u_{s_1 j_1} u_{s_2 j_1} \iff u u^t = 1$

$\sqcup$: $u^t u = 1$

$\uparrow \in P(0,1)$: $1 = \delta_p(0, s_1) = \sum_{j_1} \delta_p(0, j_1)\, u_{s_1 j_1} = \sum_{j_1} u_{s_1 j_1}$

$\downarrow \in P(1,0)$: $\sum_{j_1} u_{j_1 s_1} = 1$

$\sqcap\!\!\!\sqcup \in P(2,2)$: $\delta_{s_1 = s_2} u_{s_1 i_1} u_{s_2 i_2} = \delta_{i_1 = i_2} u_{s_1 i_1} u_{s_2 i_1} \iff u_{ik} u_{jk} = 0$ if $i \neq j$, $u_{ki} u_{kj} = 0$ if $i \neq j$

$X \in P(2,2)$: $u_{ij} u_{kl} = u_{kl} u_{ij}$

Hence: associated to NC is $G = (A, u)$ with $u = \bar{u}$, $uu^t = u^t u = 1$, $\sum_i u_{ik} = \sum_k u_{kj} = 1$, $u_{ik} u_{jk} = u_{ki} u_{kj} = 0$ if $i \neq j$ (i.e. $u_{ij}^2 = \sum_k u_{ij} u_{ik} = u_{ij}$ projection)

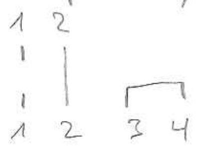
Thus $\mathcal{C}(S_n^+) \to A$. It can be checked, that $\mathcal{C}(S_n^+)$ fulfills all relations for NC, hence $A \to \mathcal{C}(S_n^+)$. See §5.
⌊

• As $NC_2 \subseteq \mathcal{C} \subseteq P$ $\forall\, \mathcal{C}$ category, we have $S_n \subseteq G \subseteq O_n^+$ for all easy QG.

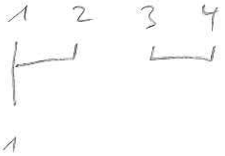
Thm 4.9.

§4 Definition of easy QG

4.1 Definition: (a) A <u>partition</u> $p \in P(k,l)$ is a decomposition of $k+l$ finite points (k of which are "upper", l are "lower") into disjoint subsets, the <u>blocks</u>.
We represent them pictorially by:



$\in P(2,4)$ or



$\in P(4,1)$

The set of all partitions (i.e. the union of all $P(k,l)$) is denoted by P.

(b) Let $p \in P(k,l)$, $q \in P(k',l')$, then $p \otimes q \in P(k+k', l+l')$ is the partition obtained from placing p and q side by side, the "<u>tensor product</u>".

$\in P(6,5)$

(c) Let $p \in P(k,l)$, $q \in P(l,m)$, then the "<u>composition</u>" $qp \in P(k,m)$ is obtained from placing p above q and removing the l middle points. Certain <u>loops</u> may appear and they are removed.

$\in P(2,1)$

(d) Let $p \in P(k,l)$, then the "<u>involution</u>" $p^* \in P(l,k)$ is obtained from turning p upside down.

$\in P(1,4)$

(e) A subset $\mathcal{C} \subseteq P$ (consisting of subsets $\mathcal{C}(k,l) \subseteq P(k,l)$) is a <u>category of partitions</u>, if it is closed under taking tensor products, composition and involution, and if the "<u>identity partition</u>" $| \in P(1,1)$ and the "<u>pair partition</u>" $\sqcap \in P(0,2)$ are in $\mathcal{C}$.

4.2 <u>Example:</u> (a) P is a category.

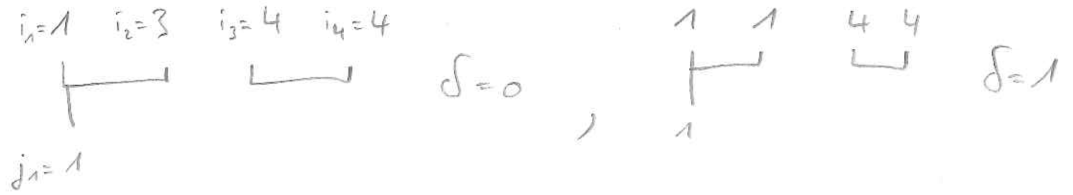
(b) The set NC of all "<u>noncrossing partitions</u>" (lines may be drawn in such a way that they do not cross) is a category.

(c) The set P_2 of all "<u>pair partitions</u>" (all blocks are of size two) is a category. Likewise NC_2, the noncrossing pair partitions.

$(k=0: (\mathbb{C}^n)^{\otimes k} = \mathbb{C})$

4.3 Definition: Let $p \in P(k,l)$. Define $T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ as the linear map given by $T_p(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1,\dots,j_l} \delta_p(i,j)\, e_{j_1} \otimes \dots \otimes e_{j_l}$, where $e_1, \dots, e_n$ is an ONB of $\mathbb{C}^n$, $i = (i_1, \dots, i_k)$, $j = (j_1, \dots, j_l)$ and

$$\begin{matrix} i_1 & \cdots & i_k \\ & p & \\ j_1 & \cdots & j_l \end{matrix} \rightsquigarrow$$ if the strings connect only equal indices with equal indices, then $\delta_p(i,j) = 1$. Otherwise $\delta_p(i,j) = 0$.

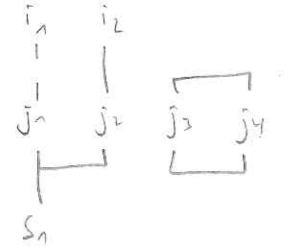


4.4 Example: Let $p = \times \in P(2,2)$. Then $T_p(e_{i_1} \otimes e_{i_2}) = e_{i_2} \otimes e_{i_1}$, the flip, since $\begin{matrix} i_1 & & i_2 \\ & \times & \\ j_1 & & j_2 \end{matrix} \Rightarrow i_1 = j_2,\ i_2 = j_1$. For $p = | \in P(1,1)$, we have $T_p = id_{\mathbb{C}^n}$.

4.5 Prop.: We have
(a) $T_p \otimes T_q = T_{p \otimes q}$
(b) $T_q T_p = n^{rl(q,p)}\, T_{qp}$
(c) $(T_p)^* = T_{p^*}$

Proof: Direct computations, but let's convince ourselves of (b) with the following example.

$T_q T_p (e_{i_1} \otimes e_{i_2}) = \sum_{j_1,j_2,j_3,j_4} \delta_p(i,j)\, T_q(e_{j_1} \otimes \dots \otimes e_{j_4}) = \sum_{s_1} \Big(\sum_{j_1,\dots,j_4} \delta_p(i,j)\, \delta_q(j,s) \Big) e_{s_1}$



$\delta_p(i,j) = \delta_q(j,s) = 1$ for $j_1 = j_2 = i_2 = s_1$
but n arbitrary choices for $j_3 (= j_4)$.
Hence $\sum_{j_1,\dots,j_4} \delta_p(i,j)\, \delta_q(j,s) = n\, \delta_{qp}(i,s)$

$\square$

~~**4.6 Prop.:** If $\mathcal{C} \subseteq P$ is a category of partitions, then $\mathrm{Span}\{T_p \mid p \in \mathcal{C}\}$ is a concrete monoidal W^*-category with $R := \{n^{\otimes k} \mid k \in \mathbb{N}_0\}$ generated by $\mathbb{C}^n$ with $\mathrm{Mor}(n^{\otimes k}, n^{\otimes l}) = \mathrm{Span}\{T_p \mid p \in \mathcal{C}\}$.~~

~~Proof: Use Prop. 4.5.~~ $\square$

$\nwarrow$ more detailed version in §3b

4.7 Prop.: To each category of partitions $\mathcal{C} \subseteq P$, we may associate a CMQG $G=(A,u)$ where A is the universal C^*-algebra generated by u_{ij}, $1 \le i,j \le n$ such that $T_p u^{\otimes k} = u^{\otimes l} T_p$ for all $p \in P(k,l)$ and $u_{ij} = u_{ij}^*$, i.e.

$$\forall s_1, \dots s_l, i_1, \dots i_k \in \{1,\dots n\}: \sum_{t_1,\dots t_k} \delta_p(t,s) u_{t_1 i_1} \dots u_{t_k i_k} = \sum_{j_1 \dots j_l} \delta_p(i,j) u_{s_1 j_1} \dots u_{s_l j_l}$$

(Replaced by 3.3b) $\quad$ ($\sum \delta_p \dots = \delta_p(0,\beta)$ if $k=0$)

In this sense $\quad P \leftrightarrow S_n \qquad P_2 \leftrightarrow O_n$

$\qquad\qquad NC \leftrightarrow S_n^+ \qquad NC_2 \leftrightarrow O_n^+$

Moreover $S_n \subseteq G \subseteq O_n^+$ for all CMQG G arising from a category $\mathcal{C}$.

Proof: By Tannaka-Krein for CQG, Th. 3.15, and using Prop. 4.6, we may find a CMQG $G'=(A',u')$ associated to $\mathrm{Span}\{T_p \mid p \in \mathcal{C}\}$. Since also (A,u) is a model for $\mathrm{Span}\{T_p \mid p \in \mathcal{C}\}$ since $T_p u^{\otimes k} = u^{\otimes l} T_p$, we have $A' \xrightarrow[u' \mapsto u]{} A$ by the universality of A'. But we also have $A \xrightarrow[u \mapsto u']{} A'$, since all relations of A are fulfilled in A' as $T_p u'^{\otimes k} = u'^{\otimes l} T_p$. Thus $A \cong A'$.

Let us now explicitly check what $T_p u^{\otimes k} = u^{\otimes l} T_p$ means:

As $u^{\otimes k} = \sum e_{\alpha_1 \beta_1} \otimes \dots \otimes e_{\alpha_k \beta_k} \otimes u_{\alpha_1 \beta_1} \dots u_{\alpha_k \beta_k} \in M_{n^k}(A)$, we see that $u^{\otimes k}$ acts on a vector $e_{i_1} \otimes \dots \otimes e_{i_k} \otimes 1$ by $u^{\otimes k}(e_{i_1} \otimes \dots \otimes e_{i_k} \otimes 1) = \sum_{\alpha_1 \dots \alpha_k} e_{\alpha_1} \otimes \dots \otimes e_{\alpha_k} \otimes u_{\alpha_1 i_1} \dots u_{\alpha_k i_k}$.

Thus $T_p u^{\otimes k}(e_{i_1} \otimes \dots \otimes e_{i_k} \otimes 1) = \sum_{t_1,\dots t_k} T_p(e_{t_1} \otimes \dots \otimes e_{t_k}) \otimes u_{t_1 i_1} \dots u_{t_k i_k} = \sum_{s_1,\dots s_l} e_{s_1} \otimes \dots \otimes e_{s_l} \otimes \Big(\sum_{t_1 \dots t_k} \delta_p(t,s) u_{t_1 i_1} \dots u_{t_k i_k}\Big)$

and $u^{\otimes l} T_p(e_{i_1} \otimes \dots \otimes e_{i_k} \otimes 1) = \sum_{j_1,\dots j_l} \delta_p(i,j) u^{\otimes l}(e_{j_1} \otimes \dots \otimes e_{j_l} \otimes 1) = \sum_{s_1,\dots s_l} e_{s_1} \otimes \dots \otimes e_{s_l} \otimes \Big(\sum_{j_1,\dots j_l} \delta_p(i,j) u_{s_1 j_1} \dots u_{s_l j_l}\Big)$

Now, let G be the CMQG associated to NC. Then, the relations for $\sqcap, \sqcup, \uparrow, \downarrow$, ⊓⊔ are fulfilled, i.e.

$\sqcap$: $\delta_{s_1 = s_2} = \delta_p(0, s_1 s_2) = \sum_{j_1 j_2} \delta_p(0, j_1 j_2) u_{s_1 j_1} u_{s_2 j_2} = \sum_{j_1} u_{s_1 j_1} u_{s_2 j_1} \iff u u^t = 1$

$\sqcup$: $u^t u = 1$

$P(0,1) \ni \uparrow$: $1 = \delta_p(0, s_1) = \sum_{j_1} \delta_p(0, j_1) u_{s_1 j_1} = \sum_{j_1} u_{s_1 j_1}$ and $\downarrow \in P(1,0)$ for $\sum_{j_1} u_{j_1 s_1} = 1$

⊓⊔: $\sum_{t_1 t_2} \delta_p(t_1 t_2, s_1 s_2) u_{t_1 i_1} u_{t_2 i_2} = \sum_{j_1 j_2} \delta_p(i_1 i_2, j_1 j_2) u_{s_1 j_1} u_{s_2 j_2} \iff u_{ik} u_{jk} = 0$, $u_{ki} u_{kj} = 0$ $\quad i \neq j$

$\qquad \shortparallel \qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\delta_{s_1 = s_2} u_{s_1 i_1} u_{s_1 i_2} \qquad \delta_{i_1 = i_2} u_{s_1 i_1} u_{s_2 i_1}$ $\quad$ Hence $u_{ij}^2 = \sum_k u_{ij} u_{ik} = u_{ij}$

$\Rightarrow$ we have $\mathcal{C}(S_n^+) \to \mathcal{C}(G)$.

But in $\mathcal{C}(S_n^+)$ all relations for $T_p u^{\otimes k} = u^{\otimes l} T_p$, $p \in NC$ are fulfilled (can be checked, see also §5), hence it is a model for this NC-category and by universality of G, we have $\mathcal{C}(G) \to \mathcal{C}(S_n^+)$, hence $\mathcal{C}(G) \cong \mathcal{C}(S_n^+)$.

Note that X yields $u_{ij}u_{kl} = \sum_{t_1,t_2} \delta_X(t_1t_2,(ki))\, u_{t_1j}u_{t_2l} = \sum_{j_1j_2} \delta_X((jl),(j_1j_2))\, u_{kj_1}u_{ij_2} = u_{kl}u_{ij}$

Thus $P \leftrightarrow S_n$. Now, since any category of partitions fulfills $NC_2 \subseteq \mathcal{C} \subseteq P$, we have $\mathcal{C}(S_n) \leftarrow \mathcal{C}(G) \leftarrow \mathcal{C}(O_n^+)$ and hence $S_n \subseteq G \subseteq O_n^+$. $\square$

4.8 Definition: [Banica, Speicher] A CMQG $S_n \subseteq G \subseteq O_n^+$ is called <u>easy</u>, if there is a category $\mathcal{C} \subseteq P$ such that $\mathrm{Hom}_G(k,l) := \{T \mid Tu^{\otimes k} = u^{\otimes l}T\} = \mathrm{span}\{T_p \mid p \in \mathcal{C}(k,l)\}$. It is <u>free</u>, if $\mathcal{C} \subseteq NC$.

4.9 Remark: Due to Tannaka-Krein, we have

$$\{\text{categories of partitions}\} \xleftrightarrow{1:1} \{\text{easy QG}\}$$

4.10 Remark: Use of easy QG.

(i) For QG:
- Easy QG's give rise to many new examples of CMQG linked to the „liberation business" and beyond. See §5.
- The philosophy of easy QG is: Everything should be visible in the combinatorics. This way, easy QG might help to get an understanding of general CMQG. This philosophy has been proven true in several examples, like in the Haar state (Weingarten calculus), a theorem by Raum-Weber 13, and the representation theory (fusion rules) by Freslon-Weber 13. (see §7)

(ii) For free probability: You can do free prob. on the easy QG (laws of characters) or use them as the right symmetries for distributions, for instance for de Finetti theorems (see §6).

Open: Find other spaces on which easy QG naturally act.

(iii) For von Neumann algebras: The von Neumann algebras associated to easy QG give rise to new examples of v. N. algebras which can be investigated. Those associated to O_n^+ seem to be close to $L F_n$ somehow. (§8)

Machine: Take a result for S_n or O_n (or U_n) and generalize it to S_n^+ or O_n^+. Then to all easy QG. Then to all CMQG.

4.11 Remark: The Haar state for $\mathcal{C} \subseteq NC$ is given by the Weingarten formula:

$$h(u_{i_1j_1} \cdots u_{i_kj_k}) = \sum_{p,q \in \mathcal{C}(0,k)} \delta_p(i)\delta_q(j) W_{kn}(p,q)$$

where $W_{kn} = (G_{kn}^{-1})$ inverse of the Gram matrix $G_{kn}(p,q) = n^{\text{blocks}(p \vee q)}$

and $p \vee q :=$ points i and j are in the same block if and only if they are in p or in q.

Lit.:
[25] Banica, Speicher, Liberation of orthogonal Lie groups, 2009
[26] Banica, Curran, Speicher, Stochastic aspects of easy QG, 2011
[27] Weber, On the classification of easy QG, 2013

§5 Classification of easy QG

Remark 4.10 motivates us to ask: How many easy QG are there, how well is this class? ~~There are~~ What can we say about a "map" of easy QG?

Lit.: [28] Banica, Curran, Speicher, Classification results for easy QG, 2010
[29] Raum, Weber, Easy QG and quantum subgroups of a semi-direct product QG, 2013
[30] Raum, Weber, The combinatorics of an algebraic class of easy QG, 2014
[31] Raum, Weber, The full classification of orthogonal easy QG, 2013

5.1 Lemma: A category $\mathcal{C} \subseteq P$ of partitions is closed under rotation, i.e. if $p \in P(k,l)$ is in $\mathcal{C}$, then also the partition obtained in $P(k-1, l+1)$ from shifting the very left upper point to the left of the lower points (it still belongs to the same block), and conversely. Likewise on the right hand side.

Example: $\sqcap\!\sqcup\sqcup \in \mathcal{C} \Rightarrow \sqcap\sqcup \in \mathcal{C} \Rightarrow \sqcap\!\sqcap \in \mathcal{C} \Rightarrow \sqcap\sqcap\sqcap \in \mathcal{C}$ etc.

Proof: Composition

does the job. □

5.2 Lemma: By $\langle p_1, \ldots, p_n \rangle$ we denote the smallest category containing $p_1, \ldots, p_n \in P$.
$NC_2 = \langle \emptyset \rangle$ (note that always $\sqcap, | \in \mathcal{C}$), $P_2 = \langle X \rangle$, $NC = \langle \uparrow, \sqcap\!\sqcap \rangle$, $P = \langle X, \uparrow, \sqcap\!\sqcap \rangle$

Proof: Let $p \in NC(0,n)$. Since p is noncrossing, there is at least one block of the form $\sqcap\!\sqcap\cdots\sqcap$. Removing it, we end up inductively again by such a block. If $p \in NC_2(0,n)$, then removing such a block $\sqcap$ (of consecutive points), we obtain a partition $p' \in \langle \emptyset \rangle$ by induction hypothesis. Composing it with $|^{\otimes \alpha} \otimes \sqcap \otimes |^{\otimes \beta}$ for suitable α, β, we obtain $p \in \langle \emptyset \rangle$. Thus $NC_2 = \langle \emptyset \rangle$.
As for $NC = \langle \uparrow, \sqcap\!\sqcap \rangle$, note that $\sqcap\!\sqcap \sqcap\!\sqcap = \sqcap\!\sqcap\!\sqcap$. Inductively, we obtain all one block partitions of even length in $\langle \uparrow, \sqcap\!\sqcap \rangle$. Since $\overbrace{\sqcap\!\sqcap\cdots\sqcap}^{2n} \circ (|\,|\,|\cdots|\,\downarrow) = \overbrace{\sqcap\!\sqcap\cdots\sqcap}^{2n-1}$, we obtain all one block partitions. By the inductive characterization of NC, we have $\langle \uparrow, \sqcap\!\sqcap \rangle = NC$. Now, using $|^{\otimes \alpha} \otimes X \otimes |^{\otimes \beta}$, we have all transpositions in a category whenever X is in there, thus we may arrange points arbitrarily. □

5.3 Example: Natural new examples of QG are ~~$\langle \uparrow \rangle$~~

$\langle \uparrow \rangle$: $C^*(u_{ij} \mid \sum_k u_{ik} = \sum_k u_{kj} = 1, u = \bar{u}, u^t u = u u^t = 1) \rightsquigarrow B_n^+$

$\langle \sqcap\!\sqcap \rangle$: $C^*(u_{ij} \mid u = \bar{u}, u u^t = u^t u = 1, u_{ik} u_{jk} = u_{ki} u_{kj} = 0, i \neq j) \rightsquigarrow H_n^+$

Actually, H_n^+ is the free QG version of $H_n = \mathbb{Z}_2 \wr S_n$ of Example 1.1.

5.4 Theorem: There are exactly seven free easy QG, i.e. seven categories $\mathcal{C} \subseteq NC$:

$$\begin{array}{ccccccc} \langle \uparrow \rangle & \supseteq & \langle |\sqcap\!\!\sqcap| \rangle & \supseteq & \langle \uparrow\otimes\uparrow \rangle & \supseteq & \langle \emptyset \rangle = NC_2 \\ \cap & & \cap & & & & \cap \\ \langle \uparrow, \sqcap\!\sqcap \rangle = NC & \supseteq & \langle \uparrow\otimes\uparrow, \sqcap\!\sqcap \rangle & & \supseteq & & \langle \sqcap\!\sqcap \rangle \end{array}$$

Proof: 1.) Let $\uparrow, \sqcap\!\sqcap \in \mathcal{C}$. Then $NC \overset{5.2}{=} \langle \uparrow, \sqcap\!\sqcap \rangle \subseteq \mathcal{C} \subseteq NC$, i.e. $\mathcal{C} = NC$.

2.) $\uparrow \in \mathcal{C}, \sqcap\!\sqcap \notin \mathcal{C}$. Then $\langle \uparrow \rangle \subseteq \mathcal{C}$. Now, all blocks of partitions in $\mathcal{C}$ are of length one or two (if $p \in \mathcal{C}$ has a block of length at least three, we use rotation such that $p \in P(0,n)$ and we use $\downarrow^{\otimes\alpha} \otimes 1 \otimes \downarrow^{\otimes\beta}$ in order to erase all points not belonging to that block and shortening it to $\sqcap\!\sqcap\!\sqcap \in \mathcal{C}$. Then $\sqcap\!\sqcap = \sqcap\!\sqcap\!\sqcap\,\sqcap\!\sqcap\!\sqcap \in \mathcal{C}$ ↯) and these partitions may be generated in $\langle \uparrow \rangle$.
$\Rightarrow \mathcal{C} = \langle \uparrow \rangle$

3.) $\uparrow \notin \mathcal{C}, \uparrow\otimes\uparrow \notin \mathcal{C}, \sqcap\!\sqcap \notin \mathcal{C}$. Then all blocks have size exactly two and $\mathcal{C} = NC_2$

4.) $\uparrow \notin \mathcal{C}, \uparrow\otimes\uparrow \notin \mathcal{C}, \sqcap\!\sqcap \in \mathcal{C}$. All blocks in $\mathcal{C}$ have even size and thus $\mathcal{C} = \langle \sqcap\!\sqcap \rangle$.

5.-7.) [...]

$\square$

5.5 Theorem: There are exactly six easy groups, i.e. categories with $\times \in \mathcal{C}$.

Proof: $\mathcal{C}_0 := \mathcal{C} \cap NC \subseteq NC$ is a category and $\mathcal{C} = \langle \mathcal{C}_0, \times \rangle$.

Now $\langle |\sqcap\!\!\sqcap|, \times \rangle = \langle \uparrow\otimes\uparrow, \times \rangle$ since $|\sqcap\!\!\sqcap| = \substack{||\sqcap \\ |\times|} \in \langle \uparrow\otimes\uparrow, \times \rangle$. $\square$

5.6 Theorem: We list all halfliberated easy QG, i.e. $\divideontimes \in \mathcal{C}$ but $\times \notin \mathcal{C}$:

$\langle \divideontimes \rangle$ (corresponding to O_n^* with $u_{ij}u_{kl}u_{mn} = u_{mn}u_{kl}u_{ij}$; $O_n \subseteq O_n^* \subseteq O_n^+$), $\langle \divideontimes, \uparrow\otimes\uparrow \rangle$, $\langle \divideontimes, \sqcap\!\sqcap \rangle$, $\langle \divideontimes, \sqcap\!\sqcap, h_s \rangle$, $s \geq 3$ for $h_s = \sqcap\!\sqcap\!\sqcap\!\sqcap \cdots \sqcap \in P(0,2s)$.
$\updownarrow H_n^{(s)}$

Proof: 1.) $\sqcap\!\sqcap \notin \mathcal{C}$. All blocks have length one or two. If $\uparrow\otimes\uparrow \notin \mathcal{C}$ they have length two and $\langle \divideontimes \rangle \subseteq \mathcal{C} \subseteq \langle \divideontimes \rangle$ by describing $\langle \divideontimes \rangle$ explicitly. Similar considerations for $\uparrow\otimes\uparrow \in \mathcal{C}$, 2.) $\sqcap\!\sqcap \in \mathcal{C}$ etc. $\square$

Note: If $|\sqcap\!\!\sqcap| \in \mathcal{C}$, then $\substack{\times| \\ \divideontimes \\ \times|} = |\times$ is in $\mathcal{C}$ which yields $\times \in \mathcal{C}$.

5.7 Definition: A category $\mathcal{C} \subseteq P$ is called <u>hyperoctahedral</u>, if $\sqcap\!\sqcap \in \mathcal{C}$, $\uparrow\otimes\uparrow \notin \mathcal{C}$.

5.8 Theorem: There are exactly 13 non-hyperoctahedral categories.

Proof: 1.) $\mathcal{C} \subseteq NC$: only $\langle \sqcap\!\sqcap \rangle$ is hyperoctahedral, the other six of Th. 5.4 are non-hyp.

2.) $\mathcal{C} \not\subseteq NC, \times \in \mathcal{C}$: only $\langle \times, \sqcap\!\sqcap \rangle$ is hyperoct., the other five are non-hyp.

3.) $\mathcal{C} \not\subseteq NC, \times \notin \mathcal{C}$: We have $\sqcap\!\sqcap \notin \mathcal{C}$ and $\divideontimes \in \mathcal{C}$, both by case studies of $\sqcap\ldots\sqcap\ldots\sqcap\ldots\sqcap\ldots$

$\left(\sqcap\!\sqcap \in \mathcal{C} \overset{\text{non-hyp.}}{\Rightarrow} \uparrow\otimes\uparrow \in \mathcal{C} \Rightarrow |\sqcap\!\!\sqcap| = \sqcap\!\sqcap\!\sqcap \in \mathcal{C} \Rightarrow \cdots \sqcap\!\!\sqcap = \sqcap\!\sqcap\!\sqcap \in \mathcal{C} \Rightarrow \times \in \mathcal{C} \text{ ↯} \right)$

(use the pair partition to infer that $\sqcap\ldots\sqcap\ldots$ is of length at most 8. Then case by case)

$\square$

5.9 <u>Theorem</u>: Let $\mathcal{C} \subseteq P$ be a category of partitions. Then either

(a) $\mathcal{C}$ is non-hyperoctahedral (and hence one of the 13 cases of 5.8)

(b) $\mathcal{C}$ is hyperoctahedral and $\not\in \mathcal{C}$ (then $\mathcal{C} = \langle \pi_k \rangle$ or $\mathcal{C} = \langle \pi_\ell, \ell \in \mathbb{N} \rangle$ for $\pi_k = \; = a_1 \dots a_k a_k \dots a_1 a_1 \dots a_k a_k \dots a_1$)

(c) $\mathcal{C}$ is hyperoctahedral and $\in \mathcal{C}$. Then we may associate a normal subgroup $F(\mathcal{C}) \subseteq \mathbb{Z}_2^{*\infty}$ (or rather $F_n(\mathcal{C}) \subseteq \mathbb{Z}_2^{*n}$ for all $n \in \mathbb{N}$) to $\mathcal{C}$ and the CMQG associated to $\mathcal{C}$ is $C^*\big(\underbrace{\mathbb{Z}_2^{*n} / F_n(\mathcal{C})}_{u_{g_1}, \dots u_{g_n}}\big) \otimes \underbrace{C(S_n)}_{v_{ij}}$ with $u = (u_{g_i} v_{ij})$

Proof: If $\mathcal{C}$ is hyperoctahedral, every partition $p \in \mathcal{C}$ is of even length and every block has size at least two (otherwise $p = \uparrow \otimes p_0$ up to rotation, i.e. $\uparrow \otimes p_0 \otimes \bar{p}_0 \otimes \uparrow \in \mathcal{C}$, $\uparrow \otimes \uparrow = |$)

Moreover, we may connect neighboring blocks by composing with $|^{\otimes \alpha} \otimes \; \otimes |^{\otimes \beta}$.

In the sequel, we view partitions $p \in P(0,n)$ as words $p = a\, b\, b\, a\, c\, d\, d\, a \dots$

(b) Partitions of the form $abaZ$ are not in $\mathcal{C}$, for some subword Z.
(otherwise $abaZ = abaZ_1 b Z_2$ and $(abaZ_1 b Z_2)(\bar{Z}_2 \bar{b} \bar{Z}_1 \bar{a} \bar{b} \bar{a}) \in \mathcal{C}$ yields $abaaba \in \mathcal{C}$, that is in a rotated version)

Thus, if $p = X_1 a X_2 a X_3 \in \mathcal{C}$ and b appears in X_2, it appears an even number of times. (Otherwise $p = X_1 a Z_1 b Z_2 b Z_3 b \dots b Z_{k+1} a X_3 \Rightarrow X_1' a b^k a X_3' \in \mathcal{C} \Rightarrow k$ even)
connect it to a; may assume $Z_i = \emptyset$; connect it to a

Hence p is of the form

colored Dyck path.

a b c c b d d a a e e b b a

color string: $\alpha\, \beta \quad \beta \quad \alpha\, \alpha \quad \beta\, \beta\, \alpha$ $\qquad$ nheight$(p) = 2$.

(if $(c_k, \dots, c_t)$ is a section of the color string, let $C_{[k,t]} = (d_1, \dots d_N)$ be all mutually different colors appearing an odd number of times in $(c_k, \dots c_{k+t})$.
nheight$(p) :=$ maximal length N such that $C_{[k_1,t_1]} = C_{[k_2,t_2]}$ for some $k_1 < t_1 < k_2 < t_2$)

Now (i) $p \in \mathcal{C}$, $p \in P(0,n)$ with nheight$(p) = k \Rightarrow \pi_k \in \mathcal{C}$
(ii) $p \in P(0,n)$ avoiding $abaZ$ etc., nheight$(p) \le k \Rightarrow p \in \langle \pi_k \rangle$

$\Rightarrow \mathcal{C} = \langle \pi_{\sup\{\text{nheight}(p) \mid p \in \mathcal{C}\}} \rangle = \{p \mid \text{nheight}(p) \le k\}$

(c) $\mathcal{C}_{se} := \{ p \in \mathcal{C} \mid p = a_{i_1} \dots a_{i_n},\ a_{i_j} \neq a_{i_{j+1}} \}$ $\quad \mathcal{C} = \langle \mathcal{C}_{se}, \cancel{\sqcap} \rangle$

(„2" ✓ „s" $p = a\underbrace{bb}b\,cad\underbrace{cc}a \Rightarrow a\ bcad\ a \in \langle \mathcal{C}_{se}, \cancel{\sqcap} \rangle$)

abbb adcca $\leftarrow$ using we are allowed to connect blocks

$F(\mathcal{C}) := \{ w(p,l) \in \mathbb{Z}_2^{*\infty} \mid w(p,l)$ is obtained from $p \in \mathcal{C}$ by labelling the blocks with $l = (a_1, a_2, \dots a_m) \}$

$F(\mathcal{C}) \subseteq \mathbb{Z}_2^{*\infty}$ subgroup $\left(w(p,l)^{-1} = w(p^*, l^{-1}),\ w(p,l)\,w(q,l') = w(p \otimes_\pi q, ll') \right)$
(maybe connecting blocks with $\cancel{\sqcap}$)

$F(\mathcal{C})$ normal ($a_k w(p,l) a_k = w(\ulcorner p \urcorner, l)$ maybe connect blocks to a_k)

$F(\mathcal{C})$ invariant under $\begin{cases} a_k \mapsto a_{i(k)} & 1 \le k \le n \\ a_k \mapsto a_k & k > n \end{cases}$ (use $\cancel{\sqcap\sqcap}$)

$F(\mathcal{C}) \subseteq F(\mathcal{D})$ for $\mathcal{C} \subseteq \mathcal{D}$

$H \subseteq \mathbb{Z}_2^{*\infty}$ normal subgroup, invar. under $\begin{cases} a_k \mapsto a_{i(k)} \\ a_k \mapsto a_k \end{cases}$ $\Rightarrow$ $\mathcal{C}_H := \{ p \in P \mid w(p,l) \in H \} \subseteq P$ (inverse to F; for some labelling)
hyperoct. category, $\cancel{\sqcap} \in \mathcal{C}$.

Thus $\{ \mathcal{C}$ hyp.oct., $\cancel{\sqcap} \in \mathcal{C} \} \longleftrightarrow \{$ normal subgroups of $\mathbb{Z}_2^{*\infty}$, invar. under relabell. of letters $\}$

$F_n(\mathcal{C}) :=$ use only n letters. $\left[\text{e.g. } \mathbb{Z}_2^{*n} / F_n(\langle \cancel{\sqcap} \rangle) = \mathbb{Z}_2^{*n},\quad \mathbb{Z}_2^{*n} / F_n(\langle h_s, \sqcap\sqcap \rangle) = \mathbb{Z}_2^{*n} / \langle (ab)^s = e \rangle \right]$
for $s=2$: $\mathbb{Z}_2^{\oplus n}$

~~$\mathcal{C}(G) \cong C^*(\mathbb{Z}_2^{*n} / F_n(\mathcal{C})) \rtimes \mathcal{C}(S_n)$~~

5.10 Remark: (a) Th. 5.9(c) holds true more generally: $G = (A, u)$ CMQG, $S_n \subseteq G \subseteq O_n^+$, u_{ij}^2 central projection (corresponds to $\cancel{\sqcap} \in \mathcal{C}$). Then $A \cong C^*(\Gamma) \otimes \mathcal{C}(S_n)$ where $\mathbb{Z}_2^{*n} \twoheadrightarrow \Gamma$.

(b) Th. 5.9(c) shows that the world of easy QG is very rich:
To a subset $W \subseteq \mathbb{F}_\infty$ we associate the variety of groups, i.e. all groups Γ such that $w(g_1, \dots g_n) = e \ \forall w \in W$.
Fact: The lattice of varieties of groups embeds into „fully characteristic subgroups of $\mathbb{F}_\infty$" ($\varphi(\Gamma) \subseteq \Gamma \ \forall \varphi \in \mathrm{End}(\mathbb{F}_\infty)$) and they embed into subgroups of $\mathbb{Z}_2^{*\infty}$ of our setting.
There are uncountably many varieties of groups. Thus the class of easy QG is huge.

(c) We conclude:

$S_n^+ \ \dots \subseteq \ \dots H_n^+ \ \dots \subseteq \ \dots O_n^+$	7 „free"	(NC)	
$\langle \sqcap_k \rangle$ $\boxed{}$		$(\langle \sqcap_k \rangle)$ $a^2c^2 = c^2a^2$	
$H_n^{[\infty]}$		$(\cancel{\sqcap})$ $a^2b = ba^2$	
$\hat{\Gamma} \rtimes S_n$ (Γ group) $\quad H_n^*$	O_n^*	half lib.	$(\times\!\!\!|)$ $abc = cba$
$S_n \ \subseteq \ H_n \ \subseteq \ O_n$	6 groups	$(\ddot{\times})$ $ab = ba$	

(c) $\mathcal{C}_{se} := \{ p \in \mathcal{C} \mid p = a_{i_1} \dots a_{i_k},\ a_{i_j} \neq a_{i_{j+1}} \}$ $\quad \mathcal{C} = \langle \mathcal{C}_{se}, \sqcap\!\!\!\!\times \rangle$

("$\supseteq$" ✓ "$\subseteq$" $p = a\underbrace{bb}b\,cad\underbrace{cc}a \Rightarrow a\ bcad\ a \in \langle \mathcal{C}_{se}, \sqcap\!\!\!\!\times \rangle$)

abbb $\quad$ adcca $\quad \leftarrow$ using we are allowed to connect blocks

$F(\mathcal{C}) := \{ w(p,l) \in \mathbb{Z}_2^{*\infty} \mid w(p,l)$ is obtained from $p \in \mathcal{C}$ by labelling the blocks with $l = (a_1, a_2, \dots a_m) \}$

$F(\mathcal{C}) \subseteq \mathbb{Z}_2^{*\infty}$ subgroup $\left(w(p,l)^{-1} = w(p^*, l^{-1}),\ w(p,l)\,w(q,l') = w(p \otimes_\pi q, ll') \right)$
(maybe connecting blocks with $\sqcap\!\!\!\!\times$)

$F(\mathcal{C})$ normal $\left(a_k\, w(p,l)\, a_k = w(\ulcorner p \urcorner, l) \right.$ maybe connect blocks to $a_k \left. \right)$

$F(\mathcal{C})$ invariant under $\begin{cases} a_k \mapsto a_{i(k)} & 1 \leq k \leq n \\ a_k \mapsto a_k & k > n \end{cases}$ (use)

$F(\mathcal{C}) \subseteq F(\mathcal{D})$ for $\mathcal{C} \subseteq \mathcal{D}$

$H \subseteq \mathbb{Z}_2^{*\infty}$ normal subgroup, invar. under $\begin{cases} a_k \mapsto a_{i(k)} \\ a_k \mapsto a_k \end{cases} \Rightarrow \mathcal{C}_H := \{ p \in P \mid w(p,l) \in H$ for some labelling $\} \subseteq P$
inverse to F
hyperoct. category, $\sqcap\!\!\!\!\times \in \mathcal{C}$.

Thus $\{ \mathcal{C}$ hyp.oct., $\sqcap\!\!\!\!\times \in \mathcal{C} \} \leftrightarrow \{$ normal subgroups of $\mathbb{Z}_2^{*\infty}$, invar. under identif. of letters $\}$

$F_n(\mathcal{C}) :=$ use only n letters. $\left[\text{e.g. } \frac{\mathbb{Z}_2^{*n}}{F_n(\langle \sqcap\!\!\!\!\times \rangle)} = \mathbb{Z}_2^{*n},\quad \frac{\mathbb{Z}_2^{*n}}{F_n(\langle h_s, \sqcap\!\sqcap \rangle)} = \frac{\mathbb{Z}_2^{*n}}{\langle (ab)^s = e \rangle} \right]$
for $s = 2$: $\mathbb{Z}_2^{\oplus n}$

L

~~Then $C(G) \cong C^*(\mathbb{Z}_2^{*n} / F_n(\mathcal{C})) \rtimes C(S_n)$~~

5.10 <u>Remark</u>: (a) Th. 5.9(c) holds true more generally: $G = (A, u)$ CMQG, $S_n \subseteq G \subseteq O_n^+$, u_{ij}^2 central projection (corresponds to $\sqcap\!\!\!\!\times \in \mathcal{C}$). Then $A \cong C^*(\Gamma) \otimes C(S_n)$ where $\mathbb{Z}_2^{*n} \twoheadrightarrow \Gamma$.

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There are uncountably many varieties of groups. Thus the class of easy QG is huge.

(c) We conclude:

$S_n^+ \ \dots \subseteq \dots\ H_n^+ \ \dots \subseteq \dots\ O_n^+$ $\quad$ 7 "free" $\quad$ (NC)

$\langle \sqcap_k \rangle \quad$ $(\langle \sqcap_k \rangle)\ a^2b^2 = b^2a^2$

$H_n^{[\infty]}$ $\quad$ $(\sqcap\!\!\!\!\times)\ a^2b = ba^2$

$\hat{\Gamma} \rtimes S_n$ $\quad$ H_n^* $\quad$ O_n^* $\quad$ halfl. $\quad$ $(\times\!\!\!\!|)\ abc = cba$

Γ group

$S_n \quad \subseteq \quad H_n \quad \subseteq \quad O_n$ $\quad$ 6 groups $\quad$ $(\ddot{\times})\ ab = ba$